The Nazarov-Sodin constant and critical points of Gaussian fields

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Random Waves in Oxford, June 2018

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Preliminaries

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Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a stationary Gaussian field with zero-mean, unit variance and covariance function $\kappa : \mathbb{R}^2 \to [-1, 1]$ and spectral measure ρ, i.e. for $x, y \in \mathbb{R}^2$

$$\kappa(x) = \mathbb{E}(f(y)f(y+x)) = \int_{\mathbb{R}^2} e^{it\cdot x} d\rho(t)$$

Basic assumptions:

1. $\kappa \in C^{4+}(\mathbb{R}^2)$ (which implies $f \in C^{2+}(\mathbb{R}^2)$ a.s.)
2. $\nabla^2 f(0)$ is a non-degenerate Gaussian vector

We are interested in the geometry of the level sets

$$\{f = \ell\} := \{x \in \mathbb{R}^2 \mid f(x) = \ell\}$$

for $\ell \in \mathbb{R}$.

Previous results

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For $\Omega \subset \mathbb{R}^2$ let $N_{LS}(\ell, \Omega)$ be the number of components of $\{f = \ell\}$ in Ω.

Theorem (Nazarov-Sodin 2016)

If f is ergodic then there exists $c_{NS}(\rho) \geq 0$ such that

$$N_{LS}(0, R \cdot \Omega)/(Area(\Omega) R^2) \to c_{NS}(\rho)$$

a.s. and in L^1.

Theorem (Kurlberg-Wigman 2018)

If ρ has compact support then there exists $c_{NS}(\rho) \geq 0$ such that

$$\mathbb{E}(N_{LS}(0, [0, R]^2)) = c_{NS}(\rho) \, R^2 + O(R)$$

Moreover $c_{NS}(\rho)$ is continuous in ρ (w.r.t. the w^*-topology).

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Moreover $c_{NS}(\rho, \ell)$ is continuous in ρ (w.r.t. the w^*-topology) for each $\ell \in \mathbb{R}$.

Previous results

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For $\Omega \subset \mathbb{R}^2$ let $N_{ES}(\ell, \Omega)$ be the number of components of $\{f \geq \ell\}$ in Ω.

Theorem (Nazarov-Sodin 2016)

If f is ergodic then there exists $c_{ES}(\rho, \ell) \geq 0$ such that

$$N_{ES}(\ell, R \cdot \Omega)/(Area(\Omega)R^2) \to c_{ES}(\rho, \ell)$$

a.s. and in L^1.

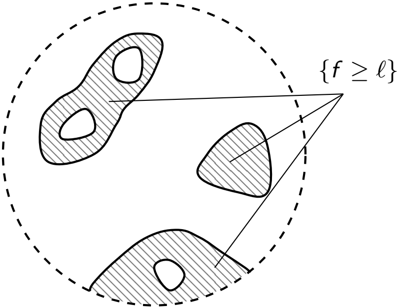
Theorem (Kurlberg-Wigman 2018)

If ρ has compact support then there exists $c_{ES}(\rho, \ell) \geq 0$ such that

$$\mathbb{E}(N_{ES}(\ell, [0, R]^2)) = c_{ES}(\rho, \ell)\, R^2 + O(R)$$

Moreover $c_{ES}(\rho, \ell)$ is continuous in ρ (w.r.t. the w^*-topology) for each $\ell \in \mathbb{R}$.

Level sets and excursion sets



$$\begin{aligned}\#\{\text{Components of } \{f = \ell\}\} \approx & \#\{\text{Components of } \{f \geq \ell\}\} \\ & + \#\{\text{Components of } \{f \leq \ell\}\}\end{aligned}$$

Corollary

$$c_{NS}(\rho, \ell) = c_{ES}(\rho, \ell) + c_{ES}(\rho, -\ell)$$

Critical points

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Definition

If f is aperiodic we say that a saddle point x is *lower connected* if it is in the closure of only one component of $\{f < \ell\}$. We say that x is *upper connected* if it is in the closure of only one component of $\{f > \ell\}$.

(When f is periodic, we use a different definition for lower/upper connected saddles.)

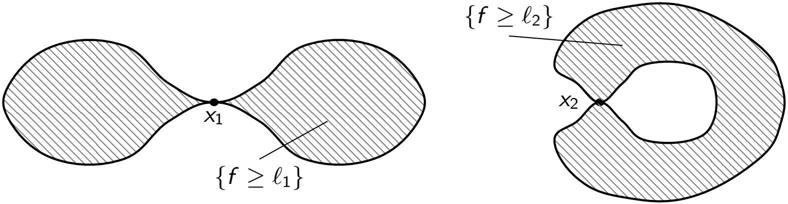

Figure: x_1 is a lower connected saddle and x_2 is an upper connected saddle.

Proposition

Let f satisfy the basic assumptions. There exists a function $p_{s^-} : \mathbb{R} \to [0, \infty)$ such that the following holds. Let $\Omega \subset \mathbb{R}^2$ and let $N_{s^-}[\ell, \infty)$ denote the number of lower connected saddles of f in Ω with level above ℓ. Then

$$\mathbb{E}[N_{s^-}[\ell, \infty)] = \text{Area}(\Omega) \int_{\ell}^{\infty} p_{s^-}(x)\, dx.$$

Analogous statements hold for local maxima, local minima, upper connected saddles and saddles with the densities p_{m^+}, p_{m^-}, p_{s^+} and p_s respectively. These functions can be chosen to satisfy $p_{s^-} + p_{s^+} = p_s$, and such that p_{m^+}, p_{m^-} and p_s are continuous.

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Theorem

Let f be a Gaussian field satisfying the basic assumptions, and let p_{m^+}, p_{m^-}, p_{s^+}, p_{s^-} denote the critical point densities defined above. Then

$$c_{NS}(\rho, \ell) = \int_{\ell}^{\infty} p_{m^+}(x) - p_{s^-}(x) + p_{s^+}(x) - p_{m^-}(x)\, dx \tag{1}$$

$$c_{ES}(\rho, \ell) = \int_{\ell}^{\infty} p_{m^+}(x) - p_{s^-}(x)\, dx \tag{2}$$

and hence c_{NS} and c_{ES} are absolutely continuous in ℓ. In addition c_{NS} and c_{ES} are jointly continuous in (ρ, ℓ) provided ρ has a fixed compact support.

Proof: Intuition
Local extrema

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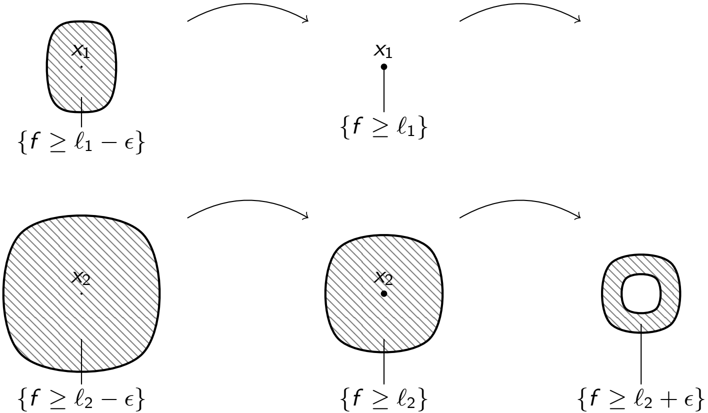


Figure: On raising the level through the local maximum x_1, the number of level set components decreases by one. On passing through the local minimum x_2, the number of level set components increases by one.

Proof: Intuition

Lower connected saddle points

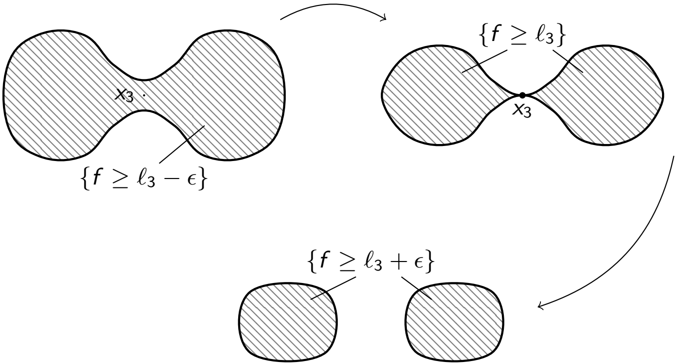


Figure: On raising the level through the lower connected saddle point x_3, the number of level set components increases by one.

Proposition (Cheng-Schwartzman 2017)

Let f be the random plane wave (RPW) so that $\kappa(t) = J_0(|t|)$ (the 0-th Bessel function), then

$$p_{m^+}(x) = p_{m^-}(-x) = \frac{1}{4\sqrt{2}\pi^{3/2}} \left((x^2-1)e^{-\frac{x^2}{2}} + e^{-\frac{3x^2}{2}} \right) \mathbb{1}_{x\geq 0}$$

$$p_s(x) = \frac{1}{4\sqrt{2}\pi^{3/2}} e^{-\frac{3x^2}{2}}.$$

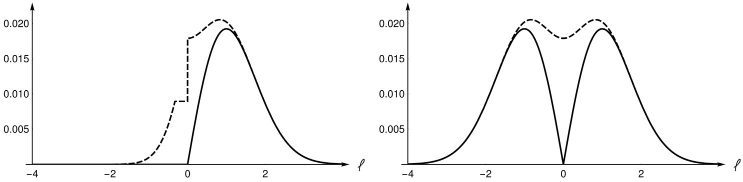
Substituting these expressions into the main integral equality and considering the number of ‘flip points’ (see Kurlberg-Wigman 2018) shows that

Corollary

Let f be the RPW and $\ell \geq 0$, then

$$\frac{1}{4\pi}\ell\,\phi(\ell) \leq c_{ES}(\ell) \leq c_{NS}(\ell) \leq \frac{1}{4\pi}\,\phi(\ell)\left(\sqrt{2}\,\phi(\sqrt{2}\ell) + \ell\left(2\Phi(\sqrt{2}\ell)-1\right)\right)$$

Bounds on c_{NS} and c_{ES} in the isotropic case



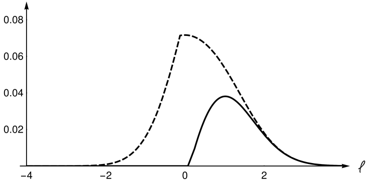
(a) $c_{ES}(\rho, \ell)$ for the RPW

(b) $c_{NS}(\rho, \ell)$ for the RPW

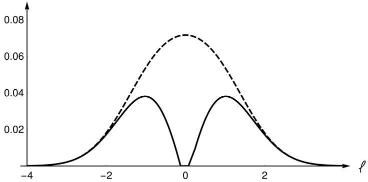
Figure: Lower bounds (solid) and upper bounds (dashed) for $c_{ES}(\rho, \ell)$ and $c_{NS}(\rho, \ell)$ respectively for the RPW.

The bound on $c_{ES}(\rho, \ell)$ for $\ell < 0$ is a result of the equality $c_{NS}(\rho, \ell) = c_{ES}(\rho, \ell) + c_{ES}(\rho, -\ell)$ and the fact that $c_{ES}(\rho, \ell)$ is non-decreasing for $\ell < 0$ (this part is specific to the RPW).

Similar results hold for all isotropic fields satisfying the basic assumptions. (The general expression for upper and lower bounds becomes more complicated, but depends only on the derivatives of κ at 0.)



(a) $c_{ES}(\rho, \ell)$ for the Bargmann-Fock field.



(b) $c_{NS}(\rho, \ell)$ for the Bargmann-Fock field.

Figure: Lower bounds (solid) and upper bounds (dashed) for $c_{ES}(\rho, \ell)$ and $c_{NS}(\rho, \ell)$ respectively, where ρ is the spectral measure of the Bargmann-Fock field.

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Proposition

Let f be the Gaussian field with spectral measure $\rho = \alpha\delta_0 + \frac{\beta}{2}(\delta_K + \delta_{-K}) + \frac{\gamma}{2}(\delta_L + \delta_{-L})$ where $\beta, \gamma > 0$, $\alpha = 1 - \beta - \gamma \geq 0$ and $K, L \in \mathbb{R}^2$ are linearly independent. Then

$$c_{NS}(\ell) = |K \times L| \cdot \mathbb{P}\left(|Y_1 - Y_2| \leq \ell + X_0 \leq Y_1 + Y_2\right),$$
$$c_{ES}(\ell) = |K \times L| \cdot \mathbb{P}\left(|Y_1 - Y_2| \leq |\ell + X_0| \leq Y_1 + Y_2\right),$$

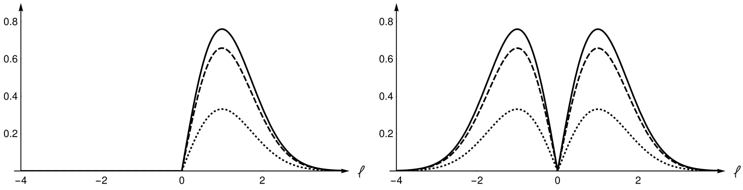
$\times$ denotes the cross product, $X_0 \sim \mathcal{N}(0, \alpha)$, $Y_1 \sim \mathrm{Ray}(\sqrt{\beta})$, $Y_2 \sim \mathrm{Ray}(\sqrt{\gamma})$ and X_0, Y_1, Y_2 are independent.
If $c_{NS}(\ell) \neq 0$ then $N_{LS,R}(\ell)/(\pi R^2)$ converges in L^1 to a non-constant random variable and hence does not converge a.s. to a constant, and this statement also holds for c_{ES} and $N_{ES,R}(\ell)/(\pi R^2)$. Furthermore

$$p_{m^+}(x) = p_{m^-}(-x) = |K \times L| \cdot p_{X_0 + Y_1 + Y_2}(x)$$
$$p_{s^-}(x) = p_{s^+}(-x) = |K \times L| \cdot p_{X_0 + |Y_1 - Y_2|}(x)$$

where p_Z denotes the probability density of a random variable Z.

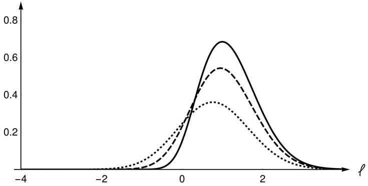
Consequences of main results

Derivation of c_{NS} and c_{ES} for 4 point spectral measures

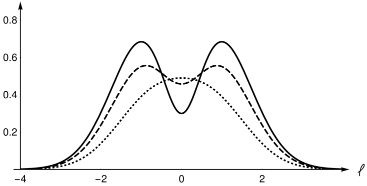


(a) $c_{ES}(\ell)$ (b) $c_{NS}(\ell)$

Figure: The functions $c_{ES}(\ell)$ (left) and $c_{NS}(\ell)$ (right) with $\alpha = 0$ for $\beta - \gamma = 0$ (solid), $\beta - \gamma = 0.5$ (dashed) and $\beta - \gamma = 0.9$ (dotted) respectively.



(a) $c_{ES}(\ell)$



(b) $c_{NS}(\ell)$

Figure: The functions $c_{ES}(\ell)$ (left) and $c_{NS}(\ell)$ (right) with $\beta = \gamma$ for $\alpha = 0.1$ (solid), $\alpha = 0.3$ (dashed) and $\alpha = 0.6$ (dotted) respectively.

Extensions/open questions

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1. Characterising p_{s-} (or p_{s+})
2. Higher dimensions
3. Continuous differentiability of c_{NS}
4. Bimodality

References

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[2] P. Kurlberg and I. Wigman. "Variation of the Nazarov-Sodin constant for random plane waves and arithmetic random waves". (2018).

[3] F. Nazarov and M. Sodin. "Asymptotic laws for the spatial distribution and the number of connected components of zero sets of Gaussian random functions". (2016).