Two questions on nodal sets

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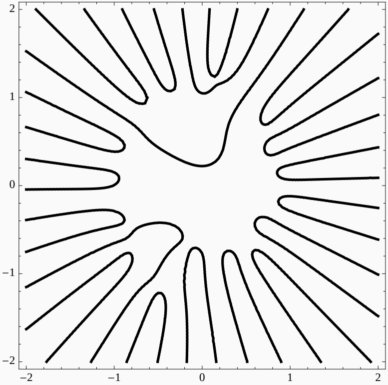
Part-1

A random non-crossing matching from random polynomials

Zeros of the real part of Kac polynomial

Kac polynomial

$P_n(z) = a_0 + a_1 z + \ldots + a_n z^n$ where a_k are i.i.d. $N(0,1)$ random variables. Let $\mathcal{Z}$ be the zero set of $\Re(P_n(z))$.



- $\mathcal{Z}$ consists of n disjoint curves (there is no z such that $\Re(P_n)$ and $\nabla\Re(P_n)$ vanish simultaneously).
- For large $|z|$ we have $\Re(P_n(z)) \sim a_n r^n \cos(n\theta)$. Hence the rays $\theta = \frac{\pi k}{n}$, $0 \leq k \leq 2n-1$ are asymptotically close to the zeros.
- The topology of the zero set can be captured as a non-crossing matching or as a random tree.
- Question: What is the structure of this random tree? In particular, how large is $\mathcal{D}_n$, the depth of the tree.

- Let $Z_{k,n}$ be the number of positive real roots of

$$r \mapsto \sum_{\ell=0}^{n} a_\ell \cos(\ell\theta_k)\, r^\ell, \quad \text{where } \theta_k = \frac{\pi}{n}(k+\frac{1}{2}).$$

Then, $\mathcal{D}_n \leq \max_{0 \leq k \leq 2n-1} Z_{k,n}$.

- $Z_{k,n}$ is like the number of positive roots of a Kac polynomial, hence about $\log n$. More precisely, one can prove that

$$\mathbb{P}\{Z_{k,n} \geq b \log n\} \leq n^{-C_b}, \quad \text{where } C_b \to \infty \text{ as } b \to \infty.$$

- Take b large enough so that $C_b > 3$. Then $\mathcal{D}_n \leq b \log n$ with probability at least $1 - \frac{1}{n^2}$.

- Then, $\mathbb{E}[\mathcal{D}_n] \lesssim \log n$.

Question

$\mathbb{E}[\mathcal{D}_n] \gtrsim \log n$?

- Analogous question for Weyl polynomials:

$$Q_n(z) = \sum_{k=0}^{n} a_k \frac{z^k}{\sqrt{k!}}.$$

One can show $\mathbb{E}[\mathcal{D}_n] \lesssim \sqrt{n}$.

Question

Show that $\mathbb{E}[\mathcal{D}_n] \gtrsim \sqrt{n}$.

- Random trees naturally occurring in probability tend to have depth of order $\sqrt{n}$. Eg. Uniform random tree with vertex set $\{1, 2, \ldots, n\}$.

- Possible source of difficulty: Lack of semi-locality.

Part-2

Discrete nodal length

Discrete nodal length

- Let $G = (V, E)$ be a finite graph. Its Laplacian is the $V \times V$ matrix

$$L(i,j) = \begin{cases} -1 & \text{if } i \sim j, \\ \deg(i) & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

- This is the correct analogue of continuous Laplacian. For example, for $f : V \mapsto \mathbb{R}$,

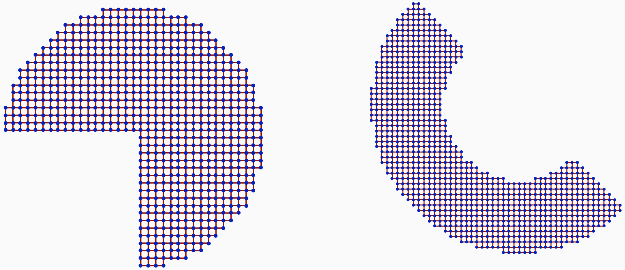
$$\langle Lf, f \rangle = \sum_{i \sim j} (f(i) - f(j))^2.$$

This is analogous to $\langle -\Delta f, f \rangle = \int \|\nabla f\|^2$.

- Let $0 = \lambda_1 \leq \ldots \leq \lambda_n$ be the eigenvalues of L and let $f_1, \ldots, f_n$ be the corresponding eigenvectors.
- For $f : V \mapsto \mathbb{R}$, define its *discrete nodal length*

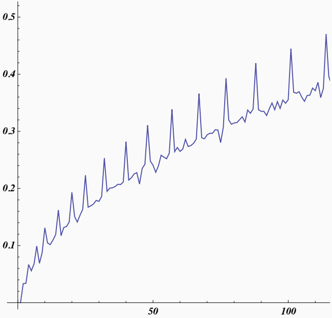
$$S(f) = \frac{1}{n} \#\{(i,j) : i \sim j \text{ and } f(i)f(j) < 0\}.$$

- In what follows, fix a bounded region $\Omega \subseteq \mathbb{R}^2$ and let G_n be the subgraph $\frac{1}{n}\mathbb{Z}^2 \cap \Omega$.



Discrete nodal length

- In such examples, we expect $S(f_k) \asymp \sqrt{\lambda_k}$ (*Yau conjecture*).



- Reason: $n^2 \lambda_k(G_n) \to \lambda_k$ (with appropriate boundary condition) and similarly for eigenfunctions...

- But this is only for fixed k. For $k = k_n$ growing with n, there is no analogue in the continuum. But in discrete situation we can consider all eigenfunctions.

Plots of $S(f_k)$ versus λ_k

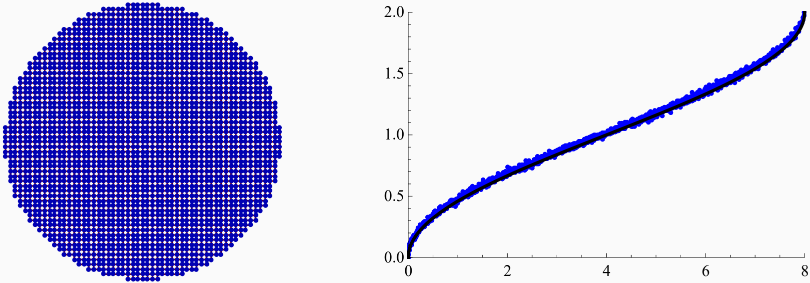


Figure: A discrete disk with nearly 2500 vertices. The black curve is $\frac{4}{\pi}\arcsin(\sqrt{x/8})$.

Relationship to Yau conjecture: $\arcsin(\sqrt{x}) \sim \sqrt{x}$ as $x \to 0$.

Plots of $S(f_k)$ versus λ_k

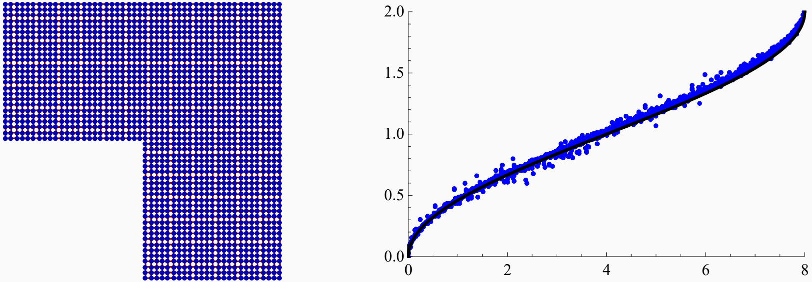


Figure: L-shaped region with nearly 1875 vertices. The black curve is $\frac{4}{\pi}\arcsin(\sqrt{x/8})$.

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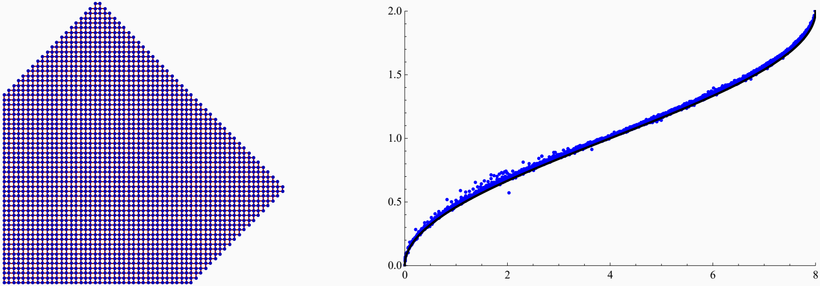

Figure: A pentagonal region with nearly 2300 vertices. The black curve is $\frac{4}{\pi}\arcsin(\sqrt{x/8})$.

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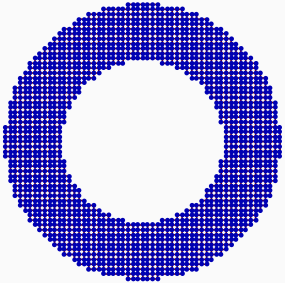

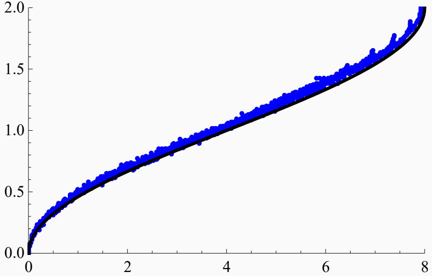

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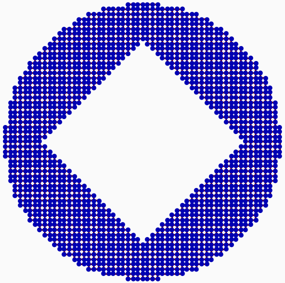

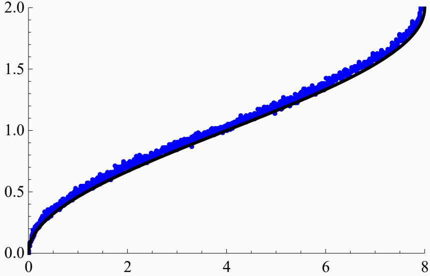

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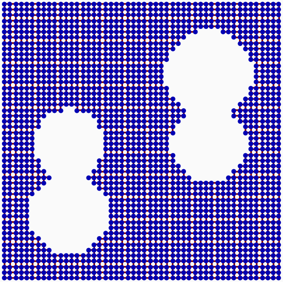

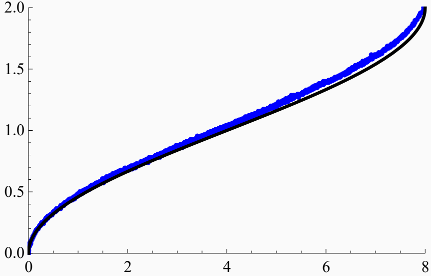

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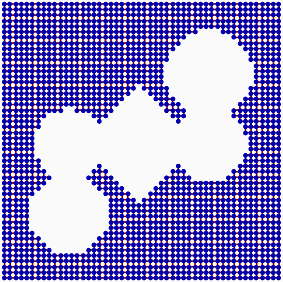

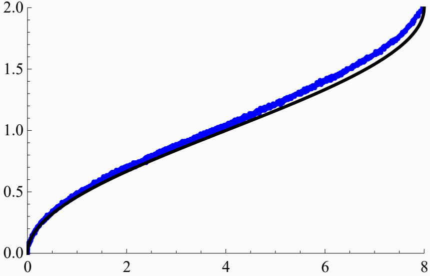

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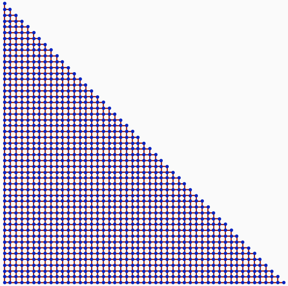
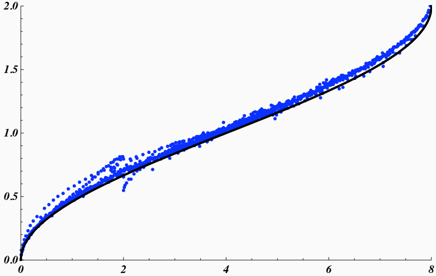




Explicit case of the square

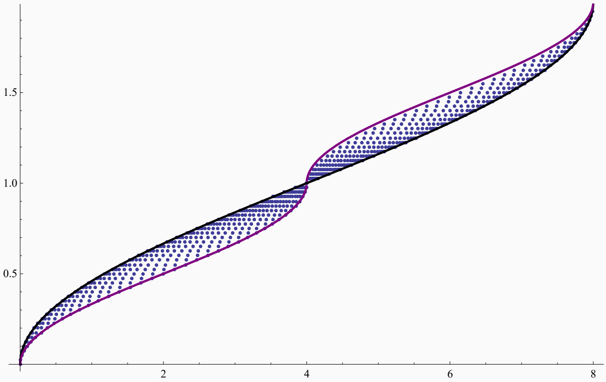
Let G_n the subgraph of $\mathbb{Z}^2$ with vertex set $[n] \times [n]$. With slight modification (identify opposite sides), we have eigenvalues and eigenfunctions for $0 \leq k, \ell \leq n-1$.

$$\lambda_{k,\ell} = 4\sin^2\left(\frac{\pi k}{n}\right) + 4\sin^2\left(\frac{\pi \ell}{n}\right),$$
$$\varphi_{k,\ell}(p,q) = \cos\left(\frac{2\pi k}{n}p\right)\cos\left(\frac{2\pi \ell}{n}q\right).$$
$$S(\varphi_{k,\ell}) = 2\frac{k}{n} + 2\frac{\ell}{n}.$$

Simple calculation: If $4[\sin^2(\pi x) + \sin^2(\pi y)] = t$, then

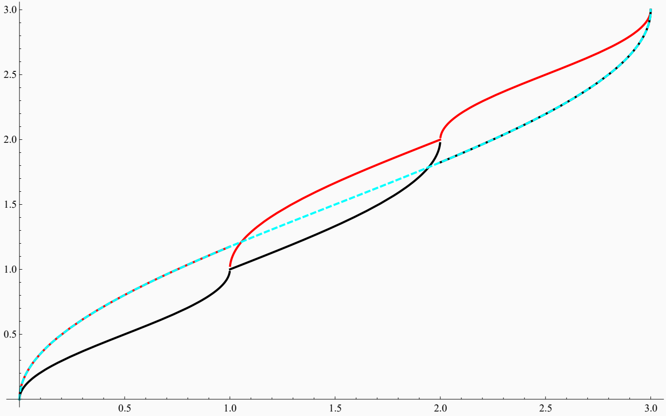
$$\frac{1}{\pi}\arcsin(\sqrt{t/4}) \leq 2(x+y) \leq \frac{2}{\pi}\arcsin(\sqrt{t/8}) \quad \text{if } 0 \leq t \leq 4,$$
$$\frac{2}{\pi}\arcsin(\sqrt{t/8}) \leq 2(x+y) \leq 1 + \frac{1}{\pi}\arcsin(\sqrt{(t-4)/4}) \quad \text{if } 4 \leq t \leq 8.$$

Explicit case of the square



In this case, there is no curve describing $S(\varphi_\lambda)$ as a function of λ but only these envelopes (the entire region between the curves is filled.

In three dimensions



Questions?

- Generically, does $S(\varphi_\lambda)$ follow the curve $\frac{2}{\pi}\arcsin(\sqrt{\lambda/8})$? If so, in what sense? (analogy with random matrix).
- If not, does it fill out a region bounded by arcsin curves?
- In higher dimensions, a similar picture (one or more arcsine curves) is seen, but not in random graphs.