Level repulsion for arithmetic toral point scatterers

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Random Waves in Oxford
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Brief intro to quantum chaos

Detect classical integrability vs chaos in terms of spectral properties of "quantized Hamiltonians".
Simple example setup:

- Classical dynamics given by "billiards" (geodesic flow) on compact manifold M.
- "Quantized Hamiltonian": Laplacian $-\Delta$ acting on $L^2(M)$.

With

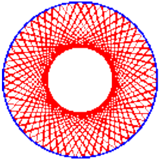
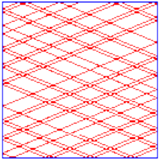
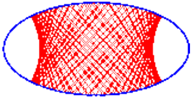
$$-\Delta\psi_i = \lambda_i\psi_i$$

what can we say about

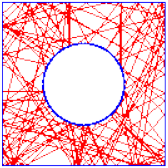
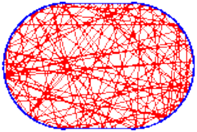
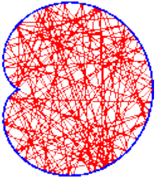
- Eigenvalues — in particular gaps between them?
- (Eigenfunctions ψ_i? Quantum ergodicity etc.)

Examples: billiards with classical integrability/chaos

Integrable = simple; trajectories have structure:

Chaos — particle bounces "everywhere, from every direction":

Tell difference by looking at gaps between eigenvalues ("spacing distribution".)

The universality conjecture

- Spacing distribution: first **order** eigenvalues such that

$$\lambda_1 \leq \lambda_2 \leq \dots.$$

Define spacing density function $P(s)$ (if exists) so that

$$\lim_{N\to\infty} \frac{|\{\lambda_i \leq N : \lambda_{i+1} - \lambda_i \in (a,b)\}|}{|\{\lambda_i \leq N\}|} = \int_a^b P(s)\, ds$$

Remark: implicit rescaling so that $|\{\lambda_i \leq N\}| \sim N$.

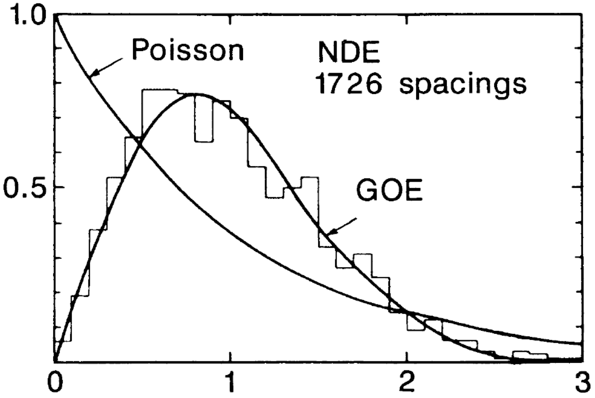
- The spacing statistics (generically) falls into two classes.
  - Berry-Tabor: If the classical system is integrable, the spacing statistics are Poissonian ("random")

$$P(s) = e^{-s}.$$

  - Bohigas-Giannoni-Schmit: If the classical system is chaotic, the spacing statistics are given by random matrix theory. "Nonrandom", eigenvalues "repel":

$$P(s) \approx \frac{\pi}{2} s \cdot \exp(-\frac{\pi}{4} s^2)$$

Poisson vs RMT (GOE)



(NDE: "nuclear data ensemble", experimental data for neutron absorbtion in heavy atomic nuclei.)

Systems with intermediate statistics

To study transition between integrability and chaos, Šeba proposed "perturburbing Laplacian by delta potential at x_0":

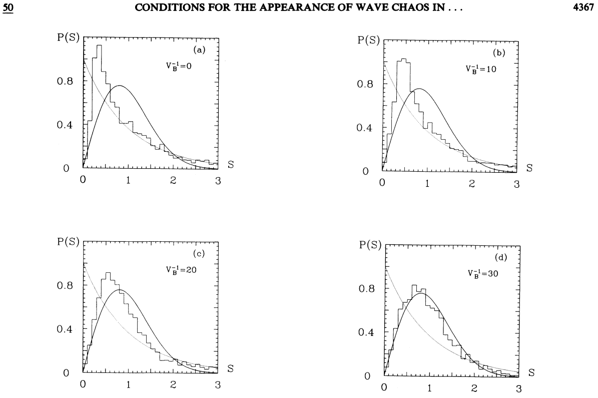
$$H := -\Delta + \alpha \delta_{x_0}$$

on rectangles with Dirichlet boundary conditions.

- View as "Sinai billiard with shrinking obstacle".
- Mathematical setup: von Neumann theory of self adjoint extensions.
  - Roughly, let Δ act on smooth functions vanishing at x_0, then find self adjoint extension.
  - "Old" eigenfunctions: regular Laplace eigenfunctions vanishing at x_0.
  - "New" eigenfunctions: given by Green's function (have singularity at x_0.)
  - One parameter family of extensions (cf. α), roughly giving "strength" of perturbation.
- Model **appears** to have level repulsion.

Some results

- Shigehara later found that level repulsion is quite subtle, need to carefully adjust parameters with λ.



Some results

- Shigehara later found that level repulsion is quite subtle, need to carefully adjust parameters with λ.
- Shigehara-Cheon: for $d = 3$ this issue goes away.
- Bogomolny-Gerland-Schmit: obtained level repulsion and Poisson tails **provided** the unperturbed spectrum has Poisson spacings. Subtle points:
  - For Dirichlet boundary conditions (and x_0 "generic"), get $P(s) \sim s \log^4 s$
  - For periodic boundary condition, get $P(s) \sim (\pi\sqrt{3}/2)s$ for small s. (**Not** the GOE constant!)

We'll consider $d = 3$ with periodic boundary conditions, i.e.,

$$H := -\Delta + \alpha\delta_{x_0}$$

acting on $3d$ tori. WLOG, from now on $x_0 = 0$.

Toral point scatterers

Work with "arithmetic torus": $\mathbb{T} = \mathbb{R}^3/2\pi\mathbb{Z}^3$.
Before perturbation: $-\Delta$ acts on $L^2(\mathbb{T})$, spectrum given by

$$S := \{m \in \mathbb{Z} : m = a^2 + b^2 + c^2, a, b, c \in \mathbb{Z}\}$$

Eigenspace decomposition:

$$L^2(\mathbb{T}) = \oplus_{m \in S} V_m$$

where (multiplicities!)

$$\dim(V_m) = r_3(m) := |\{v \in \mathbb{Z}^3 : |v|^2 = m\}|$$

Toral point scatterer — introduce perturbation:

$$H = -\Delta + \alpha \cdot \delta, \quad \alpha \in \mathbb{R}^\times$$

where δ is Dirac delta supported (say) at $x = 0 \in \mathbb{T}$.
Again α is parameter controlling "strength" of perturbation.

"New" vs "old" eigenvalues

Perturbation "tiny" (rank one); V_m splits into two eigenspaces:

- Functions vanishing at δ-support remain eigenfunctions ("boring"):

$$V_m^{\text{old}} := \{\psi \in V_m : \psi(0) = 0\}$$

- Each V_m also "gives birth" to

$$V_m^{\text{new}} = \text{Span}(\psi_{\lambda_m}^{\text{new}})$$

with the "new" eigenvalue λ_m being a solution of

$$\sum_{n \in S} r_3(n) \left(\frac{1}{n - \lambda_m} - \frac{n}{n^2 + 1} \right) = 0$$

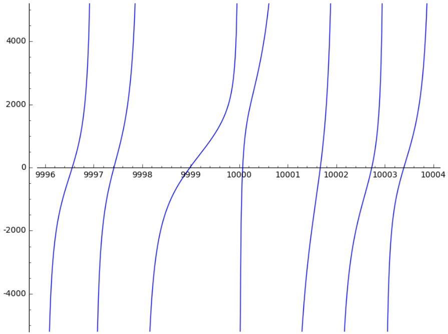
and the "new" eigenfunction is given by Green's function:

$$\psi_{\lambda_m}^{\text{new}} := \sum_{v \in \mathbb{Z}^3} \frac{e^{i\langle v, x\rangle}}{|v|^2 - \lambda_m}, \quad x \in \mathbb{T}$$

Spectral equation

New eigenvalues are solutions of

$$G(\lambda) := \sum_n r_3(n) \left(\frac{1}{n-\lambda} - \frac{n}{n^2+1} \right) = 0$$



Note: new eigenvalues **interlace** with the unperturbed eigenvalues.

Some convenient notation

- If $r_3(m) > 0$, let λ_m denote largest solution to $G(\lambda) = 0$ such that $\lambda < m$.
- Caveat: λ_m is **not** the m-th eigenvalue.
- Weyl's law for new eigenvalues:

$$|\{m < T : r_3(m) > 0\}| \sim \frac{5}{6} T$$

  so won't bother with rescaling.
- Given m such that $r_3(m) > 0$, let m_+ denote smallest $n > m$ such that $r_3(n) > 0$ — "nearest right neighboor in unperturbed spectrum". Similarly, let m_- denote left neighboor.
- Define associated spacing

$$s_m = \lambda_{m_+} - \lambda_m$$

Repulsion results for this model

Repulsion between new and old eigenvalues

Theorem (Rudnick-Ueberschär)

$$\sum_{m \leq T} (\lambda_{m_+} - m) \sim \frac{1}{2} \sum_{m \leq T} (m_+ - m)$$

Remarks:

- Result holds for any $3d$ tori.
- However, cannot rule out s_m alternating between $o(1)$ and $m - m_- - o(1)$.
  - If so, get $P(s) = \frac{1}{2}\delta_0(s) + ?(s)$, i.e., lots of mass at $s = 0$ — no level repulsion.

Main result

Can't prove that the spacing distribution (between new eigenvalues) exists, but... can show that small gaps are very rare.

Theorem

Given any small $\gamma > 0$, we have

$$\frac{|\{m \leq T : r_3(m) > 0, s_m < \epsilon\}|}{|\{m \leq T : r_3(m) > 0\}|} = O_\gamma(\epsilon^{4-\gamma})$$

as $T \to \infty$ (and $\epsilon > 0$ small.)

Upshot: result suggests essentially cubic order repulsion

$$P(s) \sim s^{3-\gamma}, \quad s \to 0$$

What is the truth?

Some numerics

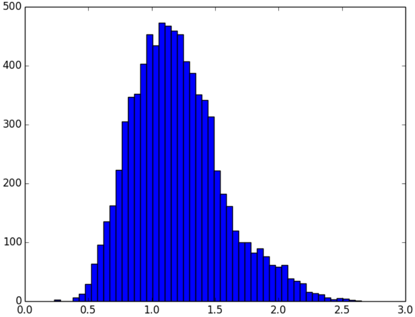
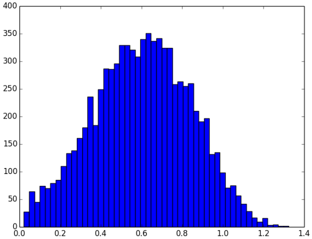


Figure: Histogram illustration of the distribution of s_m, for $m \leq 10000$ (and $r_3(m) > 0$.)

Suggests $P(s) = s^\infty$ for s small (!?!?)
Turns out: most small s_m "comes from" m such that $4^l | m$ and l "big".

Gaps along $m = 4^l k$



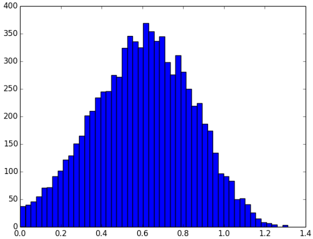


Figure: Histogram illustration of the distribution of s_m (for m such that $r_3(m) > 0$), along the progressions $\{m = 4^{10} \cdot k : k \leq 10000\}$ (left) and for $\{m = 4^{20} \cdot k : k \leq 10000\}$ (right).

Possibly $P(s) \sim s^4$ is the truth.

Proof idea

Put $\delta = \lambda - m$ and rewrite

$$G(\lambda) := \sum_n r_3(n) \left(\frac{1}{n-\lambda} - \frac{n}{n^2+1} \right) = 0$$

as

$$\frac{r_3(m)}{\delta} = G_m(\delta) + \text{error}$$

where

$$G_m(\delta) := \sum_{0<|n-m|\leq m^{1/2}} \frac{r_3(n)}{n-m-\delta}$$

Enough to consider $|\delta| < 1/10$, let's assume that $-1/10 < \delta_1 < 0 < \delta_2 < 1/10$ are two nearby solutions. Simple lemma: if $G_m'(\delta) \leq B_m$ for $|\delta| < 1/10$, then

$$s_m = \delta_2 - \delta_1 > \sqrt{r_3(m)/B_m}$$

Get repulsion if $r_3(m)$ tiny, or B_m big, is rare.

Bounding the derivative via L-functions

Easy:

$$B_m \ll \sum_{k \neq 0} \frac{r_3(m+k)}{k^2}$$

What do we know about $r_3(n)$?

- "Morally", $r_3(n) \sim \sqrt{n}$ (but: Siegel zero issue!)
- Also: powers of 2 obstruction: $r_3(n) = 0$ if $n \equiv 7 \mod 8$. Further, $r_3(4^l n) = r_3(n)$ so r_3 can be quite small if nonzero.
- Amazing formula (by Gauss): If n is squarefree and $n \not\equiv 7 \mod 8$, then

$$r_3(n) \sim \sqrt{n} L(1, \chi_{-4n})$$

Granville-Soundararajan: large (and small) values $L(1, \chi)$ extremely rare:

$$|\{d : d < T : L(1, \chi_{-d}) > x\}| \ll T e^{-e^{cx}}$$

Easy consequence: for any $k > 0$,

$$|\{m < T : B_m > \sqrt{m} x\}| \ll_k T/x^k$$

Proof sketch, cont'd

- Upshot of derivative bound: $B_m \ll \sqrt{m}$ "always holds" (don't have to worry about large B_m-values.)
- Recall separation lemma

$$s_m = \delta_2 - \delta_1 > \sqrt{r_3(m)/B_m}$$

so done if $r_3(m)/\sqrt{m}$ not too small. Two "enemies":

  - Siegel zeros. Very rare!
  - Powers of four: $r_3(4^l m_0) = r_3(m_0)$. But m's divisible by large powers of four also very rare.
- Upshot: almost all m such that $s_m < \epsilon$ "come from" m divisible by 4^l and $4^l \sim (1/\epsilon)^4$.
- Only about proportion ϵ^4 of these.

The end

Thank you!

Gauss' amazing formula

If n is squarefree and $n \not\equiv 7 \mod 8$, then

$$r_3(n) = \pi^{-1} \mu_n \sqrt{n} L(1, \chi_{-4n})$$

where $\mu_n = 16$ for $n \equiv 3 \mod 8$, and $\mu_n = 24$ for $n \equiv 1, 2, 5, 6 \mod 8$;

$$L(1, \chi_{-4n}) = \sum_{m=1}^{\infty} \chi_{-4n}(m)/m$$

where χ_{-4n} is defined via the Kronecker symbol, namely

$$\chi_{-4n}(m) := \left(\frac{-4n}{m} \right).$$

Remark: $\sqrt{n}L(1, \chi_{-4n})$ is essentially a class number (of an imaginary quadratic field). Formula is amazing relation between $r_3(n)$ (the number of ways to express n in terms of the ternary form $x^2 + y^2 + z^2$) and the *number* of classes of binary quadratic forms.