Decoupling of distant local events for Gaussian fields

Alejandro Rivera

(joint work with Hugo Vanneuville)

The characters...

- Take f smooth Gaussian field on $\mathbb{R}^2$ with covariance $\kappa(x-y) = \mathbb{E}\left[f(x)f(y)\right]$ and assume that, $\kappa(0) = 1$ and $\lim_{|x|\to+\infty} \kappa(x) = 0$.

Disclaimer : I will always assume κ is C^∞ and that for distinct $x_1, \ldots, x_k$, $(f(x_1), \ldots, f(x_k))$ is non-degenerate.

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- Set $\mathcal{D}_\pm = \{\pm f \geq 0\}$.

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- Set $\mathcal{D}_\pm=\{\pm f\geq 0\}$.
- Color $\mathcal{D}_+$ in black and $\mathcal{D}_-$ in white.

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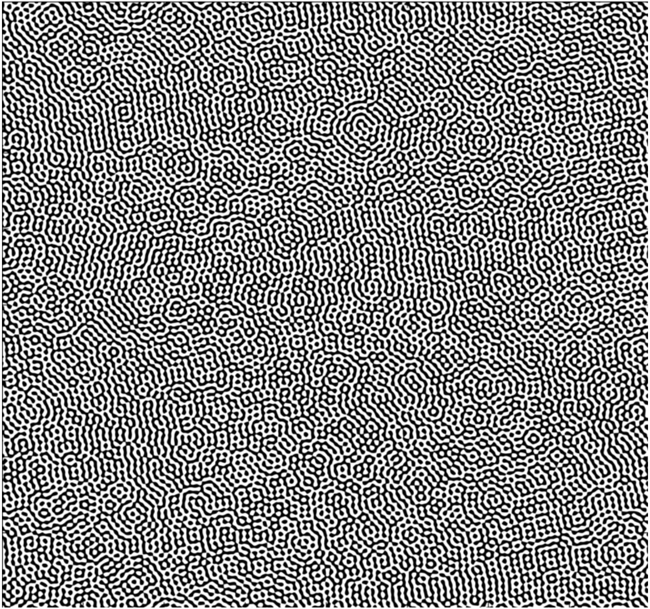
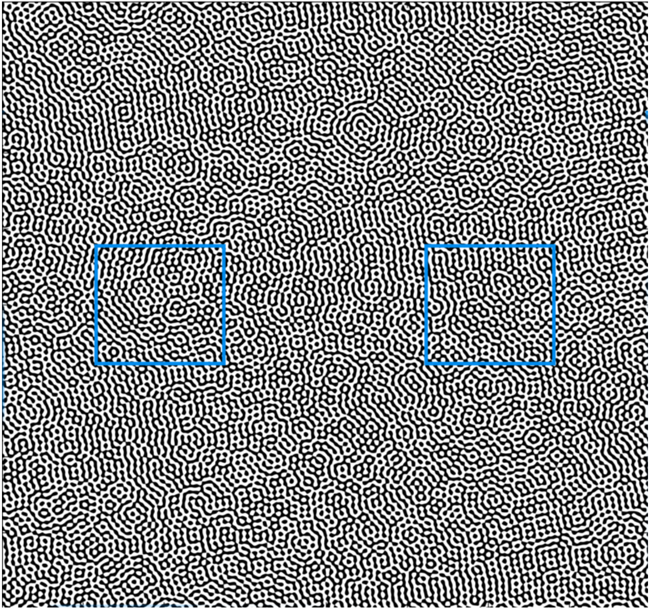
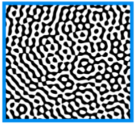
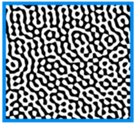
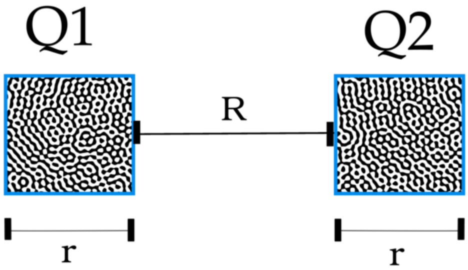


The question...





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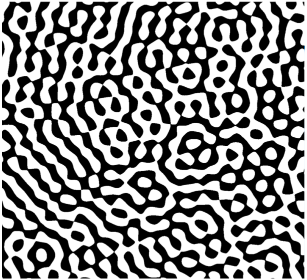
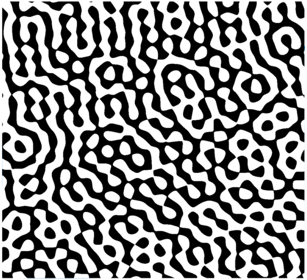
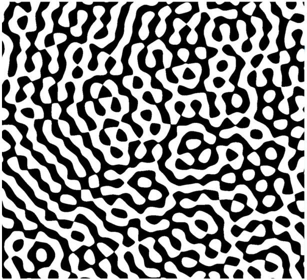
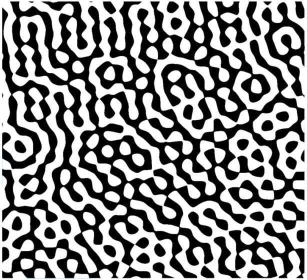
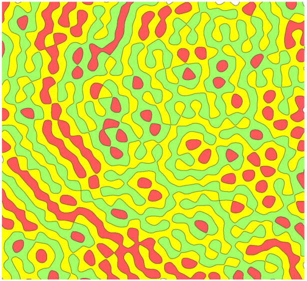
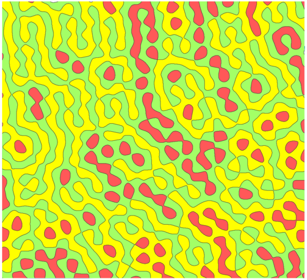



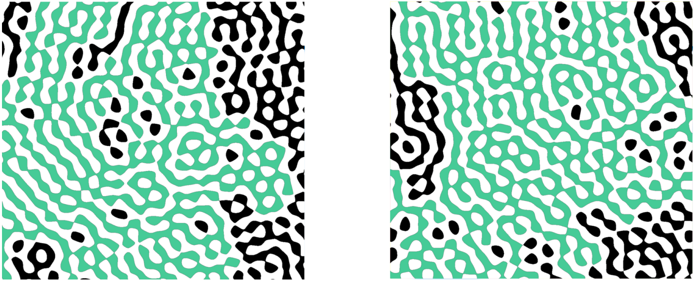
Is $\mathcal{D}_+ \cap \mathcal{Q}_1$ asymptotically independent from $\mathcal{D}_+ \cap \mathcal{Q}_2$?

Is $\mathcal{D}_+ \cap \mathcal{Q}_1$ asymptotically independent from $\mathcal{D}_+ \cap \mathcal{Q}_2$?
NO : unique continuation issues

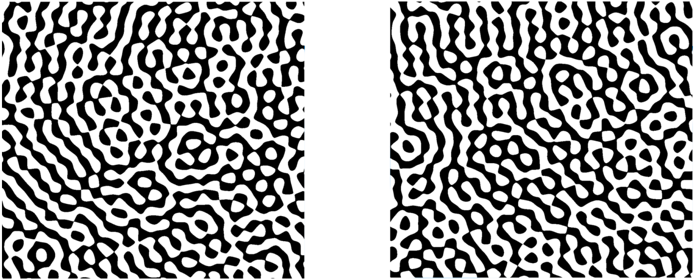
 

Are the number of nodal domains asymptotically independent from each other ?

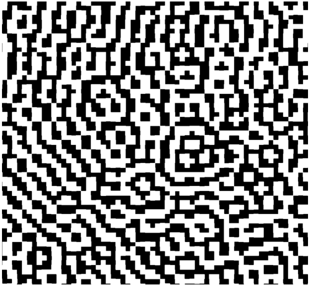


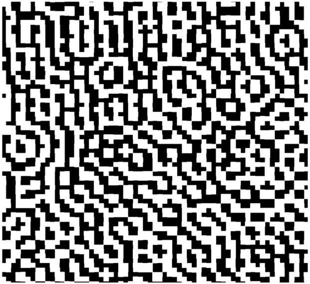
Are percolation events asymptotically independent from each other ?

Possible solutions...



Discretize at scale $\varepsilon << 1$.





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- Set $\eta = \sup_{x \in \mathcal{Q}_1,\ y \in \mathcal{Q}_2} |\kappa(x-y)|$.

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$$\mathcal{Q}_1^\varepsilon = \{x_i\}_{i \in I},\ \ \mathcal{Q}_2^\varepsilon = \{y_j\}_{j \in J}\,.$$

- Apply finite-dimensional arguments to control correlations.
- Justify that discretized events approximate continuous events.

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Theorem (see Chapter 1 of Pit82)

For any $A^\varepsilon \in \mathcal{F}_1^\varepsilon$ *and* $B^\varepsilon \in \mathcal{F}_2^\varepsilon$,

$$|\mathbb{P}[A^\varepsilon \cap B^\varepsilon] - \mathbb{P}[A^\varepsilon]\mathbb{P}[B^\varepsilon]| \leq C(r/\varepsilon)^4 \eta$$

where $C < +\infty$ *is an absolute constant.*

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... see also [NSV05], [BG16] and [BM17].
In particular, A^ε and B^ε are asymptotically independent when

$$\eta(R) \leq C(r/\varepsilon)^{-4-\delta}$$

for some $\delta > 0$. Of course, ε must tend to 0 to approximate topological events properly.

The punchline...

Theorem (RV18)

Let A (resp. B) be either a crossing event or a component counting event on $\mathcal{Q}_1$ (resp. $\mathcal{Q}_2$).

$$|\mathbb{P}[A \cap B] - \mathbb{P}[A]\mathbb{P}[B]| \leq Cr^4\eta$$

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In particular, A and B are asymptotically independent when

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More precisely,

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"There is a continuous black path inside $\mathcal{Q}$ joining two given sides of $\mathcal{Q}$."

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...and a **component counting event** on a box $\mathcal{Q}$ is an event measureable with respect to the random variable

"Number of connected components of $\mathcal{D}_+$ contained inside $\mathcal{Q}$."

Heuristics...

- Discretize : Define $X = f|_{\mathcal{Q}_1^\varepsilon}$ and $Y = f|_{\mathcal{Q}_2^\varepsilon}$.

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- The influence of i on A is

$$I_i(A) := \mathbb{P}\left[\mathrm{Piv}_i(A) \mid X_i = 0\right] .$$

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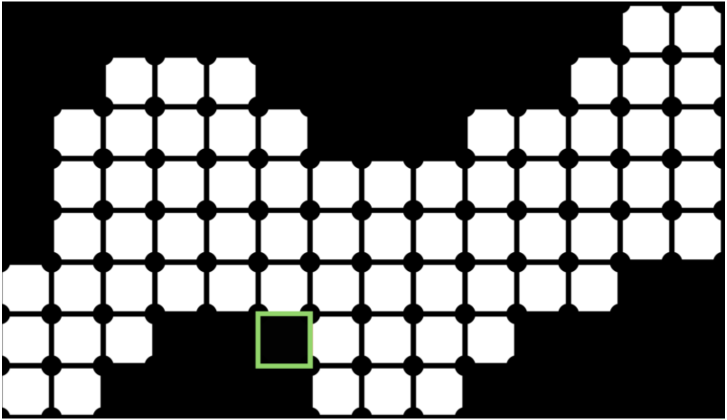

Not pivotal

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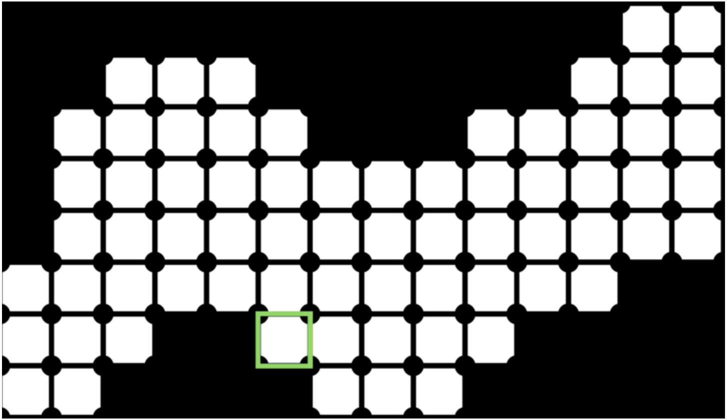

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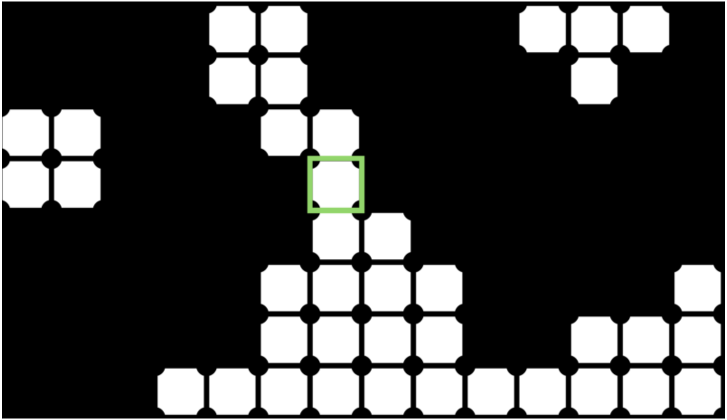

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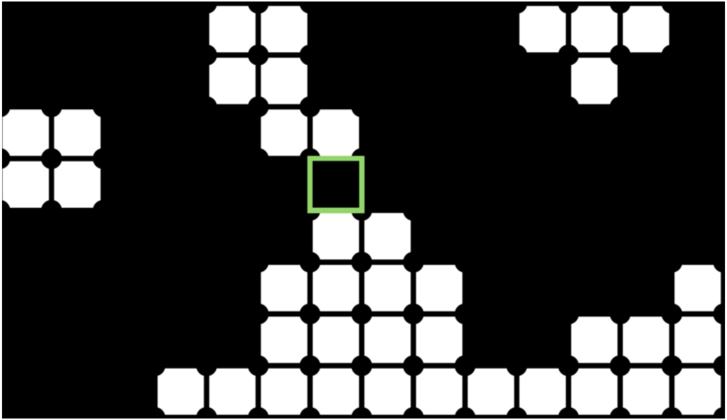

Pivotal

An ε-saddle point at x_i means $(f(x_i), \nabla_{x_i} f)$ is ε-small.

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$$I_i(A) \approx \varepsilon^2 \,.$$

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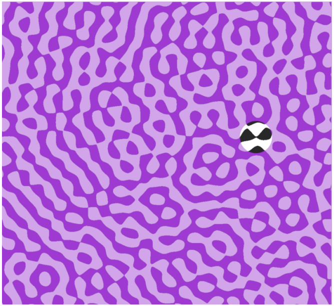
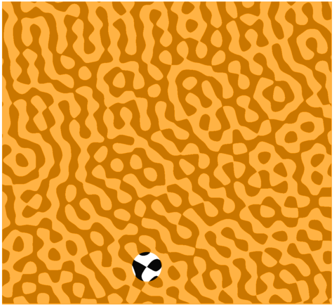
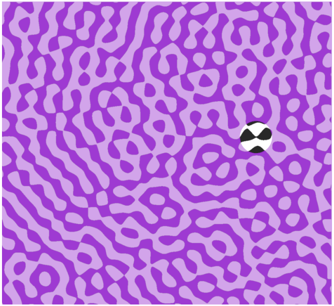
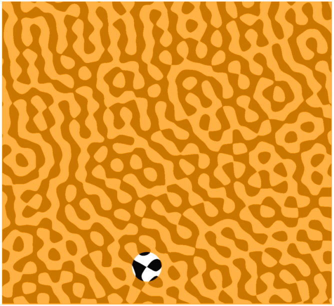




Conclusion :

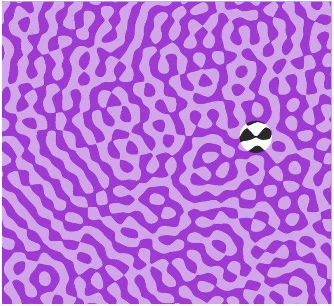


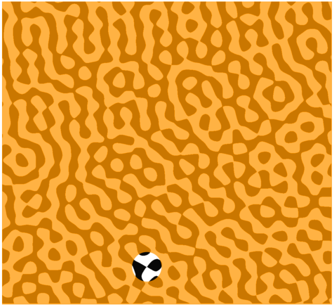


The correlation of A^ε and B^ε should be :

$$\approx \sum_{i \in \mathcal{Q}_1^\varepsilon,\ j \in \mathcal{Q}_2^\varepsilon} I_i(A^\varepsilon) \times \kappa(x_i - y_j) \times I_j(B^\varepsilon)$$

Conclusion :

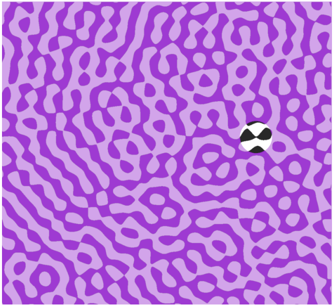


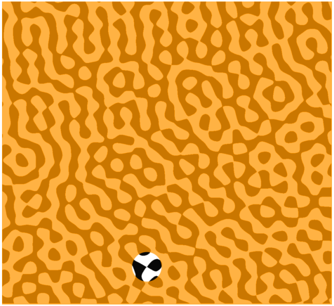


The correlation of A^ε and B^ε should be :

$$\leq C(r/\varepsilon)^2 \times (r/\varepsilon)^2 \times \varepsilon^2 \times \eta \times \varepsilon^2 \,.$$

Conclusion :





So that :

$$|\mathbb{P}[A^\varepsilon \cap B^\varepsilon] - \mathbb{P}[A^\varepsilon]\mathbb{P}[B^\varepsilon]| \leq C r^4 \eta \,.$$

Two applications...

For $R > 0$ let $N(R)$ be the number of connected components of $\mathcal{D}_+$ contained inside the box $[-R, R]^2$.

Theorem (NS15)

Under some mild condition on κ, there exists $\nu = \nu(\kappa) > 0$ such that

$$\frac{N(R)}{R^2} \xrightarrow{p.s \ and \ L^1} \nu \, .$$

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However : there is no control of the speed of convergence.

Under the same assumptions as for the previous result :

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Theorem (RV18)

Assume that $|\kappa(x)| \leq C|x|^{-\alpha}$ *for some* $\alpha > 4$*. Then, for all* $\varepsilon > 0$ *and* $0 < \delta < \alpha - 4$*,*

$$\mathbb{P}\left[N(R) \leq (\nu - \varepsilon)R^2\right] \leq C R^{4-\alpha+\delta}\,.$$

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Assume that $|\kappa(x)| \leq C|x|^{-\alpha}$ *for some* $\alpha > 4$. *Then, for all* $\varepsilon > 0$ *and* $0 < \delta < \alpha - 4$,

$$\mathbb{P}\left[N(R) \leq (\nu - \varepsilon)R^2\right] \leq CR^{4-\alpha+\delta}\,.$$

Note that this is only a lower concentration result.

Let $\mathrm{Cross}(R)$ be the event that there exists a black path in the box $[0, 2R] \times [0, R]$ joining the left and right sides.

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Theorem (BG16, BM17, RV18)

Assume that $\kappa(x) \geq 0$ and $|\kappa(x)| \leq C|x|^{-\alpha}$ for some $\alpha > 4$. Then, there is $c = c(\kappa) > 0$ such that for each $R \geq 1$,

$$\mathbb{P}[\mathrm{Cross}(R)] \in [c, 1 - c] \,.$$

What's next ?

- Decoupling in any dimension, for general topological events : work in progress with Dmitry Beliaev and Stephen Muirhead.

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- What about vector valued fields ?
- What about topological functionals of the fields ?

Thanks for listening!

References :

- BG16 : *Percolation of random nodal lines*, by Vincent Beffara and Damien Gayet
- BM17 : *Discretization schemes for level sets of planar Gaussian fields*, Dmitry Beliaev and Stephen Muirhead
- NS05 : *Transportation to random zeroes by the gradient flow*, by Fedor Nazarov and Mikhail Sodin
- NS15 : *Asymptotic laws for the spatial distribution and the number of connected components of zero sets of Gaussian random functions* by Fedor Nazarov and Mikhail Sodin
- Pit82 : *Gaussian Stochastic Processes*, Vladimir I. Piterbarg, Transl. of Math. Monographs, Vol 148
- RV18 : *Quasi-independence for nodal lines* by Alejandro Rivera and Hugo Vanneuville

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