The phase transition for level sets of smooth planar Gaussian fields

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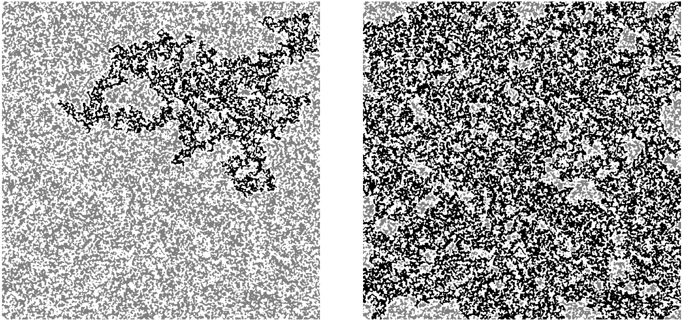

Credit: Dmitry Beliaev

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Two main aspects to this conjecture:

- Existence and sharpness of the phase transition (exponential decay of crossing probabilities, polynomial critical window).
- Scaling limits at the critical level (RSW estimates, convergence of crossing probabilities, convergence to CLE).

Gaussian fields and percolation

Let f be a stationary, centred, continuous Gaussian field on $\mathbb{R}^2$ with covariance kernel

$$\kappa(x) = \mathbb{E}[f(0)f(x)] \ , \quad x \in \mathbb{R}^2.$$

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Define the level sets and (lower-)excursion sets of f by

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$$\mathcal{L}_\ell = \{x : f(x) = \ell\} \quad \text{and} \quad \mathcal{E}_\ell = \{x : f(x) \leq \ell\}.$$

We say that $\mathcal{L}_\ell$ or $\mathcal{E}_\ell$ *percolate* if almost surely they have an unbounded connected component.

Gaussian fields and percolation

By monotonicity, there exists a *critical level* $\ell_c \in [-\infty, \infty]$ such that $\mathcal{E}_\ell$ percolates if $\ell > \ell_c$ and does not if $\ell < \ell_c$.

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Under mild conditions on κ it is natural to expect that $\ell_c = 0$.

In fact, we expect a *phase transition* at $\ell_c = 0$:

- If $\ell \leq 0$, then almost surely the connected components of $\mathcal{E}_\ell$ are bounded;
- If $\ell > 0$, then almost surely $\mathcal{E}_\ell$ has a unique unbounded connected component.

Previous results

In 1983, Molchanov & Stepanov showed that if κ is absolutely integrable then $\ell_c < \infty$, i.e., there exists a $\ell^* < \infty$ such that $\mathcal{E}_\ell$ percolates at every level $\ell \geq \ell^*$.

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Together, these results show that if correlations are (i) positive, and (iii) integrable, then

$$0 \leq \ell_c < \infty.$$

Previous results

Recently, advances in percolation theory have inspired a flurry of new results:

- In 2016, Beffara & Gayet proved that, if κ is (i) positive, (ii) symmetric, and (iii) decays polynomially with exponent $\gamma > 325$, then the 'RSW estimates' hold at the zero level.

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- In 2016, Beffara & Gayet proved that, if κ is (i) positive, (ii) symmetric, and (iii) decays polynomially with exponent $\gamma > 325$, then the 'RSW estimates' hold at the zero level.
- The necessary exponent γ for RSW estimates has been subsequently reduced, first to $\gamma > 16$ [Beliaev & M, 2017], then to $\gamma > 4$ [Rivera & Vanneuville, 2017] (integrability corresponds to $\gamma > 2$).

Previous results

- It was also recently shown [Rivera & Vanneuville, 2017] that the phase transition exists for the *Bargmann-Fock* field, i.e. the Gaussian field with covariance

$$\kappa(x) = e^{-|x|^2/2}.$$

  Their argument relied on exact Fourier-type computations on the covariance kernel κ.

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The existence of the spectral density guarantees that f is non-degenerate and ergodic, and also that $\kappa(x) \to 0$ as $|x| \to \infty$.

On the other hand, this assumption is weaker than the condition that the covariance kernel κ is absolutely integrable, and also holds for the band-limited kernels.

Our results

The existence of the spectral density is fundamental to our analysis because it permits a *white-noise representation* of f:

$$f \overset{d}{=} q \star W$$

where $q := \mathcal{F}[\rho] \in L^2(\mathbb{R}^2)$, W is a planar white-noise, and $\star$ denotes convolution.

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In fact, the existence of this representation is *equivalent* to the existence of ρ^2.

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- (Symmetry) The function q is symmetric under (i) reflection in the x-axis, and (ii) rotation by $\pi/2$ about the origin.
- (Positivity) The function $q \geq 0$ is positive.
- ('Integrable correlations') There exists $\gamma > 2$ and $c > 0$ such that, for every $|x| > 1$,

  $$|q(x)| < c|x|^{-\gamma},$$

  and, for every multi-index α such that $|\alpha| \leq 2$,

  $$|\partial^\alpha q(x)| < c|x|^{-\gamma}.$$

Our results: Existence of the phase transition

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Theorem

Under the stated conditions:

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- *If* $\ell > 0$*, then almost surely* $\mathcal{E}_\ell$ *has a unique unbounded connected component.*

Our results: Existence of the phase transition

Another way to state this result is in terms of the 'ε'-thickened nodal set:

Theorem

Let $\mathcal{N}_\varepsilon = \{|f| \leq \varepsilon\}$. Then under the stated conditions:

- *If $\varepsilon = 0$, then almost surely the connected components of $\mathcal{N}_\varepsilon$ are bounded;*
- *If $\varepsilon > 0$, then almost surely $\mathcal{N}_\varepsilon$ has a unique unbounded connected component.*

Our results: Sharpness of the phase transition

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One can take, for instance, D to be a rectangle and η and η' to be opposite edges.

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Define a *quad* Q to be a simply-connected piece-wise smooth compact domain $D \subset \mathbb{R}^2$ and two disjoint boundary arcs η and η'.

One can take, for instance, D to be a rectangle and η and η' to be opposite edges.

For each quad Q and level ℓ, let $\mathrm{Cross}_\ell(Q)$ denote the event that there is a connected component of $\mathcal{E}_\ell$ that crosses Q, i.e., whose intersection with Q intersects both η and η'.

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Theorem

Under the stated conditions, the following hold for every Q:

- *If $\ell < 0$, there exist $c_1, c_2 > 0$ such that, for all $s \geq 1$,*

$$\mathbb{P}\left(f \in \mathrm{Cross}_\ell(sQ)\right) < c_1 e^{-c_2 \log^2(s)}.$$

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- *If $\ell = 0$,*

$$\inf_{s>0} \mathbb{P}(f \in \mathrm{Cross}_0(sQ)) > 0 \quad \textit{and} \quad \sup_{s>0} \mathbb{P}(f \in \mathrm{Cross}_0(sQ)) < 1.$$

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Theorem

Suppose that, in addition to the above assumptions, there exists a constant $c > 0$ *such that, for every* $|x| > 1$ *and for every multi-index* α *such that* $|\alpha| \leq 2$,

$$|\partial^\alpha q(x)| < c e^{-|x|(\log |x|)^2}.$$

Then, for each $\ell < 0$ *there exist* $c_1, c_2 > 0$ *such that, for all* $s \geq 1$,

$$\mathbb{P}\left(f \in \mathrm{Cross}_\ell(sQ)\right) < c_1 e^{-c_2 s}.$$

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Theorem

Under the stated conditions, there exist $0 < c_1 < c_2 < \infty$ such that, for every quad Q,

$$\limsup_{s\to\infty} \mathbb{P}\left[f \in \mathrm{Cross}_{s^{-c_2}}(sQ)\right] < 1$$

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$$\lim_{s\to\infty} \mathbb{P}\left[f \in \mathrm{Cross}_{s^{-c_1}}(sQ)\right] = 1.$$

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We can show only that it is strictly positive and at most 1 (which is roughly all that is known in percolation outside some special lattices).

The percolation universality class

A major unresolved question raised by our work is to determine how rapidly correlations must decay in order for the analogy with percolation to be valid.

That is, for which κ do our results hold (and more refined results, such as the convergence of the nodal set to $CLE(6)$)?

The percolation universality class

According to the 'Harris criterion', the percolation universality class consists of all κ satisfying

$$\int_{x \in B_R} \int_{y \in B_R} \kappa(x-y)\, dxdy \ll R^{5/2};$$

for positive κ, this equates to polynomial decay of order $\gamma > 3/2$.

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The *random plane wave* satisfies the HC since

$$\int_{x \in B_R} \int_{y \in B_R} J_0(|x-y|)\, dxdy = O(R).$$

despite correlations decaying only at rate $R^{-1/2}$ (for which the LHS is a priori $O(R^{7/2})$)

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3. ($\ell_c = 0$) Use ideas from randomised algorithms (i.e. the OSSS inequality) to deduce a qualitative description of the phase transition at $\ell = 0$.
4. (Sharpness) Bootstrap the previous step to give a quantitative description of the phase transition.

Elements of the proof: Quasi-independence

To prove QI, we couple f to the R-dependent field

$$f_R = q_R \star W = (q\chi_R) \star W$$

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Standard arguments (Kolmogorov, BTIS) then give that

$$\mathbb{P}[|f - f_R|_{C^0(B_R)} > \varepsilon] < \delta$$

for $\varepsilon \approx R^{1-\gamma}$ and $\delta \approx e^{-c_2(\log R)^2}$.

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Since crossing events separated by a distance R are independent for f_R, it remains to find a bound on

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By monotonicity and the bound on $\|f - f_R\|_{C^0(B_R)}$, it is enough to find a bound on

$$|\mathbb{P}[f \in \mathrm{Cross}_\ell(RQ)] - \mathbb{P}[f \pm \varepsilon \in \mathrm{Cross}_\ell(RQ)]|$$

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Theorem (Cameron–Martin)

Let H be the RKHS of f. Then for every $h \in H$, the Radon–Nikodym derivative of the law of $f + h$ with respect to the law of f is

$$\exp\left\{\langle f, h\rangle - \frac{1}{2}\mathbb{E}[\langle f, h\rangle^2]\right\}.$$

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Corollary

For every $h \in H$ and event A,

$$|\mathbb{P}[f \in A] - \mathbb{P}[f \pm h \in A]| \leq \frac{\|h\|_H \sqrt{\mathbb{P}[f \in A]}}{\sqrt{\log 2}}.$$

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This gives the identity

$$\|h\|_H^2 = \int_{x\in\mathbb{R}^2} |\hat{h}(x)|^2/\rho^2(x)\, dx,$$

which means that, if $h \approx \mathbb{1}_{B_R}$, then $\|h\|_H^2 \approx \int_{x\in B(1/R)} \rho^{-2}(x)\, dx$.

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Proposition

Suppose $\rho(0) > 0$. Then there exist $c, R_0 > 0$ such that, for every $R > R_0$, monotonic event A that depends on $f|_{B_R}$, and $\varepsilon > 0$,

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$$|\mathbb{P}[f \in A] - \mathbb{P}[f \pm \varepsilon \in A]| \leq cR\varepsilon\sqrt{\mathbb{P}[f \in A]}.$$

From here, it is easy to deduce QI if $\gamma > 2$,

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- Sufficient symmetry, guaranteed by our assumptions;
- The QI property; and
- Positive associations, which is equivalent in the Gaussian setting to $\kappa \geq 0$ (and so is implied by $q \geq 0$).

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Let $\mathcal{A}$ be a *random algorithm* that determines A, i.e. a procedure that reveals the coordinates and stops once the value of $\mathbb{1}_A$ is known. The *revealment* $\delta_i(\mathcal{A})$ of the i^{th} coordinate for the algorithm $\mathcal{A}$ is the probability that the coordinate is revealed.

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Consider a finite-dimensional product space and an event A. The OSSS inequality bounds the variance of A in terms of the 'influence' and the 'revealement' of each coordinate.

The *influence* $I_i(A)$ of the i^{th} coordinate on A is defined as the probability that resampling the coordinate modifies $\mathbb{1}_A$.

Let $\mathcal{A}$ be a *random algorithm* that determines A, i.e. a procedure that reveals the coordinates and stops once the value of $\mathbb{1}_A$ is known. The *revealment* $\delta_i(\mathcal{A})$ of the i^{th} coordinate for the algorithm $\mathcal{A}$ is the probability that the coordinate is revealed.

Theorem (O'Donnell, Saks, Schramm, Servedio, 2005)

$$\mathrm{Var}(\mathbb{1}_A) \leq \sum_i \delta_i(\mathcal{A}) I_i(A).$$

Elements of the proof: Sharp thresholds via OSSS

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Moreover, if A is increasing (w.r.t. the X_i),

$$\mathbb{E}[X_i \mathbb{1}_{\{f+\ell\in A\}}] \geq c I_i(\{f+\ell \in A\})$$

for $c = \sup_{a\geq 0}\mathbb{P}\left[Z \geq a\right]/\mathbb{E}\left[Z\mathbb{1}_{Z\geq a}\right] < \infty$.

Elements of the proof: Sharp thresholds via OSSS

Applying the OSSS inequality, for any algorithm $\mathcal{A}$ that determines $\mathrm{Cross}_\ell(RQ)$,

$$\frac{d}{d\ell}\mathbb{P}\left[\mathrm{Cross}_\ell(RQ)\right] \geq \frac{\mathrm{Var}\big(\mathbb{1}_{\mathrm{Cross}_\ell(RQ)}\big)}{\sup_i \delta_i(\mathcal{A})}.$$

Hence, in order to demonstrate $\ell_c = 0$, i.e. to show that

$$\frac{d}{d\ell}\mathbb{P}\left[\mathrm{Cross}_\ell(RQ)\right]\Big|_{\ell=0} \to \infty\ , \quad \text{as } R \to \infty,$$

we need only exhibit an algorithm $\mathcal{A}$ for $\mathrm{Cross}_\ell(RQ)$ such that

$$\sup_i \delta_i(\mathcal{A}) \to 0, \quad \text{as } R \to \infty.$$

Elements of the proof: Sharp thresholds via OSSS

To approximate f by a finite-dimensional field, we couple W to a discretised white-noise W^ε at scale $\varepsilon > 0$ by setting

$$\eta_v = \varepsilon^{-1} \int_{x \in v + [-\varepsilon/2, \varepsilon/2]^2} dW(x) \ , \quad v \in \varepsilon\mathbb{Z}^2,$$

(η_v are i.i.d. standard Gaussians), and defining

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On any compact set, we can approximate f by the *finite-dimensional* Gaussian field

$$f_r^\varepsilon = q_r \star W^\varepsilon.$$

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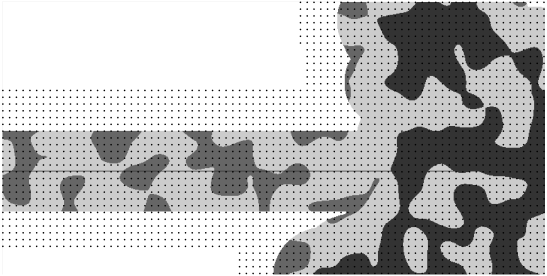
Let Q be a rectangle, and define $\mathcal{A}$ to be the algorithm that picks a random horizontal line, and reveals η_v in the r-neighbour of this line and of all 'blocking' clusters that intersect this line.

This determines the event $\{f_r^\varepsilon \in \mathrm{Cross}_0(R)\}$.



Credit: Dmitry Beliaev

Elements of the proof: Sharp thresholds via OSSS

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The latter 'one-arm event' can be controlled thanks to the RSW estimates.

Elements of the proof: Bootstrapping

The upshot of the OSSS analysis is a ‘qualitative’ description for the phase transition: for any quad Q and $\ell > 0$,

$$\mathbb{P}\left[f \in \mathsf{Cross}_\ell(RQ)\right] \to 1 \quad \text{as } R \to \infty.$$

The final step is to convert this into a quantitative description of the sharp phase transition.

Elements of the proof: Bootstrapping

Let Q be the 3×1 rectangle, and define

$$a_R = \mathbb{P}\left[f_R \notin \mathrm{Cross}_\ell(RQ)\right];$$

The goal is to upgrade the qualitative statement $a_R \to 0$, to the quantitative statement that $a_R \leq e^{-c \log^2(R)}$.

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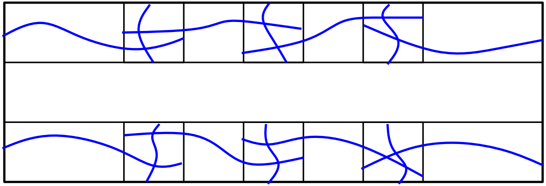
This is implied from the following functional inequality

$$a_{3R} \leq c_1 a_R^2 + R^{2-\beta}\sqrt{(a_R)^2} + e^{-c_2 \log^2(R)},$$

which is deduced from the event below.



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Thank you!