UNIVERSITÉ DU LUXEMBOURG

Second Order Results for Nodal Sets of Gaussian Random Waves

Giovanni Peccati (Luxembourg University)

Joint works with:
F. Dalmao, G. Dierickx, D. Marinucci,
I. Nourdin, M. Rossi and I. Wigman

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Introduction

- ⋆ In recent years, proofs of **second order results in the high-energy limit** (like central and non-central limit theorems) for local quantities associated with **random waves on surfaces**, like the flat 2-torus, the sphere or the plane (but not only!). Works by J. Benatar, V. Cammarota, F. Dalmao, D. Marinucci, I. Nourdin, G. Peccati, M. Rossi, I. Wigman.

- ⋆ Common feature: the asymptotic behaviour of such local quantities is dominated (in L^2) by their projection on a fixed **Wiener chaos**, from which the nature of the fluctuations is inherited.

- ⋆ 'Structural explanation' of **cancellation phenomena** first detected by Berry (plane, 2002) and Wigman (sphere, 2010).

Vignette: Wiener Chaos

- $\star$ Consider a generic separable Gaussian field $\mathbb{G} = \{G(u) : u \in \mathscr{U}\}$.
- $\star$ For every $q = 0, 1, 2...$, set

$$P_q := \overline{\mathbf{v.s.}}\Big\{ p\big(G(u_1), ..., G(u_r)\big) : d^\circ p \leq q \Big\}.$$

Then: $P_q \subset P_{q+1}$.

- $\star$ Define the family of orthogonal spaces $\{C_q : q \geq 0\}$ as $C_0 = \mathbb{R}$ and $C_q := P_q \cap P_{q-1}^{\perp}$; one has

$$L^2(\sigma(\mathbb{G})) = \bigoplus_{q=0}^{\infty} C_q.$$

- $\star$ C_q = qth **Wiener chaos** of $\mathbb{G}$.

A RIGID ASYMPTOTIC STRUCTURE

For fixed $q \geq 2$, let $\{F_k : k \geq 1\} \subset C_q$ (with unit variance).

- ⋆ *Nourdin and Poly (2013)*: If $F_k \Rightarrow Z$, then Z has necessarily a density (and the set of possible laws for Z does not depend on $\mathbb{G}$).
- ⋆ *Nualart and Peccati (2005)*: $F_k \Rightarrow Z \sim \mathscr{N}(0,1)$ if and only if $\mathbb{E}F_k^4 \to 3 (= \mathbb{E}Z^4)$.
- ⋆ *Peccati and Tudor (2005)*: Componentwise convergence to Gaussian implies joint convergence.
- ⋆ *Nourdin, Nualart and Peccati (2015)*: given $\{H_k\} \subset C_p$, then F_k, H_k are asymptotically independent if and only if $\mathbf{Cov}(H_k^2, F_k^2) \to 0$.
- ⋆ Nonetheless, there exists no full characterisation of the asymptotic structure of chaoses ≥ 3.

Berry's Random Waves (Berry, 1977)

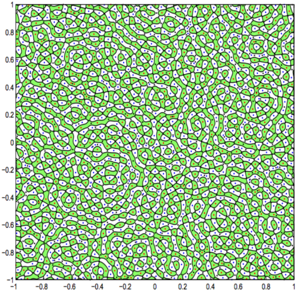
- Fix $E > 0$. The **Berry random wave model** on $\mathbb{R}^2$, with parameter E, written
$$B_E = \{B_E(x) : x \in \mathbb{R}^2\},$$
is the unique (in law) centred, isotropic Gaussian field on $\mathbb{R}^2$ such that
$$\Delta B_E + E \cdot B_E = 0, \text{ where } \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$
- Equivalently,
$$\mathbb{E}[B_E(x)B_E(y)] = \int_{\mathbb{S}^1} e^{i\sqrt{E}\langle x-y,z\rangle}\, dz = J_0(\sqrt{E}\|x-y\|).$$
(this is an *infinite-dimensional* Gaussian object).
- Think of B_E as a **"canonical"** Gaussian Laplace eigenfunction on $\mathbb{R}^2$, emerging as a **universal local scaling limit** for **arithmetic and monochromatic RWs, random spherical harmonics**... .

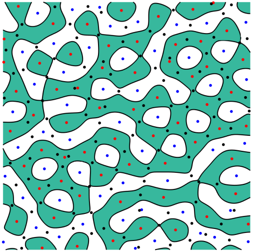
NODAL SETS

Focus on the length L_E of the **nodal set**:

$$B_E^{-1}(\{0\}) \cap \mathcal{Q} := \{x \in \mathcal{Q} : B_E(x) = 0\},$$

where $\mathcal{Q}$ is some fixed domain , as $E \to \infty$.





Images: D. Belyaev

A Cancellation Phenomenon

⋆ *Berry* (2002): an application of **Kac-Rice formulae** leads to

$$\mathbb{E}[L_E] = \text{area } \mathcal{Q} \times \sqrt{\frac{E}{8}},$$

and a legitimate guess for the order of the variance is

$$\mathbf{Var}(L_E) \asymp \sqrt{E}.$$

⋆ However, Berry showed that

$$\mathbf{Var}(L_E) \sim \frac{\text{area } \mathcal{Q}}{512\pi} \log E,$$

whereas *the length variances of non-zero level sets display the "correct" order* of $\sqrt{E}$.

⋆ Such a variance reduction "... *results from a cancellation whose meaning is still obscure...* " (Berry (2002), p. 3032).

Spherical Case

- ⋆ Berry's constants were confirmed by I. Wigman (2010) in the related model of **random spherical harmonics** — see Domenico's talk.

- ⋆ Here, the Laplace eigenvalues are the integers

$$n(n+1), \quad n \in \mathbb{N}.$$

Picture: A. Barnett

Arithmetic Random Waves (Oravecz, Rudnick and Wigman, 2007)

- Let $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2 \simeq [0,1)^2$ be the 2-dimensional flat torus.
- We are again interested in real (random) **eigenfunctions** of Δ, that is, solutions of the **Helmholtz equation**

$$\Delta f + Ef = 0,$$

for some adequate $E > 0$ (**eigenvalue**).
- The eigenvalues of Δ are therefore given by the set

$$\{E_n := 4\pi^2 n : n \in S\},$$

where

$$S = \{n : n = a^2 + b^2;\, a, b \in \mathbb{Z}\}.$$

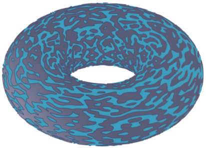
- For $n \in S$, the dimension of the corresponding eigenspace is $\mathcal{N}_n = r_2(n) := \#\Lambda_n$, where $\Lambda_n := \{(\lambda_1, \lambda_2) : \lambda_1^2 + \lambda_2^2 = n\}$

Arithmetic Random Waves (Oravecz, Rudnick and Wigman, 2007)

We define the **arithmetic random wave** of order $n \in S$ as:

$$f_n(x) = \frac{1}{\sqrt{\mathcal{N}_n}} \sum_{\lambda \in \Lambda_n} a_\lambda e^{2i\pi\langle \lambda, x\rangle}, \ x \in \mathbb{T},$$

where the a_λ are i.i.d. complex standard Gaussian, except for the relation $a_\lambda = \overline{a_{-\lambda}}$.



Picture: J. Angst & G. Poly

We are interested in the behaviour, as $\mathcal{N}_n \to \infty$, of the **total nodal length**

$$\mathscr{L}_n := \text{length}\, f_n^{-1}(\{0\}).$$

Nodal Lengths and Spectral Measures

- ⋆ Crucial role played by the set of **spectral probability measures** on $\mathbb{S}^1$

$$\mu_n(dz) := \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \delta_{\lambda/\sqrt{n}}(dz), \quad n \in S$$

(invariant with respect to $z \mapsto \overline{z}$ and $z \mapsto i \cdot z$.)

- ⋆ The set $\{\mu_n : n \in S\}$ is relatively compact and its adherent points are an **infinite strict subset** of the class of invariant probabilities on the circle (see Kurlberg and Wigman (2015)).

Another Cancellation

- ⋆ *Rudnick and Wigman* (2008): For every $n \in S$, $\mathbb{E}[\mathcal{L}_n] = \frac{\sqrt{E_n}}{2\sqrt{2}}$. Moreover, $\textbf{Var}(\mathcal{L}_n) = O\left(E_n/\mathcal{N}_n^{1/2}\right)$. Conjecture: $\textbf{Var}(\mathcal{L}_n) = O(E_n/\mathcal{N}_n)$.
- ⋆ *Krishnapur, Kurlberg and Wigman* (2013): if $\{n_j\} \subset S$ is such that $\mathcal{N}_{n_j} \to \infty$, then

$$\textbf{Var}(\mathcal{L}_{n_j}) = \frac{E_{n_j}}{\mathcal{N}_{n_j}^2} \times c(n_j) + O(E_{n_j} R_5(n_j)),$$

where

$$c(n_j) = \frac{1 + \widehat{\mu}_{n_j}(4)^2}{512};\ R_5(n_j) = \int_{\mathbb{T}} |r_{n_j}(x)|^5 dx = o\left(1/\mathcal{N}_{n_j}^2\right).$$

- ⋆ Two phenomena: (i) **cancellation**, and (ii) **non-universality**.

Next Step: Second Order Results

⋆ For $E > 0$ and $n \in S$, define the normalized quantities

$$\widetilde{L}_E := \frac{L_E - \mathbb{E}(L_E)}{\mathbf{Var}(L_E)^{1/2}} \quad \text{and} \quad \widetilde{\mathscr{L}_n} := \frac{\mathscr{L}_n - \mathbb{E}(\mathscr{L}_n)}{\mathbf{Var}(\mathscr{L}_n)^{1/2}}.$$

⋆ **Question** : Can we explain the above cancellation phenomena and, as $E, \mathcal{N}_n \to \infty$, establish limit theorems of the type

$$\widetilde{L}_E \xrightarrow{\mathbf{LAW}} Y, \quad \text{and} \quad \widetilde{\mathscr{L}}_{n'_j} \xrightarrow{\mathbf{LAW}} Z?$$

($\{n'_j\} \subset S$ is some subsequence)

A Common Strategy

- ⋆ **Step 1**. Let $V = f_n$ or B_E, and $L = L_E$ or $\mathscr{L}_n$. Use the representation (based on the coarea formula)

$$L = \int \delta_0(V(x)) \|\nabla V(x)\| \, dx, \quad \text{in } L^2(\mathbb{P}),$$

to deduce the **Wiener chaos expansion** of L.

- ⋆ **Step 2**. Show that exactly **one chaotic projection** $L(4) := \mathrm{proj}(L \,|\, C_4)$ dominates in the high-energy limit – thus accounting for the cancellation phenomenon.

- ⋆ **Step 3**. Study by "bare hands" the limit behaviour of $L(4)$.

Fluctuations for Berry's Model

Theorem (Nourdin, P., & Rossi, 2017)

1. **(Cancellation)** *For every fixed* $E > 0$*,*

$$\mathrm{proj}(L_E \,|\, C_{2q+1}) = 0, \quad q \geq 0,$$

   and $\mathrm{proj}(\widetilde{L}_E \,|\, C_2)$ *reduces to a "negligible boundary term", as* $E \to \infty$*.*

2. **(4^{th} chaos dominates)** *Let* $E \to \infty$*. Then,*

$$\widetilde{L}_E = \mathrm{proj}(\widetilde{L}_E \,|\, C_4) + o_{\mathbb{P}}(1).$$

3. **(CLT)** *As* $E \to \infty$*,*

$$\widetilde{L}_E \Rightarrow Z \sim N(0,1).$$

Theorem

Define, for $B = B_1$*:*

$$\mathbf{L}_r := \mathrm{length}(B^{-1}(\{0\}) \cap \mathrm{Ball}(0, r)).$$

Then,

1. $\mathbb{E}[\mathbf{L}_r] = \frac{\pi r^2}{2\sqrt{2}}$;
2. *as* $r \to \infty$, $\mathrm{Var}(\mathbf{L}_r) \sim \frac{r^2 \log r}{256}$;
3. *as* $r \to \infty$, $$\frac{\mathbf{L}_r - \mathbb{E}[\mathbf{L}_r]}{\mathrm{Var}(\mathbf{L}_r)^{1/2}} \Rightarrow Z \sim N(0, 1).$$

Theorem (Marinucci, P., Rossi & Wigman, 2016)

1. **(Exact Cancellation)** *For every fixed $n \in S$,*

$$\mathrm{proj}(\mathscr{L}_n \,|\, C_2) = \mathrm{proj}(\mathscr{L}_n \,|\, C_{2q+1}) = 0, \quad q \geq 0.$$

2. **(4^{th} chaos dominates)** *Let $\{n_j\} \subset S$ be such that $\mathcal{N}_{n_j} \to \infty$. Then,*

$$\widetilde{\mathscr{L}_{n_j}} = \mathrm{proj}(\widetilde{L}_{n_j} \,|\, C_4) + o_{\mathbb{P}}(1).$$

3. **(Non-Universal/Non-Gaussian)** *If $|\widehat{\mu}_{n_j}(4)| \to \eta \in [0,1]$, where $\widehat{\mu}_n(4) = \int z^4 \mu_n(dz)$, then*

$$\widetilde{\mathscr{L}_{n_j}} \Rightarrow M(\eta) := \frac{1}{2\sqrt{1+\eta^2}} \left(2 - (1-\eta)Z_1^2 - (1+\eta)Z_2^2\right),$$

where Z_1, Z_2 independent standard normal.

Theorem (Dalmao, Nourdin, P. & Rossi, 2016)

For $\widehat{T}$ an independent copy, consider

$$I_n := \#[T_n^{-1}(\{0\}) \cap \widehat{T}_n^{-1}(\{0\})].$$

1. *As $\mathcal{N}_n \to \infty$,*
$$\mathrm{Var}(I_n) \sim \frac{E_n^2}{\mathcal{N}_n^2} \frac{3\widehat{\mu}_{n_j}(4)^2 + 5}{128\pi^2}$$
2. *If $|\widehat{\mu}_{n_j}(4)| \to \eta \in [0,1]$, then*
$$\widetilde{I}_{n_j} \Rightarrow J(\eta) := \frac{1}{2\sqrt{10+6\eta^2}} \left(\frac{1+\eta}{2} A + \frac{1-\eta}{2} B - 2(C-2) \right)$$

with A, B, C independent s.t. $A \overset{\text{law}}{=} B \overset{\text{law}}{=} 2X_1^2 + 2X_2^2 - 4X_3^2$ and $C \overset{\text{law}}{=} X_1^2 + X_2^2$, where (X_1, X_2, X_3) is standard Gaussian.

Elements of Proof (BRW)

- In view of Green's identity, one has that

$$\mathrm{proj}(L_E \mid C_2) = \frac{1}{2\sqrt{E}} \int_{\partial \mathcal{Q}} B_E(x) \langle \nabla B_E(x), n(x) \rangle \, dx,$$

  where $n(x)$ is the outward unit normal at x (variance bounded).
- The term $\mathrm{proj}(\widetilde{L}_E \mid C_4)$ is a l.c. of 4^{th} order terms, among which

$$V_E := \sqrt{E} \int_{\mathcal{Q}} H_4(B_E(x)) dx,$$

  for which one has that

$$\mathbf{Var}(V_E) \quad = \quad \frac{24}{E} \int_{(\sqrt{E}\mathcal{Q})^2} J_0(\|x-y\|)^4 dxdy \sim \frac{18}{\pi^2} \log E,$$

  using e.g. $J_0(r) \sim \sqrt{\frac{2}{\pi r}} \cos(r - \pi/4)$, $r \to \infty$.
- In the proof, one cannot *a priori* rely on the "full correlation phenomenon" seen in Domenico's talk.

Elements of Proof (ARW)

- ⋆ Write $\mathscr{L}_n(u) = \text{length}\, f_n^{-1}(u)$. One has that

$$\begin{aligned} \text{proj}(\mathscr{L}_n(u) \,|\, C_2) &= c e^{-u^2/2} u^2 \int_{\mathbb{T}} (f_n(x)^2 - 1) dx \\ &= c \frac{e^{-u^2/2} u^2}{\mathcal{N}_n} \sum_{\lambda \in \Lambda^n} (|a_\lambda|^2 - 1) \end{aligned}$$

  (this is the dominating term for $u \neq 0$; it verifies a CLT).
- ⋆ Prove that $\text{proj}(\mathscr{L}_n \,|\, C_4)$ has the form

$$\sqrt{\frac{E_n}{\mathcal{N}_n^2}} \times Q_n,$$

  where Q_n is a quadratic form, involving sums of the type

$$\sum_{\lambda \in \Lambda_n} (|a_\lambda|^2 - 1) c(\lambda, n)$$

- ⋆ Characterise $\text{proj}(\mathscr{L}_n \,|\, C_4)$ as the dominating term, and compute the limit by Lindeberg and continuity.

Further Results

- ⋆ *Benatar and Maffucci* (2017) and *Cammarota* (2017): fluctuations on nodal volumes for ARW on $\mathbb{R}^3/\mathbb{Z}^3$.
- ⋆ The nodal length of random spherical harmonics verifies a **Gaussian CLT** (*Marinucci, Rossi, Wigman* (2017)).
- ⋆ Analogous non-central results hold for nodal lengths on **shrinking balls** (*Benatar, Marinucci and Wigman*, 2017).
- ⋆ **Quantitative versions** are available: e.g. (Peccati and Rossi, 2017)

$$\mathbf{Wass}_1(\widetilde{\mathscr{L}_n}, M(\widehat{\mu}_n(4))) = \inf_{X\sim L, Y\sim M} \mathbb{E}|X-Y| = O\left(\frac{1}{\mathcal{N}_n^{1/4}}\right).$$

Beyond Explicit Models (W.I.P.)

- ⋆ Suppose $\{K_\lambda : \lambda > 0\}$ is a collection of covariance kernels on $\mathbb{R}^2$ such that, for $\lambda \to \infty$, some $r_\lambda \to \infty$ and every α, β,

$$\sup_{|x|,|y| \le r_\lambda} | \, \partial^\alpha \partial^\beta (K_\lambda(x,y) - J_0(\|x-y\|))| := \eta(\lambda) = o(1)$$

- ⋆ Let $Y_\lambda \sim K_\lambda$ and $B \sim J_0$.

- ⋆ Typical example: $Y_\lambda = \frac{1}{\sqrt{2\pi}} \times$ Canzani-Hanin's **pullback random wave** (dim. 2) at a point of isotropic scaling (needs $r_\lambda = o(\lambda)$).

Beyond Explicit Models (W.I.P.)

- ⋆ Write $\mathbf{L}(\Upsilon_\lambda, r_\lambda) := \mathrm{length}\{\Upsilon_\lambda^{-1}(\{0\}) \cap \mathrm{Ball}(0, r_\lambda)\}$, and $\mathbf{L}_r := \mathrm{length}(B_1 \cap \mathrm{Ball}(0, r))$.
- ⋆ Then, one can couple Υ_λ and B on the same probability space, in such a way that, if $r_\lambda \eta(\lambda)^\beta \to 0$ (say, $\beta \simeq 1/30$),

$$\left| \frac{\mathbf{L}(\Upsilon_\lambda, r) - \mathbb{E}\mathbf{L}(\Upsilon_\lambda, r)}{\mathrm{Var}(\mathbf{L}_{r_\lambda})^{1/2}} - \frac{\mathbf{L}_{r_\lambda} - \mathbb{E}\mathbf{L}_{r_\lambda}}{\mathrm{Var}(\mathbf{L}_{r_\lambda})^{1/2}} \right| \to 0,$$

  in L^2.
- ⋆ For instance, if $\eta(\lambda) = O(1/\log \lambda)$ (expected for pullback waves coming from manifolds with no conjugate points), then the statement is true for $r_\lambda = (\log \lambda)^\beta$, $\beta \simeq 1/30$.

Bibliographic Remarks

- ⋆ The use of Wiener chaos for studying excursions of random fields appears in seminal works e.g. by Azaïs, Kratz, Léon and Wschebor (in the 90s).
- ⋆ Starting from seminal contributions by Marinucci and Wigman (2010, 2011): geometric functionals of random Laplace eigenfunctions on compact manifolds can be studied by detecting specific **domination effects**.
- ⋆ Such geometric functionals include: **lengths of level sets, excursion areas, Euler-Poincaré characteristics, # critical points, # nodal intersections**. See several works by Cammarota, Dalmao, Marinucci, Nourdin, Peccati, Rossi, Wigman, ... (2010–2018).
- ⋆ Further examples of previous use of Wiener chaos in a close setting: Sodin and Tsirelson (2002) (Gaussian analytic functions), Azaïs and Leon's proof (2011) of the Granville-Wigman CLT for zeros of trigonometric polynomials.

THANK YOU FOR YOUR ATTENTION!