C10.1b – – 16HT
Brownian Motion in Complex Analysis

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Contents

Course Description

C10.1b: Brownian Motion in Complex Analysis – Dr T Cass – 16HT

Subsections

1. Recommended Prerequisites

   The third year course on Martingales through measure (B10a) and the second year course on Complex variable. C10.1 Stochastic Differential Equations would be very desirable but is not (quite) essential.

2. Aims & Objectives

   Randomness plays a key feature in the behaviour of many high dimensional systems and so is intimately connected with applications. However, it also plays a key role in our understanding of many aspects of pure mathematics. This course will look at the deep interaction between 2 dimensional Brownian motion and complex analysis. At the core of these interactions is the conformal invariance of Brownian motion observed by Lévy and the relationship with Harmonic Functions (based on Martingales) first observed by Kakutani and Doob.

   Since that time there have been many developments and connections. The Hardy spaces of Fefferman and Stein, Value Distribution Theory, and most recently the stochastic Loewner equation (a topic of current and very exciting research).

   We will use martingales to examine and prove some deep theorems due to Nevanlinna, from value distribution theory for complex variable theory.

3. Synopsis

   Brownian motion. Continuous martingales and Lévy's characterisation in terms of Brownian motion. Conformal invariance of Brownian motion. Brownian motion tangles about two points and a proof of Picard's theorems. Harmonic functions on the disk and the solution of the Dirichlet problem. The Burkholder Davis inequalities and Hardy spaces. Harmonic and superharmonic functions - via probability. [Nevanlinna's Theorems].

4. Method of Examination

   Examination questions.

5. Reading

   For background material,

   H. A,. Priestley, Complex Analysis.

   Rogers & Williams, and Diffusions, Markov Processes and Martingales.

   The lecture notes of C10a.

   There isn't a perfect book for this course and we will refer to research papers to a limited extent.

   McKean, Stochastic Integrals, (1969) and recently reprinted [9]. Hard, short, with much relevant material and some mistakes! Excellent for the able!

   K. E. Petersen, Brownian Motion, Hardy Spaces and Bounded Mean Oscillation, [12].

   T. K. Carne, Brownian Motion and Nevanlinna Theory, [4]

6. Further Reading

   Richard F. Bass, Probabilistic Techniques in Analysis, [2]

   Lars Ahlfors, Complex Variable, [1]

   Jean-Claude Gruet, Nevanlinna Theory, Fuchsian Functions and Brownian Motion Windings, [6]

   Key historical research papers (very non-trivial) on Hardy Spaces, BMO and and Martingales can be found in [7, 5, 3].

Introduction

Like the counting numbers, probability theory is a topic that obviously relates in a fairly direct way to the world we live in. Non-mathematicians have an experience and intuition of it in quite complex settings. However, and even with this experience, non-mathematicians would find it very hard indeed to appreciate the universal role the counting numbers play almost all parts of mathematics.

Probability theory only became a rigorous tool in mathematics after the work of Kolmogorov[1] which appeared in the late 1920's, culminating in his decisive text [Grundbegriffe der Wahrscheinlichkeitsrechnung, Ergebnisse der Math., v. 2, no. 3, Springer, Berlin, 1933]. Since that time its importance has grown significantly. It is easy to recognise some of it's many applications outside mathematics - but much more difficult to imagine how it could add to our understanding of purely mathematical topics not directly related to probability theory.

This course is intended to expose some of the links between probability and Complex Analysis. Complex analysis is a deep and always surprising area of mathematics - which is taught less often than it used to be. in part because, quite correctly, fashions change and the opportunities for research have declined. But, as is often the case with deep and excellent areas of mathematics, the area has recently re-invigorated itself with different priorities and emphasis. Stochastic Loewner Equations were recently introduced by Oded Schramm and exploit the same conformal invariance of Brownian motion we mention here along with the stochastic differential equations of Professor Lyons's course to solve a variety of open problems of contemporary interest in Conformal Field Theory. The original results continue to be part of the general methodology and toolbox of analysts and probabilists in much the same way that homology and algebraic topology are standard tools for topologists and algebraists.

Before beginning our story one should mention that, without question, the flow of information between probability and complex function theory is bidirectional. Perhaps the most important tool in probability theory is the humble martingale. The concept and the major theorems were developed by Joe Doob (who sadly died in 2004[2]). Now, a martingale can be viewed as an infinite dimensional generalisation of the concept of solution to the Laplace equation on the disk - and Doob's thesis was on the behaviour of analytic functions near the boundary of the disk and his results on martingales can be seen as generalisations of much more classical results. It seems very likely that Doob's

[1] http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Kolmogorov.html
[2] http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Doob.html

understanding of complex function theory - together with an exposure to probability because he was working in a statistics department (because no major math department offered him a job (he was Jewish)) lead to the development of martingales as we know them today.

Part I

Conformal Invariance and Brownian Motion

Chapter 1

Revision on Complex Variable

The complex plane $\mathbb{C}$ comprises the algebra of points $z = x + iy$, where x and y are real numbers and $i^2 = -1$. The smooth functions $f(z) = u(x, y) + iv(x, y)$ mapping $\mathbb{C}$ to $\mathbb{C}$ can be identified with smooth function from $\mathbb{R}^2 \to \mathbb{R}^2$.Within the class of smooth functions, polynomials $\sum_{n=0}^{N} a_n z^n$ form a special subclass.

Definition 1 *We say that a function $f : D \to \mathbb{C}$ defined on an open subset D of $\mathbb{C}$ and given by $f = u + iv$ with $u, v : \mathbb{R}^2 \to \mathbb{R}^2$ is analytic on D if it satisfies the Cauchy Riemann equations*[1]

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0$$

everywhere on D.

The class of analytic funtions is closed under composition and addition, and so, since $f(z) = z$ is clearly analytic, polynomials are in this class. In general these functions f are quite remarkable; they are smooth; infinitely differentiable in the complex sense; and have an exact local power series representations. The most remarkable results include Cauchy's theorem, the integral representation formula, Rouche's theorem and its connection with winding numbers.

Definition 2 *A open subset X of a topological space Y is said to be disconnected if there exist open subsets U and V of Y such that $U \cap V = \emptyset$ and $U \cup V = X$. If X is not disconnected we say it is connected.*

Recall that continuous, integer valued functions defined on a connected set are constant.

Definition 3 *A domain is an open, connected subset of $\mathbb{C}$.*

Definition 4 *A curve $\gamma : [a, b] \to \mathbb{C}$ is said to be closed or is called a loop if $\gamma(a) = \gamma(b)$*

[1]The notation $z = x + iy = (x, y)$ and $f(z) = f(x, y) = u(x, y) + iv(x, y)$ will allow us to transfer between $\mathbb{C}$ and $\mathbb{R}^2$. $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ then refer to partial deriatives of f taking values in $\mathbb{R}^2$. See Priestly, Section 1.10 and Theorem 5.3.

If $\gamma : [a, b] \to \mathbb{C}$ is a (continuously differentiable) curve which does not pass through $0 \in \mathbb{C}$, a continuous choice of the argument on γ is a continuous map $\theta : [a, b] \to \mathbb{R}$ such that $\gamma(t) = |\gamma(t)| e^{i\theta(t)}$. The quantity $\theta(b) - \theta(a)$ measures the angle turned through by γ.

Definition 5 *We call* $(\theta(b) - \theta(a))/2\pi$ *the winding number* $\Gamma(\gamma, 0)$ *of* γ *about* 0, *and we may define* $\Gamma(\gamma, w)$ *for* $w \in \mathbb{C}$ *similarly.*

One can check that this definition is well defined by taking another continuous choice of the argument ϕ and observing that $\theta(t) - \phi(t)$ is an integer multiple of 2π.Then, the continuous function $(\theta - \phi)/2\pi$ is integer valued and since $[a, b]$ is connected it must be constant giving $\theta(b) - \theta(a) = \phi(b) - \phi(a)$. If γ is a closed curve then $\Gamma(\gamma, w)$ is integer valued and we can describe it using an integral representation formula (see chapter 3). We call a closed curve $\gamma : [a, b] \to D \subseteq \mathbb{C}$ simple if it does not cross itself, i.e. $\gamma(s) = \gamma(t)$ for s and t distinct if and only if s and t are the endpoints a and b. A contractible loop is one which can be continuously deformed to a constant loop.

Theorem 6 (Taylor's Theorem) *If D is a domain, γ is a simple closed curve that is contractible in D, winding around z once, and f is an analytic function defined on D then*[2]

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(\gamma)}{\gamma - z} d\gamma \\ f^{(n)}(z) &= \frac{n!}{2\pi i}\int_\gamma \frac{f(\gamma)}{(\gamma - z)^{n+1}} d\gamma \end{aligned}$$

1.1 Values taken by Entire functions

Definition 7 *We say an analytic function $f(z)$ defined on the whole of $\mathbb{C}$ is an entire function.*

One of our analytical objectives is to study the range of an entire function. This branch of complex analysis is known as Nevanlinna theory, after the Finnish mathematician Rolf Nevanlinna. We can find some easy results in complex analysis as follows but, as we will see in later sections, much more is true.

Corollary 8 *If f is an entire function and $|f(z)| < M < \infty$ for all $z \in \mathbb{C}$ then f is constant.*

Proof. Suppose that $|f(z)| < M$ on $|z| \leq R$ then, putting $\gamma(t) = Re^{2\pi it}$, one has for $|z| < R$ that

$$\begin{aligned} f^{(n)}(z) &= \frac{n!}{2\pi i}\int_\gamma \frac{f(\gamma)}{(\gamma - z)^{n+1}} d\gamma \\ &= n!\int_0^1 \frac{f\left(Re^{2\pi it}\right) Re^{2\pi it}}{\left(Re^{2\pi it} - z\right)^{n+1}} dt \end{aligned}$$

[2]The notation $\int_\gamma f(\gamma) d\gamma$ refers to $\int_{t=0}^1 f(\gamma(t))\gamma(t)$ where $\gamma : [0, 1] \to \mathbb{C}$. This (standard) notation will not cause us confusion. See Priestley, Section 10.3.

and

$$|f'(z_0)| \leq \frac{MR}{|R-r|^2}$$

for $|z_0| < r$. Applying the mean value theorem we have

$$\begin{aligned} |f(z_0) - f(z_1)| &\leq \frac{MR}{|R-r|^2}|z_0 - z_1| \\ &\leq \frac{2MRr}{|R-r|^2} \end{aligned}$$

if for $|z_1| < r$ as well. If f is entire then we can let $R \to \infty$ and conclude that f is constant. ■

The quantitative estimates we derived in the proof are of independent interest. For example (as you would see in the functional analysis course C4a), we could also conclude that any family of analytic functions on a common domain with a common bound are equicontinuous. By the Stone-Weierstrass and then Arzela-Ascoli theorems the restrictions of these functions to a compact set K will always be relatively compact in the uniform norm as a subset of $C(K)$.

So we have seen that an entire function that is not constant must take many values (it cannot be bounded). However, the most obvious entire functions are polynomials. We have the famous

Theorem 9 (Fundamental Theorem of Algebra) *If $P(z)$ is a polynomial of degree d then*

$$P(z) = w$$

has exactly d complex solutions (when multiplicity is taken into account).

Proof. It is enough to treat the case where P is monic and $w = 0$ so that

$$P(z) = z^d + a_{d-1}z^{d-1} + \ldots + a_0.$$

Let $\gamma(t) = Re^{2\pi it}$ then using either a simple application of the Residue Theorem or by applying the Argument Principle it follows that the number of zeros inside the disk $|z| = R$ is given by

$$\frac{1}{2\pi i}\int_\gamma \frac{P'(\gamma)}{P(\gamma)}d\gamma, \tag{1.1}$$

provided there is no zero of P on the curve γ. Choose

$$R > 1 + d\max\{a_i | i = 0, \ldots, d-1\}.$$

Now, if $|z| > 1$ then

$$\begin{aligned} |P(z)| &> |z|^d - d\max\{|a_i| | i = 0, \ldots, d-1\}|z|^{d-1} \\ &= (|z| - d\max\{|a_i| | i = 0, \ldots, d-1\})|z|^{d-1} \end{aligned}$$

and by our choice of R,if $|z| > R$ then

$$|P(z)| > |z|^{d-1} > 1$$

and is not zero. Define

$$P_\theta = \theta z^d + (1-\theta) P$$

then the same argument applies to show that there are no zeros of of P_θ on the curve γ providing we choose

$$R > 1 + d \max \{|a_i| | i = 0, \ldots, d-1\}$$

and consider only $\theta \in [0,1]$. We conclude that, for

$$R > 1 + d \max \{|a_i| | i = 0, \ldots, d-1\},$$

the function

$$n(\theta, R) := \frac{1}{2\pi i} \int_\gamma \frac{P'_\theta(\gamma)}{P_\theta(\gamma)} d\gamma. \tag{1.2}$$

is defined for every $\theta \in [0,1]$.It is easy to prove that it is also continuous. As $n(\theta, R)$ counts the number of zeros of P_θ it is also integer valued. The intermediate value theorem tells us that any integer valued continuous function is constant. A trivial computation shows that $n(1, R) = d$ so $n(0, R) = d$ and we have proved the theorem. ■

Problem 10 *Polynomials take all values an equal number of times. Can we say something similar about analytic functions? Unlike polynomials, they can omit values.*

Lemma 11 *The function e^z is an entire function that omits the point zero.*

Proof. The series for e^z converges at every point $z \in \mathbb{C}$ and so e^z is certainly entire. The absolute convergence of that series representation allows one to conclude

$$\begin{aligned} e^z e^w &= e^{z+w} \\ e^{-z} &= 1/e^z \end{aligned}$$

In particular, if e^z had a zero it would also have a pole and it does not have the latter. ■

Theorem 12 (Picard) *An entire function f can omit at most one value.*

We will give a proof of this based on the conformal invariace of two dimensional Brownian motion and the fact that Brownian motion 'tangles' around two points!

A much deeper theorem of Nevanlinna explains how all values are taken essentially the same number of times and controls the total deficiency. This also has a probabilistic proof.

1.2 Harmonic functions and the real parts of analytic functions

Definition 13 *If $f : \mathbb{C} \to \mathbb{C}$ is differentiable then we define operators (the ∂ and $\bar{\partial}$ (dee and dee-bar) operators)*

$$\begin{aligned} \bar{\partial} f &: \frac{\partial}{\partial \bar{z}} f := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y}\right) \\ \partial f &: \frac{\partial}{\partial z} f := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y}\right) \end{aligned}$$

Lemma 14 *A function f is analytic if $\bar{\partial} f = 0$ and in that case its derivative in the complex sense is given by ∂f.*

Note that ∂f and $\bar{\partial} f$ are defined for and smooth real or complex valued f defined on an open set in $\mathbb{C}$ and not just for analytic functions.

Definition 15 *A twice differentiable (real or complex) function f is harmonic if*

$$\left(\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2\right) f = 0$$

which we write $\Delta f = 0$ where $\Delta := \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2$.

Remark 16 *Recall from Itô's formula that if f is harmonic and B_t is 2 dimensional Brownian Motion then $f(B_t)$ is a local martingale:*

$$\begin{aligned} f(B_t) - f(B_0) &= \int_0^t \frac{\partial f}{\partial x}(B_s)\, dB_s^1 + \frac{\partial f}{\partial y}(B_s)\, dB_s^2 + \frac{1}{2}\int \Delta f(B_s)\, ds \\ &= \int_0^t \frac{\partial f}{\partial x}(B_s)\, dB_s^1 + \frac{\partial f}{\partial y}(B_s)\, dB_s^2 \end{aligned}$$

Exercise 17 *If f is twice continuously differentiable then $\frac{\partial}{\partial x}\frac{\partial}{\partial y} f = \frac{\partial}{\partial y}\frac{\partial}{\partial x} f$ and one observes that*

$$\partial\bar{\partial} f = \bar{\partial}\partial f = \frac{1}{4}\Delta f.$$

In particular, if f is harmonic then ∂f is analytic.

Lemma 18 *If the function f is analytic then it and its real and imaginary parts u and v are harmonic.*

Proof. Suppose that f is analytic, then a calculation shows that $\frac{\partial}{\partial z} f = f'$, since f' is also analytic and so $\frac{\partial}{\partial z} f$ satisfies the Cauchy Riemann equations. Hence $\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} f = 0$. ■

There is a strong converse to this statement.

Theorem 19 *Suppose that D is a simply connected domain, $z_0 \in D$, and that u is a harmonic function on D. Then there is a harmonic function v on D such that $f := u + iv$ is analytic. If we specify that $v(z_0) = c$ then v is unique.*

Proof. Since u is harmonic, $\bar{\partial}\partial u = 0$ and hence $g := 2\partial u$ is analytic. Let

$$f(z) := \int_{\gamma_z} g(\gamma_z)\, d\gamma_z + u(z_0)$$

where $\gamma_z : [0,1] \to D$ is a smooth path in D that starts at z_0 and finishes at z. By Cauchy's theorem the value of the integral will be the same if performed over a different path θ_z which is homotopic to γ_z and has the same end points. (Integration round a simply connected loop is zero). As D is simply connected all paths beginning at z_0 and terminating at z have this property, so the definition of f is well defined (it does not depend on γ). It is obvious to see that f is

analytic and a simple computation using the definition of the ∂ operator shows that

$$\begin{aligned} \operatorname{Re} f(z) &= \int_0^1 \nabla u(\gamma_z(t))\,\dot{\gamma}_z(t)\,dt + u(z_0) \\ &= \int_0^1 \frac{d}{dt} u(\gamma_z(t))\,dt + u(z_0) = u(z)\,. \end{aligned}$$

The uniqueness follows from considering the difference between two extensions $f_1 - f_2$. Such a function is analytic and purely imaginary and hence constant (by the Cauchy Riemann equations), since $f_1(z_0) - f_2(z_0) = 0$ it follows that $f_1 - f_2 \equiv 0$. ■

Chapter 2

Brownian motion

We know from the course C10a Stochastic Differential Equations that for each $d \geq 1$ there is a process B_t, a d-dimensional canonical Brownian motion defined on some filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and more precisely, we can conclude from Professor Lyons's notes that if

$$B_t = \left(B_t^1, \ldots B_t^d\right)$$

then the co-ordinates are independent 1-dimensional Brownian motions. It is clear they are adapted to the same filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. The converse is also true: if B_t^i are independent one dimensional Brownian motions adapted to the same filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ then $B_t = \left(B_t^1, \ldots B_t^d\right)$ is a d-dimensional Brownian motion.

Random variables taking values in in the complex plane $\mathbb{C}$ can be identified with random variables in $\mathbb{R}^2$ using our identification $x + iy = (x, y)$.

Definition 20 *Canonical complex Brownian motion $Z_t = X_t + iY_t$ is a stochastic process in $\mathbb{C}$,starting at zero, and adapted to a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ whose real and imaginary parts X_t and Y_t are independent real 1-dimensional Brownian motions.*

A process Z_t is a canonical complex Brownian motion if and only if (X_t, Y_t) is canonical 2-dimensional real Brownian motion on the (x, y)-plane. A fundamental connection between complex variable theory and complex Brownian motion comes from Paul Lévy's theorem that, if $f(z)$ is an analytic function and Z_t is complex Brownian motion then $f(Z_t)$ is (up to a change of time) a complex Brownian motion as well.

2.1 Lévy's characterisation for martingales

Now Professor Lyons has in his notes the characterisation of the bracket process associated to two continuous real martingales (and later extended to local martingale and semi-martingales):

Theorem 21 *Let $(M)_{t\geq 0}$ and $(N_t)_{t\geq 0}$ be two continuous, square-integrable martingales, and let*

$$\langle M,N\rangle_t = \frac{1}{4}\left(\langle M+N\rangle_t - \langle M-N\rangle_t\right)$$

be called the bracket (or covariation) process of M and N. Then, $\langle M,N\rangle$ is the unique increasing, adapted, continuous process with $\langle M,N\rangle_0 = 0$ and such that $M_tN_t - \langle M,N\rangle_t$ is a martingale. Moreover,

$$\sum_{i=1}^{n}\left(M_{t_i} - M_{t_{i-1}}\right)\left(N_{t_i} - N_{t_{i-1}}\right) \xrightarrow{P} \langle M,N\rangle_t$$

as $m(D) = \max\left(t_i - t_{i-1}\right) \to 0$ and $D = \{0 = t_0 < ... < t_n = t\}$. Note that $\langle M,N\rangle = \langle N,m\rangle$ and if M and N are independent, $\langle M,N\rangle = 0$.

and this bracket process is used to formulate the correction term needed to get a fundamental theorem of calculus for Itô integration against a (semi-)martingale; the resulting formula is known as Itôs formula:

Theorem 22 *(Itô's formula) Let $X_t = \left(X_t^1, ..., X_t^d\right)$ be a continuous semi-martingale in $\mathbb{R}^d$. Let $f \in C^2\left(\mathbb{R}^d, \mathbb{R}\right)$ then*

$$f\left(X_t\right) - f\left(X_0\right) = \sum_{i=1}^{d}\int_0^t \frac{\partial f}{\partial x_i}\left(X_s\right) dX_s^i + \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}\left(X_s\right) d\left\langle X^i, X^j\right\rangle_s .$$

In the special case of d-dimensional Brownian motion the bracket process has a very simple form and one gets (in vector notation)

$$f\left(B_t\right) - f\left(B_0\right) = \int_0^t \nabla f\left(B_s\right) dB_s + \int_0^t \frac{1}{2}\Delta f\left(B_s\right) ds.$$

If we let

$$M_t^f = f\left(B_t\right) - f\left(B_0\right) - \int_0^t \frac{1}{2}\Delta f\left(B_s\right) ds,$$

then M^f is a local martingale for every $f \in C^2\left(\mathbb{R}^d, \mathbb{R}\right)$ and

$$\left\langle M^f, M^g\right\rangle_t = \int_0^t \langle \nabla f, \nabla g\rangle\left(B_s\right) ds.$$

One of the key consequences of the Itô formula is the Lévy characterisation of Brownian motion (that any martingale with the bracket process of Brownian motion is Brownian motion)

Theorem 23 *(Lévy characterisation) Let $M_t = \left(M_t^1, ..., M_t^d\right)$ be an adapted, continuous adapted process on a filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ satisfying the usual conditions[1] with $M_0 = 0$. Then $(M)_{t\geq 0}$ is a Brownian motion if and only if*

[1] These are standard regularity conditions on the filtered space which we will take for granted for now on. The reasons why we need them are not obvious, and have probably been glossed over by your previous courses. You do not need to mention them in your own work. For a proper discussion look in Rogers and Williams.

1. *Each* M_t^i *is a continuous local martingale.*
2. *For any* $i, j \in \{1, ...d\}$ *the process* $M_t^i M_t^j - \delta_{ij} t$ *is a martingale, i.e.* $\left\langle M_t^i, M_t^j \right\rangle = \delta_{ij} t$

and the application that every real continuous local martingale admits a time change to a Brownian motion:

Theorem 24 *(Dubins and Schwarz) Let* $(M)_{t \geq 0}$ *be a continuous local martingale on* $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ *satisfying* $M_0 = 0$ *and* $\langle M \rangle_\infty = \infty$*. Let*

$$T(t) = \inf \{s : \langle M \rangle_s > t\},$$

then $T(t)$ *is a stopping time for each* $t \geq 0$*. Moreover* $B_t = M_{T(t)}$ *is an* $\left\{\mathcal{F}_{T(t)}\right\}-$ *Brownian motion and* $M_t = B_{\langle M \rangle_t}$*.*

We need to rewrite these theorems for the complex case. The first point to note is that the definition of a martingale does not change - the definition of conditional expectation makes sense for vector valued variables and processes providing they are integrable.

In finite dimensions constructing this conditional expectation it is very simple. Suppose that $X \in V$ where V is a finite dimensional vector space.

Lemma 25 *If* X *is a random variable on* $(\Omega, \mathcal{F}, \mathbb{P})$ *with values in* V *and* $(e_i)_{i=1}^d$ *is any basis for* V *then*

$$\mathbb{E}[X|\mathcal{G}] = \sum_{i=1}^d \mathbb{E}\left[x^i|\mathcal{G}\right] e_i$$

where the x^i *are determined by the relation* $X(\omega) = \sum_{i=1}^d x^i(\omega) e_i$ *for all* ω*.*

Definition 26 *A complex martingale is a process* M_t *in* $\mathbb{C}$ *or* $\mathbb{C}^d$ *that is a vector martingale when* $\mathbb{C}$ *or* $\mathbb{C}^d$ *is regarded as a real vector space.*

However there are changes brought on by the complex structure. It is clear that if M is a complex martingale then so is its complex conjugate $\bar{M}_t$. The bracket process is a bit more tricky to define in this setting - because there is a choice! Should it be bilinear or sesquilinear

$$\begin{aligned} \langle \alpha M, \beta N \rangle &= \\ & \quad ?\alpha\beta \langle M, N \rangle \\ & \quad ?\alpha\bar{\beta} \langle M, N \rangle . \end{aligned}$$

There is no correct answer and we adopt the first convention.

Definition 27 *Let* M *and* N *be two continuous square integrable complex martingales then* $\langle M, N \rangle_t$ *is the unique bounded variation, adapted, and continuous* $\mathbb{C}$*-valued process with initial value zero so that*

$$M_t N_t - \langle M, N \rangle_t$$

is a martingale.[2]

[2]Getoor and Sharpe take the alternative approach where $M_t N_t - \left\langle M, \bar{N} \right\rangle_t$ is a martingale.

Lemma 28 *If $M_t = R_t + iS_t$, $\tilde{M}_t = \tilde{R}_t + i\tilde{S}_t$ where R_t, S_t etc. are real valued martingales then*

$$\langle M, M\rangle_t = \left\langle R, \tilde{R}\right\rangle_t - \left\langle S, \tilde{S}\right\rangle_t + i\left(\left\langle R, \tilde{S}\right\rangle_t + \left\langle \tilde{R}, S\right\rangle_t\right)$$

Definition 29 *A continuous complex (local) martingale M_t is a conformal (local) martingale if $\langle M, M\rangle_t \equiv 0$.*

Whenever we talk about conformality we refer to the complex case.

Remark 30 *It is obvious that a square integrable martingale M_t is conformal if and only if its square M_t^2 is also a conformal martingale.*

Lemma 31 *Complex Brownian motion is a conformal martingale.*

Proof. Let $Z_t = X_t + iY_t$ be complex Brownian motion adapted to a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ whose real and imaginary parts X_t and Y_t. Then X_t and Y_t are independent real 1-dimensional Brownian motions over $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. So

$$\begin{aligned}\langle Z, Z\rangle_t &= \langle X, X\rangle_t - \langle Y, Y\rangle_t + 2i\langle X, Y\rangle_t \\ &= t - t + 0 \\ &= 0.\end{aligned}$$

■

The process $\left\langle Z, \bar{Z}\right\rangle_t$ plays the role of the quadratic variation for conformal martingales. Thus conformality of Z implies that $\langle X, \bar{X}\rangle = \langle Y, \bar{Y}\rangle$, which is interpreted as the X and Y components of Z travel at the same rate[3]. We have an extension of Lévy's theorem (also due to Lévy)

Lemma 32 *Let Z_t be a complex valued continuous adapted process. Then Z_t is a complex Brownian motion if and only if*

1. *Z_t is a conformal local martingale.*
2. *$\langle Z, \bar{Z}\rangle_t = 2t$.*

Proof. This is easily deduced from the real theorem. We may compute

$$\langle Z, \overline{Z}\rangle_t = \langle X, X\rangle_t + \langle Y, Y\rangle_t$$

and the forward implication is obvious. For the converse, by conformality

$$\langle X, X\rangle = \langle Y, Y\rangle_t$$

and combining this with the above equation gives $\langle X, X\rangle_t = \langle Y, Y\rangle_t = t$. We may then apply the real theorem. ■

Lemma 33 *If Z_t is a conformal martingale with continuous sample paths then $\left\langle Z, \bar{Z}\right\rangle_t$ is positive, increasing and continuous. Moreover, if $\left\langle Z, \bar{Z}\right\rangle_t$ is strictly increasing[4] and*

$$\tau(s) := \inf\left\{t \mid \left\langle Z, \bar{Z}\right\rangle_t > 2s\right\}$$

is finite for all s then $Z_{\tau(s)}$ is a complex Brownian motion on $\left(\Omega, \mathcal{F}_{\tau(s)}, \mathbb{P}\right)$.

[3]There is a discussion of conformality as a concept in the solutions to Assignment 2

[4]This extra condition is required in the complex case. Without it $\langle X, X\rangle$ might stay constant while $\langle Y, Y\rangle$ increases, and thus X would be constant while Y varies; we would not be able to time change this to 2-dimensional Brownian motion.

Proof. The first sentence is obvious since if $Z_t = X_t^1 + iX_t^2$ then it is easy to check that $\langle Z, \bar{Z}\rangle_t = \langle X^1, X^1\rangle_t + \langle X^2, X^2\rangle_t$. The rest is essentially a re-writing of the result for the real 2 dimensional case. We first note that because $\langle Z, \bar{Z}\rangle_t$ is strictly increasing τ must be continuous and hence we can deduce that $Z_{\tau(s)}$ is continuous. Next, we observe that the $\tau(s)$ are stopping times and we introduce

$$\sigma(n) = \inf\{t \geq 0 | \, |Z_t| > n\}, \quad \phi(n) = \frac{1}{2}\langle Z, \bar{Z}\rangle_{\sigma(n)}.$$

Then, it may be easily checked that $\tau(t \wedge \phi(n)) = \sigma(n) \wedge \tau(t)$ for all t, and hence that $Z_{\tau(t\wedge\phi(n))} = Z_{\sigma(n)\wedge\tau(t)}$. Since $\{\phi(n) \leq t\} = \{\sigma(n) \leq \tau(t)\} \in \mathcal{F}_{\tau(t)}$ it follows that $\phi(n)$ is a $\{\mathcal{F}_{\tau(t)}\}$ stopping time so by the optional stopping theorem (applied to the bounded martingale $Z_{t\wedge\sigma(n)}$) we have

$$E\left[Z_{\tau(t\wedge\phi(n))}|\mathcal{F}_{\tau(s)}\right] = E\left[Z_{\sigma(n)\wedge\tau(t)}|\mathcal{F}_{\tau(s)}\right] == Z_{\sigma(n)\wedge\tau(s)} = Z_{\tau(s\wedge\phi(n))} \text{ for } s \leq t,$$

and since $\phi(n) \uparrow \infty$ as $n \to \infty$ it follows that $Z_{\tau(t)}$ is a local martingale with respect to $(\Omega, \mathcal{F}_{\tau(t)}, \mathbb{P})$. By hypothesis Z is a conformal martingale so $Z^2_{\cdot\wedge\sigma(n)}$ and $Z\bar{Z}_{\cdot\wedge\sigma(n)} - \langle Z, \bar{Z}\rangle_{\cdot\wedge\sigma(n)}$ are bounded martingales and two applications of optional stopping give

$$E\left[Z^2_{\tau(t\wedge\phi(n))}|\mathcal{F}_{\tau(s)}\right] = Z^2_{\tau(s\wedge\phi(n))}$$

and

$$\begin{aligned}
& E\left[Z_{\tau(t\wedge\phi(n))}\bar{Z}_{\tau(t\wedge\phi(n))} - \langle Z, \bar{Z}\rangle_{\tau(t\wedge\phi(n))}|\mathcal{F}_{\tau(s)}\right] \\
= \; & E\left[Z_{\sigma(n)\wedge\tau(t)}\bar{Z}_{\sigma(n)\wedge\tau(t)} - (2t)\wedge\sigma(n)|\mathcal{F}_{\tau(s)}\right] \\
= \; & Z_{\sigma(n)\wedge\tau(s)}\bar{Z}_{\sigma(n)\wedge\tau(s)} - (2s)\wedge\sigma(n) \\
= \; & Z_{\tau(s\wedge\phi(n))}\bar{Z}_{\tau(s\wedge\phi(n))} - \langle Z, \bar{Z}\rangle_{\tau(s\wedge\phi(n))}
\end{aligned}$$

By comparing real and imaginary parts in these formulae we see that $\left(X^1_{\tau(t)}\right)^2 - \left(X^2_{\tau(t)}\right)^2, X^1_{\tau(t)}X^2_{\tau(t)}$ and $\left(X^1_{\tau(t)}\right)^2 + \left(X^2_{\tau(t)}\right)^2 - 2t$ are local martingales. From this we can deduce immediately that the continuous local martingales X^1 and X^2 are such that $\langle X^i, X^j\rangle_{\tau(t)} = \delta_{ij}t$ and hence by Lévy's characterisation $\left(X^1_{\tau(t)}, X^2_{\tau(t)}\right)$ is a two dimensional Brownian motion, which implies that $Z_{\tau(t)}$ is a complex Brownian motion on $(\Omega, \mathcal{F}_{\tau(s)}, \mathbb{P})$. ■

2.2 Itô's Lemma - The complex case

Theorem 34 (Ito's Lemma - Complex Variable case) *Suppose that Z is a continuous complex martingale and that f is a C^2 function then*

$$\begin{aligned}
f(Z_t) - f(Z_0) \; = \; & \int_0^t \partial f(Z_s)\; dZ_s + \int_0^t \bar{\partial} f(Z_s)\; d\bar{Z}_s \\
& + \left(\int_0^t \partial\partial f(Z_s)\; d\langle Z, Z\rangle_s + \int_0^t \bar{\partial}\bar{\partial} f(Z_s)\; d\langle \bar{Z}, \bar{Z}\rangle_s\right) \\
& + 2\int_0^t \partial\bar{\partial} f(Z_s)\; d\langle Z, \bar{Z}\rangle_s
\end{aligned}$$

The proof is a matter of checking algebra[5].

Corollary 35 *In the case Z_t is a conformal martingale we get a simplification:*

$$\begin{aligned} f(Z_t) - f(Z_0) &= \int_0^t \partial f(Z_s)\ dZ_s + \int_0^t \bar{\partial} f(Z_s)\ d\bar{Z}_s \\ &\quad +2\int_0^t \partial\bar{\partial} f(Z_s)\ d\left\langle Z, \bar{Z}\right\rangle_s \end{aligned}$$

Corollary 36 *In the case Z_t is a conformal martingale and f is harmonic then $f(Z)$ is a martingale*

$$f(Z_t) - f(Z_0) = \int_0^t \partial f(Z_s)\ dZ_s + \int_0^t \bar{\partial} f(Z_s)\ d\bar{Z}_s$$

Corollary 37 *In the case Z_t is a conformal martingale and f is analytic then $f(Z)$ is a martingale then Itô's formulae is classical:*

$$f(Z_t) - f(Z_0) = \int_0^t f'(Z_s)\ dZ_s$$

Lemma 38 *If Z_t is a conformal local martingale which always takes values in a set E and f is an analytic function on E then $f(Z_t)$ is a conformal local martingale with*

$$\begin{aligned} f(Z_t) - f(Z_0) &= \int_0^t f'(Z_s)\, dZ_s \\ \left\langle f(Z), \overline{f(Z)}\right\rangle_t &= \int_0^t f'(Z_s)\, \overline{f'(Z_s)\, d\left\langle Z, Z\right\rangle_s} \end{aligned}$$

Proof. Itô's formulae gives us the above formulae, and this confirms that $f(Z_t)$ is a local martingale. Since f^2 is also analytic $f(Z_t)^2$ is also a local martingale by Itô's formulae, and by a routine stopping argument we have that $f(Z_t)$ is a conformal local martingale ■

Corollary 39 *If Z_t is a complex Brownian motion then $e^{i\theta}Z_t$ is also a complex Brownian motion.*

Proof. Note that $z \to e^{i\theta}z$ is an analytic function with gradient one and hence $e^{i\theta}Z_t$ is a conformal martingale. Moreover, $\left\langle e^{i\theta}Z, , \overline{e^{i\theta}Z}\right\rangle_t = 2t$ and so it it is a complex Brownian motion by Lévy's characterisation. (Note: we did not define complex Brownian motion in a rotation invariant way.) ■

We are now in a position to prove conformal invariance of complex Brownian motion.

Theorem 40 (Lévy) *If f is an entire function and Z is a conformal local martingale with $\left\langle Z, \bar{Z}\right\rangle_t$ strictly increasing, then $f(Z_t)$ is a conformal local martingale. It's bracket $\left\langle f(Z), \overline{f(Z)}\right\rangle_t = \int_0^t |f'(Z_s)|^2\ d\left\langle Z, \bar{Z}\right\rangle_s$ is positive, strictly increasing and continuous and if $\tau(s) := \inf\left\{t \mid \left\langle f(Z), \overline{f(Z)}\right\rangle_t > 2s\right\}$ then $f(Z_\tau)$ is a complex Brownian motion.*

[5]See Assignment 1.

Proof. Itô's lemma gives that $f(Z_t)$ is a conformal local martingale with bracket process $\int_0^t |f'(Z_s)|^2 \; d\langle Z, \bar{Z}\rangle_s$, which is clearly finite at all stopping times $\tau < \infty$. It remains to show that this is strictly increasing to infinity. Strictly increasing will be shown in lectures, whereas convergence to infinity will be on assignment 2, in the case where Z is a complex Brownian motion. ■

A consequence of this is the following.

Corollary 41 *Complex Brownian motion is conformally invariant. In other words, if Z is a complex Brownian motion and f an entire function, then there exists a strictly increasing family of stopping times $(\tau_t)_{t\in[0,\infty)}$ of $\mathcal{F}_t = \sigma(Z_s : s \le t)$ such that $f(Z_{\tau_t})$ is a complex Brownian motion.*

Lemma 42 *Complex Brownian motion almost surely leaves every bounded set.*

Proof. Fix a radius $R > 0$. Then

$$\begin{aligned}
\mathbb{P}_0(\|Z_t\| \le R) &= \int_{r=0}^{R} \frac{1}{2\pi t} e^{-r^2/2t} \; 2\pi r dr \\
&= \left[-e^{-r^2/2t}\right]_0^R \\
&= 1 - e^{-R^2/2t} \\
&< R^2/2t, \quad t > 0
\end{aligned}$$

Put $T_i = i^2$. Then

$$\begin{aligned}
\sum_{i=1}^{\infty} \mathbb{P}_0(\|Z_{T_i}\| \le R) &< R^2 \sum_{i=1}^{\infty} \frac{1}{i^2} \\
&< \infty
\end{aligned}$$

and so by the first Borel Cantelli lemma, with probability one, for all but finitely many i one has $\|Z_{T_i}\| > R$. ■

Let A be any annulus $\{z \mid |z - z_0| \in (r, R)\}$ with $0 \in A$. Let T_A denote the first time Complex Brownian Motion Z_t leaves A.

$$T_A = \inf\{t \mid Z_t \notin A\}$$

Then T_A is a stopping time. Moreover, we have just proved that T_A is almost surely finite for almost all Brownian paths, and for those paths starting in A either $|Z_{T_A} - z_0| = r$ or $|Z_{T_A} - z_0| = R$.

Lemma 43 *For Brownian motion started at zero we have*

$$\begin{aligned}
\mathbb{P}(|Z_{T_A} - z_0| = r) &= \frac{\log|z_0| - \log|R|}{\log|r| - \log|R|} \\
\mathbb{P}(|Z_{T_A} - z_0| = R) &= \frac{\log|r| - \log|z_0|}{\log|r| - \log|R|}
\end{aligned}$$

Proof. Let $Z_t = X_t + iY_t$ and $z_0 = x + iy$ so that $R_t^2 = |Z_t - z_0|^2 = (X_t - x_0)^2 + (Y_t - y_0)^2$. Itô's formula (the real version) shows that

$$dR_t^2 = 2(X_t - x_0)\, dX_t + (Y_t - y_0)\, dY_t + 2dt = 2R_t dB_t + 2dt,$$

where $dB_t = R_t^{-1} \langle \mathbf{X}_t, d\mathbf{X}_t \rangle$, and $\mathbf{X}_t = (X_t - x_0, Y_t - y_0)$. B is then a local martinagle and since it is easy to check that $\langle B, B \rangle_t = t$ it must be a one dimensional Brownian motion by Lévy's characterisation. Another application of Itô gives

$$d \log R_t^2 = 2R_t^{-1} dB_t,$$

so that

$$\log |Z_t - z_0| = \log |z_0| + \int_0^t \frac{dB_s}{|Z_s - z_0|}$$

is a local martinagle. Then, by the definition of T_A and since $T_A$is a stopping time we see that $W_t := \log |Z_{t \wedge T_A} - z_0|$ is a bounded local martingale and hence a martingale. From the optional stopping theorem we see that

$$\begin{aligned} \log |z_0| &= \mathbb{E} \left[\log |Z_{T_A} - z_0| \right] \\ &= \mathbb{P} \left(|Z_{T_A} - z_0| = r \right) \log |r| + \\ &\quad \mathbb{P} \left(|Z_{T_A} - z_0| = R \right) \log |R| \end{aligned}$$

and if we set $p_r = \mathbb{P} \left(|Z_{T_A} - z_0| = r \right)$ then

$$\begin{aligned} \log |z_0| &= p_r \log |r| + (1 - p_r) \log |R| \\ p_r &= \frac{\log |z_0| - \log |R|}{\log |r| - \log |R|}. \end{aligned}$$

The result follows. ■

Corollary 44 (recurrence of planar Brownian motion) *If Z is complex Brownian motion started at zero, then for every $z_0 \in \mathbb{C}$ and every $\varepsilon > 0$ one has*

$$\mathbb{P} \left(\exists \ t > 0 \ \textit{such that} \ Z_t \in \{ |z - z_0| \leq \varepsilon \} \right) = 1$$

Proof. Suppose that $z_0 = 0$ then we are finished. Otherwise set $A = \{ z \mid |z - z_0| \in (\varepsilon, R) \}$. The result we have just proved shows that Z will exit A and will do so through the boundary $\{ |z - z_0| = \varepsilon \}$ with probabilty $\frac{\log |z_0| - \log |R|}{\log |r| - \log |R|}$.As the boundary is a subset of $\{ |z - z_0| \leq \varepsilon \}$ we can conclude that

$$\mathbb{P} \left(\exists \ t > 0 \text{ such that } Z_t \in \{ |z - z_0| \leq \varepsilon \} \right) \geq \frac{\log |z_0| - \log |R|}{\log |r| - \log |R|}$$

Letting $R \to \infty$ we conclude that

$$\mathbb{P} \left(\exists \ t > 0 \text{ such that } Z_t \in \{ |z - z_0| \leq \varepsilon \} \right) = 1.$$

■

Part II

Tangling, winding, and Picard's theorem

Chapter 3

Picard through Winding

Key theorems in complex analysis involve winding numbers and the argument principle, and their corollary Rouche's theorem. Consider a simply connected closed curve or contour γ and suppose that one is interested in counting the number N(γ) of solutions (including multiplicity) to $f(z) = 0$ in the domain bounded by the contour γ.Then the argument principle (a quite remarkable result) tells us that

$$N(f,\gamma) = \frac{1}{2\pi i}\int_{\gamma}\frac{f'(\gamma)}{f(\gamma)}d\gamma$$

Another related and very interesting integral is the winding number integral.

Theorem 45 *Consider a closed curve $\tau : [a,b] \to \mathbb{C}$ and for $w \notin \tau$ define.*

$$\Gamma(\tau, w) = \frac{1}{2\pi i}\int_{\tau}\frac{1}{\tau - w}d\tau$$

then $\Gamma(\tau, w)$ is an integer! $\Gamma(\tau, w)$ is known as the winding number of τ around w.

Proof. We have already seen that the winding number for closed curves is integer valued. By translating the curve we may assume $w = 0$ and define

$$h(t) = \int_{\tau[a,t]}\frac{1}{z}dz = \int_a^t \frac{\tau'(t)}{\tau(t)}dt$$

for $t \in [a,b]$. The chain rule shows that $\frac{d}{dt}\left(e^{-h(t)}\tau(t)\right) = 0$, hence $\tau(t) = e^{h(t)}\tau(a) = e^{\operatorname{Re} h(t)}e^{i\operatorname{Im} h(t)}\tau(a)$ and $\theta(t) = \arg\tau(a) + \operatorname{Im} h(t)$ gives a continuous choice of the argument of $\tau(t)$. Therefore, the total angle turned through by τ is given by

$$\operatorname{Im}\left(\int_{\tau[a,t]}\frac{1}{z}dz\right).$$

Since τ is closed we can say more, indeed we have $e^{h(b)} = 1$ which implies $h(b) = 2\pi i\Gamma(\tau, 0)$, and so

$$\Gamma(\tau, 0) = \frac{1}{2\pi i}\int_{\tau[a,t]}\frac{1}{z}dz = \frac{1}{2\pi i}\int_{\tau}\frac{1}{\tau}d\tau.$$

■

Note that it is obvious from this theorem that smoothly varying τ cannnot change $\Gamma(\tau, w)$ unless τ is varied in a way that it crosses w since the integral would vary continuously but is integer valued.

Theorem 46 *If f is an analytic function on a simply connected domain D bounded by a countour γ then the number of solutions (including multiplicity) to $f(z) = w$ in D is given by $\Gamma(\tau, w)$ where $\tau = f(\gamma)$.*

Proof. Simply make a substitution from τ to $f(\gamma)$ in $\frac{1}{2\pi i}\int_\tau \frac{1}{\tau - w} d\tau$ and apply the argument principle. ■

Theorem 47 *The function $\tau \to \Gamma(\tau, w)$, defined initially for smooth paths not going through w, by*

$$\Gamma(\tau, w) = \frac{1}{2\pi i}\int_\tau \frac{1}{\tau - w} d\tau$$

is continuous as τ varies in the uniform topology and is integer valued. It has a unique uniformly continuous extension to all continuous closed paths τ that do not go through w. It remains integer valued and is called the winding number of τ around w.

Definition 48 *A continuous function $\gamma(.)$ mapping a real interval $[t_0, t_1]$ (often $[0, 1]$) into a domain D*

$$\gamma : [t_0, t_1] \to D$$

is known as a path in D. If the initial point $\gamma(t_0)$ and final point $\gamma(t_1)$ of the path coincide the path is called a loop in D.

Definition 49 *If D is a domain, a loop γ in D is said to be contractible in D if there exist loops $\tilde{\gamma}(s, .)$ for each $s \in [0, 1]$ defined on the same interval $[t_0, t_1]$ as γ so that*

$$\begin{aligned} \tilde{\gamma}(1, t) &= \gamma(t), \ t \in [t_0, t_1] \\ \tilde{\gamma}(0, t) &= \tilde{\gamma}(0, t'), \ t, t' \in [t_0, t_1] \end{aligned}$$

and where $\tilde{\gamma}$ is jointly continuous on $[0, 1] \times [t_0, t_1]$.

So a contractible loop is one that can be continuously deformed in D into a constant loop.

Definition 50 *A domain D is simply connected if every loop in D is contractible in D.*

Lemma 51 *The ball $B(w, \varepsilon) = \{w| \ |w| < \varepsilon\}$, and more generally any convex or even starlike open subset C of $\mathbb{C}$ is simply connected.*

Proof. If C is starlike then there is a $w_0 \in C$ so that $sw + (1 - s) w_0 \in C$ for all w in C and $s \in [0, 1]$. If γ is a loop in C defined on $[t_0, t_1]$ then define $\tilde{\gamma}(s, t) = s\gamma(t) + (1 - s) w$. Clearly $\tilde{\gamma}(s, .)$ is a loop in C for every $s \in [0, 1]$, $\tilde{\gamma}$ is continuous, and $\tilde{\gamma}(0, .)$ is the trivial loop. ■

The complex plane $\mathbb{C}$ is simply connected.

Lemma 52 *A path γ with non-zero winding number around 0 is not contractible in $\mathbb{C}-\{0\}$.*

Proof. Suppose it were, and that $\gamma(s,)$ interpolates continuously, using loops that did not go through 0, between the path γ with non-trivial winding number and the trivial loop.$\hat{\gamma}(t) \equiv z_0$. Now $\Gamma(\hat{\gamma},0) = \frac{1}{2\pi i}\int_{\hat{\gamma}} \frac{1}{\hat{\gamma}} d\hat{\gamma} = 0$. The function $s \to \Gamma(\gamma(s,.),0)$ is a continuous integer valued function defined for every $s \in [0,1]$ by the intermediate value theorem is constant contradicting the fact that $\Gamma(\hat{\gamma},0) = 0$ and $\Gamma(\gamma,0) \neq 0$. ■

Proposition 53 $\mathbb{C}-\{0\}$ *is not simply connected.*

Proof. Suppose that $\gamma(t) = \exp 2\pi i t$ then

$$\Gamma(\gamma,0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{\gamma} d\gamma = \int_0^1 dt = 1.$$

and so is not contractible. ■

3.1 Winding as a stochastic process

Suppose that γ is any smooth path on $[0,t]$ missing 0, then $\Theta_t = \int_{s=0}^t \frac{1}{\gamma_s} d\gamma_s$ satisfies $\gamma_0 \exp \Theta_t = \gamma_t$ and Θ computes a branch of log along γ. It follows from Cauchy's theorem (integrals of analytic functions on D around contractible closed curves in D are always zero) that continuous deformations of γ that avoid zero and keep the end points 0 and t of γ fixed do not change the value of Θ_t. It follows that Θ_t can be defined for all continuous paths γ not going through 0. We now prove two unrelated results that together will be important to us.

Lemma 54 *Suppose that γ is a path contained entirely in the ball $B(w,\varepsilon)$ where $|w| > 2\varepsilon$ then*

$$\left|\frac{1}{2\pi}\operatorname{Im}\int_{s=0}^t \frac{1}{\gamma(s)} d\gamma(s)\right| \leq 1/3.$$

Proof. If we connect the beginning and end of γ by the chord. Then the resulting loop is contractible and has winding number zero around zero. Hence the integral along the curve equals that along the chord. Denoting $\bar{\gamma}(s) = \gamma(t)\, st^{-1} + \gamma(0)(t-s)\, t^{-1}$ an explicit computation gives:

$$\begin{aligned}\int_{s=0}^t \frac{1}{\gamma(s)} d\gamma(s) &= \int_\gamma \frac{1}{z} dz = \int_{s=0}^t \frac{1}{\bar{\gamma}(s)} d\bar{\gamma}(s) = \log(\gamma(t)/\gamma(0)) \\ &= \log(|\gamma(t)|/|\gamma(0)|) + i(\arg\gamma(t) - \arg\gamma(0)),\end{aligned}$$

so the imaginary part of that integral is just the angle made between two end points and zero. It is easy to see that the absolute value of this angle cannot be greater than $2\pi/3$. ■

Theorem 55 *Suppose that Z_t is complex Brownian motion with $Z_0 \neq 0$ and define*

$$\Theta_t = \int_{s=0}^t \frac{1}{Z_s} dZ_s$$

then the integral is well defined and

$$Z_0 \exp \Theta_t = Z_t$$

so Θ_t *is a branch of* $\log Z_t$*. The definition of* Θ_t *coincides with the definition for continuous paths given above.*

Proof. Apply Itô's formula: Θ_t and Z are conformal martingales and exp is analytic so

$$\begin{aligned} d\left(Z_t \exp\left(-\Theta_t\right)\right) &= \exp\left(-\Theta_t\right) dZ_t + Z_t d \exp\left(-\Theta_t\right) \\ &= \exp\left(-\Theta_t\right) dZ_t - Z_t \exp\left(-\Theta_t\right) d\Theta_t \\ &= \exp\left(-\Theta_t\right) dZ_t - Z_t \exp\left(-\Theta_t\right) \frac{1}{Z_t} dZ_t \\ &= 0 \end{aligned}$$

and $Z_0 \exp \Theta_t = Z_t$. Suppose that Φ_t is a second continuous function which satisfies $Z_0 \exp \Phi_t = Z_t$. Then for all t one has $\exp\left(\Theta_t - \Phi_t\right) = 1$ hence $t \to \Theta_t - \Phi_t$ is a continuous function on the (connected set) interval $[0, T]$ with values in $2\pi i \mathbb{Z}$ and thus constant. Since $\Theta_0 - \Phi_0 = 0$ we see that $\Theta_t = \Phi_t,\ t \in [0, T]$ ■

It follows from this that $\operatorname{Im}\left(\frac{1}{2\pi} \int_{s=0}^{t} \frac{1}{Z_s} dZ_s\right)$ is the continuous branch of the argument of Z_t taken along the path and keeps a continuous "track" of the winding of Z around zero.

Corollary 56 *With probability one, there are times* $T_n \to \infty$ *at which Brownian motion is unwound.*

Proof. The process $M_t = \operatorname{Im}\left(\frac{1}{2\pi} \int_{s=0}^{t} \frac{1}{Z_s} dZ_s\right)$ is a real local martingale with continuous paths. It is easy to see it is defined for all finite t and that because W is recurrent, $\langle M, M \rangle_t \to \infty$ as $t \to \infty$. By Levy, B defined by

$$B_{\langle M, M \rangle_t} := M_t$$

is a Brownian motion (run to infinity). Now there are certainly times $S_n \to \infty$ at which a real Brownian motion hits zero. Let

$$T_n(\omega) = \inf\left\{t \middle| \ \langle M(\omega), M(\omega) \rangle_t > S_n\left(B(\omega)\right)\right\}$$

then $T_n \to \infty$ almost surely and

$$B_{S_n} = M_{T_n} = 0.$$

Draw a picture! ■

Theorem 57 *[11, 6, 8, 10] With probability one*

$$\lim_{t \to \infty} \left(\left| \operatorname{Im}\left(\frac{1}{2\pi} \int_{s=0}^{t} \frac{1}{Z_s} dZ_s\right) \right| + \left| \operatorname{Im}\left(\frac{1}{2\pi} \int_{s=0}^{t} \frac{1}{Z_s - 1} dZ_s\right) \right| \right) \to \infty.$$

Using this result we may prove the following :

Corollary 58 *A non-constant entire function* $f(z)$ *cannot omit the points* 0 *and* 1

Proof. Suppose for a contraction that f does omit both the values 0 and 1. Fix $z_0 \in \mathbb{C}$ and a small ball $B(z_0, \varepsilon)$ around it so that $f(B(z_0, \varepsilon)) \subset B(w_0, \delta)$ where $w_0 = f(z_0)$ and $\delta < \min[|w_0 - 1|, |w_0 - 0|]/2$ and consider a Brownian path Z_t started at z_0. Let

$$W_t = f\left(Z_{\tau(t)}\right)$$

be the image path, so that W is another complex Brownian motion. Now let T_n be stopping times $T_n \leq T_{n+1}$ and $T_n \to \infty$ so that $Z_{T_n} \in B(z_0, \varepsilon)$. For each of these times T_n construct a loop ρ_n in two parts, first use $(Z_t)_{t \in [0, T_n]}$ and then take a chord γ_n from Z_{T_n} to z_0 (picture). Since

$$\lim_{t \to \infty} \left(\left| \operatorname{Im} \left(\frac{1}{2\pi} \int_{s=0}^{t} \frac{1}{W_s} dW_s \right) \right| + \left| \operatorname{Im} \left(\frac{1}{2\pi} \int_{s=0}^{t} \frac{1}{W_s - 1} dW_s \right) \right| \right) \to \infty$$

we know that by choosing a subsequence either

$$\lim_{n \to \infty} \left(\left| \operatorname{Im} \left(\frac{1}{2\pi} \int_{s=0}^{T_n} \frac{1}{W_s} dW_s \right) \right| \right) \to \infty$$

or

$$\lim_{n \to \infty} \left(\left| \operatorname{Im} \left(\frac{1}{2\pi} \int_{s=0}^{T_n} \frac{1}{W_s - 1} dW_s \right) \right| \right) \to \infty$$

and for simplicity we assume the first holds. Now consider $f(\rho_n)$. Since it is a loop we may consider its winding number about 0

$$\Gamma[f(\rho_n), 0] = \operatorname{Im} \left(\frac{1}{2\pi} \int_{f(\rho_n)} \frac{1}{w} dw \right).$$

If we decompose this integral into the integral along W and along γ_n then we have already seen that, because the path $f(\gamma_n)$ is constrained to lie in the chosen small ball this integral must be at most $1/3$ and hence

$$\left| \Gamma[f(\rho_n), 0] - \operatorname{Im} \left(\frac{1}{2\pi} \int_{s=0}^{T_n} \frac{1}{W_s} dW_s \right) \right| \leq 1/3.$$

Consequently, we may conclude that $\Gamma[f(\rho_n), 0] \to \infty$ with probability one. In particular there is at least one loop $\rho \in \mathbb{C}$ so that $\Gamma[f(\rho), 0] \neq 0$. Since ρ is contractible in $\mathbb{C}$, $f(\rho)$ is contractible in $\mathbb{C} - \{0\}$. Taking the two statements together leads to a contradiction to our earlier results. ■

3.2 Picard through Tangling

3.2.1 The free group

Suppose that $A = \{a, b, \ldots, e\}$ is an alphabet, and that Ξ is the set of all finite words (including the empty one) with letters in $A \cup A^{-1}$ where

$$A^{-1} := \left\{a^{-1}, b^{-1}, \ldots, e^{-1}\right\}$$

and define the product $\xi \circ \zeta$ of any two such words to be the word formed by concatenation. The empty word is the identity, and $\circ$ associative.

Definition 59 *the operation $\xi \circ c \circ c^{-1} \circ \zeta \to \xi \circ \zeta$ is said to be a cancellation. The word ψ is said to be a simplification of $\tilde{\psi}$ if it can be obtained by successive cancellations from $\tilde{\psi}$. Two words are said to be ˜equivalent if they have a common simplification.*

Lemma 60 *The notion of ˜equivalence introduced above is an equivalence relation. It respects $\circ$ and the quotient space $\Xi/\tilde{}$ is a group. Every word has a unique shortest simplification called the reduced word, and two words are equivalent if and only if they have the same reduced word. We can identify $\Xi/\tilde{}$ with the words which have no further simplification.*

Lemma 61 *The length of this reduced word is the length of the group element.*

Exercise 62 *Prove this lemma - hint you need to use induction. The problem is that it is not quite clear that the relation is transitive. There is no unique choice for how to cancel as $\xi \circ a^{-1} \circ a \circ a^{-1} \circ \zeta$*

Remark 63 *We will often drop the $\circ$ from our language.*

Definition 64 *$\mathbb{F}_A = \Xi/\tilde{}$ is known as the free group with generating set A.*

Exercise 65 *How many elements of the free group on two letters are there with length at most d.*

3.2.2 Coding planar paths

We are interested in paths τ with values in $\mathbb{C} \setminus \{0,1\}$. We want to introduce some stopping times. To do this introduce three line segments

$$\begin{aligned} J_0 &= (-\infty, 0) \\ J_1 &= (0,1) \\ J_2 &= (1,\infty), \end{aligned}$$

Let $K = \cup_j J_j$, and $K_i = K \setminus J_i$.

Definition 66 *Let $T_0 = \inf\{t | \tau(t) \in K\}$.*

Lemma 67 *If $T_0 < \infty$ then $\tau_{T_0} \in K$. We say that τ_{T_0} hits/is in section j of K according to whether $\tau_{T_0} \in J_j$*

Proof. By assumption $\tau_{T_0} \in \mathbb{C} \setminus \{0,1\}$, by the definition of T_0 and continuity of the path $\tau_{T_0} \in \overline{K}$; but $\overline{K} \cap \mathbb{C} \setminus \{0,1\} = K$. ■

Definition 68 *We define T_{k+1} recursively. If τ_{T_k} is in section j of K then $T_{k+1} = \inf\{t > T_k | \tau(t) \in K_j\}$. We refer to the time interval $[T_{k-1}, T_k]$ as the k th passage and associate to it three pieces of information. If $\tau_{T_{k-1}}$ is in section j of K we call j the source, if τ_{T_k} is in section j' of K we call j' the sink, and we say the passage is from above (+) if there is an interval $(T_k - \varepsilon, T_k)$ on which the imaginary part of τ is positive otherwise we say the passage is from below. Note that we never have $j = j'$.*

For each possible type for the k'th passage we associate a word l_k with one or two letters according to the type of passage:

01	$+$	a
01	$-$	a^{-1}
02	$+$	ab
02	$-$	$b^{-1}a^{-1}$
12	$+$	b
12	$-$	b^{-1}
21	$+$	b^{-1}
21	$-$	b
20	$+$	$b^{-1}a^{-1}$
20	$-$	ab
10	$+$	a^{-1}
10	$-$	a

and as the path evolves we concatenate these to produce a word $w_n = l_n w_{n-1}$ where w_0 is the identity or empty word. We let g_n be the reduced word ˜equivalent to w_n. We call (non-standard) g_n the tangling state of τ at the n'th passage time T_n. If $\tau : [0, S] \to \mathbb{C}\setminus\{0,1\}$ is a loop (so that $\tau(S) = \tau(0)$) which has been started on K and $T_n \leq S$ is the largest of the passage times for τ then we associate the loop with the corresponding word. The tangling state of the loop uniquely identifies the homotopy class $\pi(\tau, \mathbb{C}\setminus\{0,1\})$ of the loop τ in $\mathbb{C}\setminus\{0,1\}$ and maps the loops τ starting in K with a subgroup of the free group generated by a and b.

Exercise 69 *Prove that the range of the map $\tau \to \pi(\tau, \mathbb{C}\setminus\{0,1\})$ is closed under multiplication and contains inverses - and so is a group.*

We can extend this map to all loops. Fix some point $k \in K$. If the loop τ starts at $p \notin K$ let ρ be the straight line $k \to p$, and put $\tilde{\tau} = \rho^{-1} \circ \tau \circ \rho$ and define $\pi(\tau, \mathbb{C}\setminus\{0,1\}) \equiv \pi(\tilde{\tau}, \mathbb{C}\setminus\{0,1\})$.

Exercise 70 *Convince yourself that the extension should not depend on the choice of k.*

In fact much more is true. A basic theorem from algebraic topology which we will now assume is:

Theorem 71 *The map $\tau \to \pi(\tau, \mathbb{C}\setminus\{0,1\})$ is a continuous function from paths in $\mathbb{C}\setminus\{0,1\}$ started in K with the uniform topology to the free group on two generators (generated by a^2 and b^2).*

Remark 72 *It is pretty "obvious" that the stated group is the range of the map. It less obvious that continuous perturbations or (homotopies) of τ do not change the value of π. These perturbations may well change the number n of passages and the word w_n - but not the reduced word.*

Exercise 73 *Suppose $\tau(s) = r \exp i\theta$ where $\theta \in [0, 2\pi]$ where $r > 0$, and $r \neq 1$. How many passages does τ experience before its return to its starting position, and what is $\pi(\tau, \mathbb{C}\setminus\{0,1\})$.*

Exercise 74 *Find a continuous family of paths (drawing pictures will suffice) perturbing τ continuously as a loop in $\mathbb{C}\setminus\{0,1\}$ to a loop that contains a passage of every kind listed above. Write down the associated word w and check that its reduced from is the same as that given by τ.*

Corollary 75 *If a loop τ is contractible in $\mathbb{C}\setminus\{0,1\}$ then $\pi(\tau,\mathbb{C}\setminus\{0,1\})$ is the identity or empty word.*

Proof. A continuous function from the unit interval to a discrete space is always constant (the intermediate value theorem). ■

Although we will not use it, there is a strong converse:

Theorem 76 (The fundamental group of $\mathbb{C}\setminus\{0,1\}$ is the free group on two generators.) *Loops τ, $\tilde{\tau}$ can be deformed continuously into one another in $\mathbb{C}\setminus\{0,1\}$ if and only if $\pi(\tau,\mathbb{C}\setminus\{0,1\}) = \pi(\tilde{\tau},\mathbb{C}\setminus\{0,1\})$.*

Random Walk on the Free Group

Every element g in the free group $\mathbb{F}_2$ corresponds to a unique reduced word $w_1 \dots w_r$ which cannot be cancelled in any way ($\forall i$, $w_i \neq w_{i+1}^{-1}$), where r is the length of the shortest representation for g as a product of generators. It is the shortest representation. This provides a convenient way to draw the free group on 2 or d generators.

Consider, inductively, a sequence of trees. Start with $T(1)$: a base point 0 and 4 edges, labelled one of a, b, a^{-1}, b^{-1}. Now, given $T(n)$ with leaves labelled from a, b, a^{-1}, b^{-1},add a new node to each leaf, and to that add 3 new edges or leaves - lablled so that the four edges out of each node have different labels. This can only be done in one way. Then, we get a tree with 4.3^{n-1}vertices a distance n from the root.

The nodes of this tree correspond to the words in the free group. The random walk on the group obtained by multiplying by a, b, a^{-1}, b^{-1} with equal probabilities corresponds to the nearest neighbour walk on the group.

Proposition 77 *The random walk on the group $\mathbb{F}_2$ is transient.*

Proof. If $l_n \neq 0$ is the length of group element at the n-th time step, then

$$\begin{aligned} \mathbb{P}(l_{n+1} = l_n - 1) &= 1/4 \\ \mathbb{P}(l_{n+1} = l_n + 1) &= 3/4. \end{aligned}$$

and $\mathbb{E}(l_{n+1} - l_n | l_n) \geq 1/2$. ■

3.2.3 Brownian Motion and Tangling

If W_t is complex Brownian motion started at $k \in K$ then it is an easy excercise to see that, with probability one it remains in $\mathbb{C}\setminus\{0,1\}$ for all time, that the passage times T_n are stopping times, and that for all $n \in \mathbb{N}$, $T_n < \infty$. Our basic result is that, from some point on, the tangling state of the path at these passage times is always non-trivial. With probability one there is a last untangled time.

Theorem 78 *Suppose that W_t is complex Brownian motion started at $k \in K$ and that g_n is the tangling state of W at the n'th passage time T_n. With proability one the length of the g_n explodes to $+\infty$*

Picard will then follow essentially immediately. A key step in our argument is the following consequence of the reflection principle:

Lemma 79 (Reflection) *If (j, j', σ) is the information associated to the n-th passage then, conditional on j, j' and the history $\mathcal{F}_{T_{n-1}} = \sigma[W_t,\ t < T_{n-1}]$ of the path up to T_{n-1} the sign σ has equal probability of being $+$ and $-$.*

Proof. Let T_{n-1}be the stopping time marking the end of the $n-1$'th or previous passage and beginning on the n'th or current passage. Consider the adapted process with values in $Hom_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$

$$\begin{aligned} C_t(\omega) &: z = z, \quad t < T_{n-1} \\ C_t(\omega) &: z = \bar{z}, \quad t \geq T_{n-1} \end{aligned}$$

and let

$$Z_t = W_0 + \int_0^t C_s dW_s$$

then, because C is bounded and adapted, normal results form stochastic integration tell us that Z_t is a continuous square integrable martingale. More precisely, we can compute the bracket processes. They clearly agree with those of W if $t < T_{n-1}$. If $t \geq T_{n-1}$ then, recalling that $\langle W, W\rangle_t \equiv 0$ and $\langle W, \bar{W}\rangle_t$ is real valued the following calculations

$$\begin{aligned} \langle Z, Z\rangle_t &= \int_0^{T_{n-1}} d\langle W, W\rangle_s + \int_{T_{n-1}}^t d\langle \bar{W}, \bar{W}\rangle_s \\ &= \langle W, W\rangle_0^{T_{n-1}} + \langle \bar{W}, \bar{W}\rangle_{T_{n-1}}^t \\ &= \langle W, W\rangle_0^{T_{n-1}} + \overline{\langle W, W\rangle_{T_{n-1}}^t} \\ &= 0 \\ \langle Z, \bar{Z}\rangle_t &= \int_0^{T_{n-1}} d\langle W, \bar{W}\rangle_t + \int_{T_{n-1}}^t d\langle \bar{W}, W\rangle_t \\ &= \langle W, \bar{W}\rangle_0^{T_{n-1}} + \langle \bar{W}, W\rangle_{T_{n-1}}^t \\ &= \langle W, \bar{W}\rangle_0^{T_{n-1}} + \overline{\langle W, \bar{W}\rangle_{T_{n-1}}^t} \\ &= \langle W, \bar{W}\rangle_0^{T_{n-1}} + \langle W, \bar{W}\rangle_{T_{n-1}}^t \\ &= \langle W, \bar{W}\rangle_0^t \end{aligned}$$

demonstrate that they agree for all time. We conclude from the complex version of Levy's characterisation that Z_t is also a complex Brownian motion. However, the sample paths of Z are those of W until $T_{n-1}$and are the reflection of W in the real axis thereafter.

In particular, we see from our construction of Z that the event

$$\begin{aligned} & \left\{\omega \middle| \text{ the } n'\text{th passage of } Z \text{ has source } j \text{ and sink } j' \text{ and comes from above}, Z \in A, A \in \mathcal{F}_{T_{n-1}}\right\} \\ = & \left\{\omega \middle| \text{ the } n'\text{th passage of } W \text{ has source } j \text{ and sink } j' \text{ and comes from below}, W \in A, A \in \mathcal{F}_{T_{n-1}}\right\} \end{aligned}$$

and so the events (which are the same event) have equal probability. However, the event

$$\left\{\omega \middle| \text{ the } n'\text{th passage of } Z \text{ has source } j \text{ and sink } j' \text{ and comes from above}, Z \in A, A \in \mathcal{F}_{T_{n-1}}\right\}$$

is a measurable set of continuous paths and so its probability is the same for all Brownian motions. Hence

$$\begin{aligned} & \mathbb{P}\left[\text{the } n'\text{th passage of } W \text{ has source } j \text{ and sink } j' \text{ and comes from below}|\mathcal{F}_{T_{n-1}}\right] \\ = \; & \mathbb{P}\left[\text{the } n'\text{th passage of } W \text{ has source } j \text{ and sink } j' \text{ and comes from above}|\mathcal{F}_{T_{n-1}}\right] \end{aligned}$$

as claimed. ■

Corollary 80 (Burgess Davis) *If g_n is the tangling state of W_t after its n-th passage and $|g_n|$ the length of the reduced representation of g_n then $|g_n|$ a submartingale that converges to $+\infty$.*

Proof. We prove first the simpler result that there are random times $n_k \to \infty$ so that $|g_{n_k}| \to \infty$.

Recall that g_n is obtained by simplification from $g_{n-1}l_n$ where l_n is the one or two letter word associated to the n'th passage. By assumption, g_{n-1} is $\mathcal{F}_{T_{n-1}}$ measurable, and already in simplified form. It is either empty or it ends at the right in one of four possible ways

$$\begin{aligned} g_{n-1} &= \\ g_{n-1} &= \ldots a \\ g_{n-1} &= \ldots a^{-1} \\ g_{n-1} &= \ldots b \\ g_{n-1} &= \ldots b^{-1} \end{aligned}$$

and $g_{n-1}l_n$ only admits a simplification if the initial letter of l_n matches the inverse of this letter. If there is a simplification then $|g_n| \geq |g_{n-1}| - |l_n|$ and if there is no simplification $|g_n| = |g_{n-1}| + |l_n|$. By our symmetry result, and independently of $g_k|_{k<n}$, the probability of simplification happening is always less than a half and the expected value of $|g_n|$ given $\mathcal{F}_{T_{n-1}}$is strictly greater than $|g_{n-1}|$.

Let $S_R = \min\{n|\,|g_n| > R\}$, which is a stopping time, then $R+2-|g_{n\wedge S_R}|$ is a non-negative supermartingale and so converges almost surely. But $|g_n|$ changes value with every n so that convergence can occur only if $S_R = \min\{n|\,|g_n| > R\} < \infty$ almost surely for every R.

Now we prove the main result - that in fact the length converges to infinity. This does not follow from the fact that l is a positive submartingale with increments that are always at least 1 and less than 2 (for example if X_n is a simple random walk then $|X_n|$ has all these properties and clearly returns to zero infinitely many times).

We change the stopping times a bit. Suppose that $S_0 = T_0$ and that S_n is a passage time where j is the source and j' is the sink. Let S_{n+1} be the first passage time after S_n where the sink is different to j' and j. Suppose that h_n is the reduced word at time S_n then $h_{n+1} := h_n\,(l_{jj'})^m\, l_{\tilde{j}k}$ where $k \notin \{j, j'\}$ and $\tilde{j} \in \{j, j'\}$. ■

Suppose we consider times of return to a point on the axis which is equidistant from 0 and 1. Then on each occasion it will produce a random word.

Lemma 81 *The excursions to a neighbourhood of the midpoint are independent, identically distributed, if one looks at the changes in letters then flipping show*

that a and a-1 as well as b b-1 are equally likely. The number of changes has finite expectation. The probability of a word in a's and b's is essentially the same as the word with the roles flipped.

Lemma 82 *let l_n be the number of flips till the n'th excursion. and k_n the number of changes in the $n'th$ excursion. We show then with probablity 1/2*

$$\begin{aligned} l_{n+1} &= l_n + k_n \\ l_{n+1} &= l_n + k_n - i_n \end{aligned}$$

where i_n is the number of changes in sign that get erased. For a cancellation to happen at all, the reduced number of the current letter has to match exactly the previous and have the correct sign,

Prob at most 1/2 of removing one at least one sign change, prob 1/4 of removing at least 2 sign changes, .. so the expected number removed is at most

$$\mathbb{E}[C] = \int_0^\infty \mathbb{P}(C > \lambda)\, d\lambda$$

which gives one. On the other hand the number of sign changes in the excursion is definately greater than one.

There is a probability that that word will not be trivial. If it starts with a then it is as likely to start with b. It has a length l and the lenght of the next word is, with probability 3/4 the lenght of the old word +the new one. The expected number of letter changes is N so the while by symmetry, the number of with probability at least one half, we will see the number of letter changes increase. The problem is that we cn also (with small probability see a decrease in the length.

Part III

Hardy Spaces

One often tries to understand the largeness of a random variable X by looking at its moments - for example $\mathbb{E}\left[|X|^n\right]$. Sometimes one has many different interrelated random variables and would like to connect their behaviour. For example, suppose that $\{X_t, 0 \leq t < \infty\}$ is a one dimensional Brownian motion started at 0 and τ a stopping time. Then one might like to relate the behaviour of τ and X_τ, although one can see problems with this straight way, for instance if we let τ be the first time after T at which $|X_\tau| < 1$. Letting $T \to \infty$ one sees that $\mathbb{E}[\tau] > T$ and $\mathbb{E}[|X_\tau|] < 1$ so there does not always need to be a connection.

Definition 83 *If X_t is a real valued process, then the two sided maximal function is the increasing process defined by*

$$\begin{aligned} X_t^* &= \sup_{s<t} |X_s| \\ &= \sup_{s} |X_{s\wedge t}| \end{aligned}$$

Having seen the example, one might instead try to get a relationship between the maximal function X_τ^* of X and τ. This is possible and will be our goal for the next few lectures.

Chapter 4

The Burkholder Davis Inequalities

[1]Let Φ be a continuous non-decreasing function on $[0,\infty]$ with $\Phi(0)=0$ and satisfying the following "growth condition".

Condition 84 (Moderate Growth) *There exists a $c>0$ such that*

$$\Phi(2\lambda) \le c\Phi(\lambda), \ \lambda > 0. \tag{4.1}$$

The function $\lambda \to \lambda^p$ satisfies this condition for any choice of $p \in (0,\infty)$.

The remarkable theorems of Burkholder, Gundy, and Davis tell us:

Theorem 85 *If τ is a stopping time of the Brownian motion X then*

$$c\mathbb{E}\left[\Phi\left(\tau^{\frac{1}{2}}\right)\right] \le \mathbb{E}\left[\Phi\left(X_\tau^*\right)\right] \le C\mathbb{E}\left[\Phi\left(\tau^{\frac{1}{2}}\right)\right]$$

for some choice of c , C depending only on $c_{4.1}$.

We understood from scaling that somehow Brownian motion has $(\text{distance})^2 = (\text{time})$. But now we see this in a very striking form. All the moments (even those that are infinite) of the stopping time τ are directly comparable with the moments of $(X^*)^2$.

Exercise 86 *Let τ be the first time that planar Brownian motion started at i leaves the upper half plane. Then $\mathbb{E}[\tau^p] < \infty$ if and only if $p < 1/2$.*

To see this first note that we have already seen (using the angle function) that X_τ is distributed as

$$\frac{1}{\pi}\frac{1}{1+x^2}dx$$

and so $\mathbb{E}[|X_\tau|] = \infty$. Hence $\mathbb{E}[X_\tau^*] = \infty$ and setting $\Phi(\lambda) \equiv \lambda$ one has $\mathbb{E}\left[\tau^{\frac{1}{2}}\right] = \infty$. On the other hand, the strong Markov property shows that if τ_1 is the

[1]The proofs in this section are very close indeed to those provided by Donald Burkholder in sections 6-8 of his Wald Lecture, *Distribution Function Inequalities for Martingales*, Annals of Prob. Vol 1. No. 1 (Feb 1973) 19-42

time that Brownian motion from i hits $\mathbb{R}$ and τ_2 is the first time after τ_1 that Brownian motion hits $\mathbb{R} - i$ then τ_2 is independent of τ_1 and identical in distribution. A simple scaling argument shows that $\frac{1}{4}(\tau_1 + \tau_2)$ has the same distribution as τ_1 or τ_2. Similar arguments show that if τ_i are independent and distributed like τ_1 then

$$\sum_i (t_i \tau_i) \stackrel{\mathcal{D}}{=} \left(\sum_i t_i^{1/2} \right)^2 \tau_1$$
$$\sum_{i=1}^n \tau_i \stackrel{\mathcal{D}}{=} n^2 \tau_1.$$

and so we have an explicit scaling propeerty for the Laplace transform. If $\xi(x) = \mathbb{E}[\exp(-x\tau)]$ then for rational and then real r

$$\begin{aligned} \xi(x)^r &= \xi\left(xr^2\right) \\ \xi(x) &= \exp\left(k\sqrt{x}\right) \end{aligned}$$

and further computation shows that $\mathbb{E}[\tau^\rho] < \infty$ for all $\rho < 1/2$. We can conclude that $\mathbb{E}[(X_\tau^*)^\rho] < \infty$ for every $\rho < 1$.

There are two main steps to proving this result. It will follow from the general properties of Brownian motion that

Theorem 87 *Let $\beta > 1$ and $\delta > 0$. If τ is as stopping time of X then*

$$\mathbb{P}\left(\tau^{\frac{1}{2}} > \beta\lambda, X_\tau^* \le \delta\lambda\right) \le \frac{\delta^2}{\beta^2 - 1} \mathbb{P}\left(\tau^{\frac{1}{2}} > \lambda\right), \quad \lambda > 0 \tag{4.2}$$

$$\mathbb{P}\left(X_\tau^* > \beta\lambda, \tau^{\frac{1}{2}} \le \delta\lambda\right) \le \frac{\delta^2}{\left(\beta^2 - 1\right)^2} \mathbb{P}\left(X_\tau^* > \lambda\right), \quad \lambda > 0 \tag{4.3}$$

These can be interpreted as, and might be easier to understand as, conditional probabilities of disparate behaviour between $\tau^{\frac{1}{2}}$, X_τ^* given that one of them is large

$$\begin{aligned} \mathbb{P}\left(\tau^{\frac{1}{2}} > \beta\lambda, X_\tau^* \le \delta\lambda \middle| \tau^{\frac{1}{2}} > \lambda\right) &\le \frac{\delta^2}{\beta^2 - 1}. \\ \mathbb{P}\left(X_\tau^* > \beta\lambda, \tau^{\frac{1}{2}} \le \delta\lambda \middle| X_\tau^* > \lambda\right) &\le \frac{\delta^2}{\left(\beta^2 - 1\right)^2} \end{aligned}$$

captured in a way that also encapsulates scale invariance over λ .

It turns out that this inequality leads directly to the results we seek. We deal with this more abstract issue first.

4.1 Working with Φ

We start with a crucial lemma which explains our interest in (4.1). Suppose that Φ satisfies the growth condition (4.1) and that Φ is not identically zero.

Lemma 88 *Suppose that f and g are non-negative random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, that $\beta > 1$, $1 > \delta > 0$ and $\varepsilon > 0$ are real numbers and that*

$$\mathbb{P}\left(g > \beta\lambda, f \leq \delta\lambda \middle| g > \lambda\right) \leq \varepsilon, \quad \lambda > 0. \tag{4.4}$$

or equivalently

$$\mathbb{P}\left(g > \beta\lambda, f \leq \delta\lambda\right) \leq \varepsilon\mathbb{P}\left(g > \lambda\right), \quad \lambda > 0.$$

Let γ and η be real numbers satisfying

$$\begin{aligned} \Phi\left(\beta\lambda\right) &\leq \gamma\Phi\left(\lambda\right), \quad \lambda > 0, \\ \Phi\left(\delta^{-1}\lambda\right) &\leq \eta\Phi\left(\lambda\right), \quad \lambda > 0. \end{aligned}$$

and finally suppose that $\gamma\varepsilon < 1$. Then

$$\mathbb{E}\left(\Phi\left(g\right)\right) \leq \frac{\gamma\eta}{1-\gamma\varepsilon}\mathbb{E}\left(\Phi\left(f\right)\right)$$

Remark 89 *The growth condition ensures the existence of γ and η. If $\beta \in \left[2^{k-1}, 2^k\right]$ then put $\gamma = c^k$ etc . Note that γ is independent of the choice of δ. The above inequalities give us, through making δ small, a final control on ε.*

Proof. If $\mathbb{P}\left(g > \beta\lambda, f \leq \delta\lambda\right) \leq \varepsilon\mathbb{P}\left(g > \lambda\right)$ then $\mathbb{P}\left(g \wedge n > \beta\lambda, f \leq \delta\lambda\right) \leq \varepsilon\mathbb{P}\left(g \wedge n > \lambda\right)$ for all n. Conversely if $\mathbb{P}\left(g \wedge n > \beta\lambda, f \leq \delta\lambda\right) \leq \varepsilon\mathbb{P}\left(g \wedge n > \lambda\right)$ for all n then $\mathbb{P}\left(g > \beta\lambda, f \leq \delta\lambda\right) \leq \varepsilon\mathbb{P}\left(g > \lambda\right)$ so that we may as well assume that g is bounded and $\mathbb{E}\left(\Phi\left(g\right)\right) \leq \infty$.

Now recall that

$$\begin{aligned} \Phi\left(h\right) &= \int_0^h d\Phi\left(\lambda\right) \\ &= \int_0^\infty I_{\{\lambda<h\}} d\Phi\left(\lambda\right) \end{aligned}$$

and so using Fubini's theorem

$$\begin{aligned} \mathbb{E}\left(\Phi\left(h\right)\right) &= \int_0^\infty \mathbb{E}\left(I_{\{\lambda<h\}}\right) d\Phi\left(\lambda\right) \\ &= \int_0^\infty \mathbb{P}\left(\lambda < h\right) d\Phi\left(\lambda\right) \\ &= \int_0^\infty \mathbb{P}\left(h > \lambda\right) d\Phi\left(\lambda\right). \end{aligned}$$

Next

$$\mathbb{P}\left(g > \beta\lambda, f \leq \delta\lambda\right) < \varepsilon\mathbb{P}\left(g > \lambda\right)$$

implies that

$$\begin{aligned} \mathbb{P}\left(g > \beta\lambda\right) &= \mathbb{P}\left(g > \beta\lambda, f \leq \delta\lambda\right) + \mathbb{P}\left(g > \beta\lambda, f > \delta\lambda\right) \\ &< \varepsilon\mathbb{P}\left(g > \lambda\right) + \mathbb{P}\left(f > \delta\lambda\right) \end{aligned}$$

and therefore

$$\begin{aligned} \mathbb{E}\left(\Phi\left(\beta^{-1}g\right)\right) &\leq \varepsilon\mathbb{E}\left(\Phi\left(g\right)\right) + \mathbb{E}\left(\Phi\left(\delta^{-1}f\right)\right) \\ &\leq \varepsilon\mathbb{E}\left(\Phi\left(g\right)\right) + \eta\mathbb{E}\left(\Phi\left(f\right)\right) \end{aligned}$$

but

$$\begin{aligned}\mathbb{E}\left(\Phi\left(g\right)\right) &= \mathbb{E}\left(\Phi\left(\beta\beta^{-1}g\right)\right)\\ &\leq \gamma\mathbb{E}\left(\Phi\left(\beta^{-1}g\right)\right)\end{aligned}$$

and so

$$\mathbb{E}\left(\Phi\left(g\right)\right) \leq \varepsilon\gamma\mathbb{E}\left(\Phi\left(g\right)\right) + \eta\gamma\mathbb{E}\left(\Phi\left(f\right)\right).$$

and a simple rearrangement gives the claimed result. ■

Remark 90 *We also remark while we are working on* Φ *that*

$$\begin{aligned}\Phi\left(\lambda_1 \vee \lambda_2\right) &\leq \Phi\left(\lambda_1 \vee \lambda_2\right) + \Phi\left(\lambda_1 \vee \lambda_2\right)\\ \Phi\left(\lambda_1 + \lambda_2\right) &\leq \Phi\left(2\lambda_1\right) + \Phi\left(2\lambda_2\right) \leq c\left[\Phi\left(\lambda_1\right) + \Phi\left(\lambda_2\right)\right]\end{aligned}$$

4.2 Working with the Brownian motion

If we can now prove that for $\beta > 1$ and $\delta > 0$ and Brownian motion:

$$\mathbb{P}\left(\tau^{\frac{1}{2}} > \beta\lambda, X_\tau^* \leq \delta\lambda\right) \leq \frac{\delta^2}{\beta^2 - 1}\mathbb{P}\left(\tau^{\frac{1}{2}} > \lambda\right), \quad \lambda > 0 \tag{4.5}$$

$$\mathbb{P}\left(X_\tau^* > \beta\lambda, \tau^{\frac{1}{2}} \leq \delta\lambda\right) \leq \frac{\delta^2}{\left(\beta^2 - 1\right)^2}\mathbb{P}\left(X_\tau^* > \lambda\right), \quad \lambda > 0 \tag{4.6}$$

then we can clearly prove our main theorem.
Proof. Recall that X is a martingale starting at zero and that $\langle X, X\rangle_t = t$. Thus, if τ is a bounded stopping time then

$$\mathbb{E}\left(X_\tau^2\right) = \mathbb{E}\left(\tau\right).$$

It is enough to prove the result for $\tau \wedge n$ since, by letting $n \to \infty$ we would recover the unbounded case by applying the monotone convergence theorem. Let

$$\begin{aligned}\mu &= \inf\left\{t : \left(\tau \wedge t\right)^{\frac{1}{2}} > \lambda\right\}\\ \nu &= \inf\left\{t : \left(\tau \wedge t\right)^{\frac{1}{2}} > \beta\lambda\right\}\\ \sigma &= \inf\left\{t : X\left(\tau \wedge t\right) > \delta\lambda\right\}\end{aligned}$$

and these are all stoppping times. It is helpful to note that

$$\begin{aligned}\nu^{1/2} &= \beta\lambda,\ \tau > \left(\beta\lambda\right)^2\\ &= \infty,\ \tau \leq \left(\beta\lambda\right)^2\end{aligned}$$

and

$$\begin{aligned}\mu^{1/2} &= \lambda,\ \tau > \lambda^2\\ &= \infty,\ \tau \leq \lambda^2.\end{aligned}$$

In particular each variable takes only two values and as $\beta > 1$ one has $\mu < \tau$ whenever μ is finite.

Now, if $\tau^{\frac{1}{2}} > \beta\lambda$, then $\tau \wedge \nu = \nu = (\beta\lambda)^2$, and $\tau \wedge \mu = \mu = \lambda^2$. If $X_\tau^* \leq \delta\lambda$ then $\sigma = \infty$. Moreover $\nu \geq \mu$ on $\tau < \infty$, hence

$$\begin{aligned} & \mathbb{P}\left(\tau^{\frac{1}{2}} > \beta\lambda, X_\tau^* \leq \delta\lambda\right) \\ \leq \quad & \mathbb{P}\left(\tau \wedge \nu \wedge \sigma - \tau \wedge \mu \wedge \sigma \geq \beta^2\lambda^2 - \lambda^2\right) \\ \leq \quad & \frac{1}{\beta^2\lambda^2 - \lambda^2}\mathbb{E}\left(\tau \wedge \nu \wedge \sigma - \tau \wedge \mu \wedge \sigma\right) \end{aligned}$$

and by the L^2 isometry mentioned at the top of the proof

$$\mathbb{E}\left(\tau \wedge \nu \wedge \sigma - \tau \wedge \mu \wedge \sigma\right) = \mathbb{E}\left(X_{\tau\wedge\nu\wedge\sigma}^2 - X_{\tau\wedge\mu\wedge\sigma}^2\right)$$

but $X_{\tau\wedge\mu\wedge\sigma}^2 > 0$ and so

$$\begin{aligned} X_{\tau\wedge\nu\wedge\sigma}^2 - X_{\tau\wedge\mu\wedge\sigma}^2 \quad & \leq \quad X_{\tau\wedge\nu\wedge\sigma}^2 \\ & \leq \quad (\delta\lambda)^2 \end{aligned}$$

where the last line uses the sample path continuity of X. Since

$$X_{\tau\wedge\nu\wedge\sigma}^2 - X_{\tau\wedge\mu\wedge\sigma}^2 = 0$$

on $\{\mu = \nu\} = \{\mu = \infty\} = \left\{\tau^{\frac{1}{2}} \leq \lambda\right\}$. We have

$$\mathbb{P}\left(\tau^{\frac{1}{2}} > \beta\lambda, X_\tau^* \leq \delta\lambda\right) \leq \frac{1}{\beta^2\lambda^2 - \lambda^2}(\delta\lambda)^2\,\mathbb{P}\left(\tau^{\frac{1}{2}} > \lambda\right).$$

The other inequality has a similar proof:

$$\begin{aligned} \mu \quad &= \quad \inf\{t : |X(\tau \wedge t)| > \lambda\} \\ \nu \quad &= \quad \inf\{t : |X(\tau \wedge t)| > \beta\lambda\} \end{aligned}$$

and let $b = (\delta\lambda)^2$. On the set $\{X_\tau^* > \beta\lambda\}$ one has $\mu \leq \nu < \infty$, $|X(\tau \wedge \mu)| = \lambda$, $|X(\tau \wedge \nu)| = \beta\lambda$. Thus

$$\begin{aligned} \mathbb{P}\left(X_\tau^* > \beta\lambda, \tau^{\frac{1}{2}} \leq \delta\lambda\right) \quad & \leq \quad \mathbb{P}\left(|X(\tau \wedge \nu) - X(\tau \wedge \mu)| \geq \beta\lambda - \lambda, \tau \leq b\right) \\ & \leq \quad \mathbb{P}\left(|X(\tau \wedge \nu \wedge b) - X(\tau \wedge \mu \wedge b)| \geq \beta\lambda - \lambda\right) \\ & \leq \quad \frac{1}{(\beta-1)^2\lambda^2}\mathbb{E}\left(|X(\tau \wedge \nu \wedge b) - X(\tau \wedge \mu \wedge b)|^2\right). \end{aligned}$$

As $X(\tau \wedge \nu \wedge b)$, $X(\tau \wedge \mu \wedge b)$ are martingales with time parameter b it follows that

$$\begin{aligned} \mathbb{E}\left(|X(\tau \wedge \nu \wedge b) - X(\tau \wedge \mu \wedge b)|^2\right) \quad & = \quad \mathbb{E}\left(|X(\tau \wedge \nu \wedge b)|^2\right) - \mathbb{E}\left(|X(\tau \wedge \mu \wedge b)|^2\right) \\ & = \quad \mathbb{E}(\tau \wedge \nu \wedge b) - \mathbb{E}(\tau \wedge \mu \wedge b) \\ & \leq \quad b\mathbb{P}(\mu < \infty) \\ & = \quad (\delta\lambda)^2\,\mathbb{P}(X_\tau^* > \lambda) \end{aligned}$$

which completes the proof!! ■

4.2.1 Examples

We can use complex analysis (lecture gave outline) to compute the probability that maximal function for BM stopped on exit from a wedge exceeds a given level and so compute which moments of the exit time are finite (which depends explicitly on the angle).

Chapter 5

Conjugate function inequalities, singular integrals, and Riesz transforms

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ denote the unit circle. We are interested in some of the deeper questions relating to the analysis of functions on $\mathbb{T}$. The law of Brownian motion started at zero, on exit from the disk is, of course, the unique rotation invariant probability measure on $\mathbb{T}$ which we will refer to as Lesbegue or (normalised) angular measure $\mu_0(d\theta) := \frac{1}{2\pi} d\theta$. If we started the motion at $z \in D$ then, as mentioned in the excercises, we can use conformal invariance to compute the exit law μ_z of the Brownian motion from D which is given by Poisson's formula :

$$\mu_z(d\theta) := \frac{1}{2\pi} \frac{1 - |z|^2}{|z - e^{i\theta}|^2} d\theta := P(z, \theta)\, d\theta.$$

for example http://mathworld.wolfram.com/PoissonKernel.html. Some properties of the function $(x, y) \to P(x + iy, \theta)$ are given illumination by the following lemma

Lemma 91 *Let $\phi : \mathbb{C} \to \mathbb{C}$ denote the conformal map defined by*

$$\phi(z) = \frac{i\left(e^{i\theta} + z\right)}{e^{i\theta} - z},$$

which maps the unit disk to the upper half-plane, sending 0 to i and $e^{i\theta}$ to $i\infty$. Then,

$$2\pi P(z, \theta) = \frac{1 - |z|^2}{|z - e^{i\theta}|^2} = \operatorname{Im}(\phi(z)).$$

Proof. Exercise ■

Corollary 92 *For each θ the function $\psi : \mathbb{R}^2 \to \mathbb{R}$ define by $\psi(x,y) = P(x+iy,\theta)$ is positive, continuous on the interior, and HARMONIC. Its first two (in fact any $N \in \mathbb{N}$) derivatives are continuously bounded independently of θ for all disks of radius strictly less than 1.*

Proof. Positivity is obvious by inspection. Moreover, the previous lemma shows that $2\pi\psi$ arises as the imaginary part of an analytic function and so is harmonic on the interior any disk with radius less than 1. The continuity and boundedness properties of the derivatives may be verified by computing them directly. ■

Lemma 93 *If f is harmonic on D, and continuous then $f(Z_{t\wedge T_D})$ is a bounded martingale and so*

$$f(z) = \mathbb{E}_z\left(f\left(Z_{H_{\mathbb{T}}}\right)\right), \ z \in D.$$

Proof. Itô's formulae, on compact sets bounded local martingale ■

Theorem 94 *Every element $f \in C(\mathbb{T})$ extends in a* unique *way to a function harmonic on D and continuous on $\bar{D}$.*

Proof. Uniqueness: Suppose that g and $\tilde{g}$ were both extensions, then $g - \tilde{g}$ is harmonic on D and continuous on $\bar{D}$ and zero on the boundary $\mathbb{T}$. But, $(g-\tilde{g})(Z_{t\wedge T_D})$ is a bounded local martingale, it is also a bounded martingale and for any z in D the limit $\lim_{t\to\infty} t \wedge T_D = T_D < \infty$ so

$$\begin{aligned}
(g-\tilde{g})(z) &= \mathbb{E}_z\left[(g-\tilde{g})(Z_{t\wedge T_D})\right] \\
&= \lim_{t\to\infty} \mathbb{E}_z\left[(g-\tilde{g})(Z_{t\wedge T_D})\right] \\
&= \mathbb{E}_z\left[\lim_{t\to\infty}(g-\tilde{g})(Z_{t\wedge T_D})\right] \\
&= \mathbb{E}_z\left[(g-\tilde{g})(Z_{T_D})\right] \\
&= 0.
\end{aligned}$$

Existence: For each z in D we define

$$f(z) = \mathbb{E}_z\left[f(Z_{T_D})\right]$$

Suppose that S is a stopping time, and $Z^S(t) := Z(S+t)$, then loosely, the strong Markov property says that the law of Z^S is that of a Brownian motion started at Z_S. If $S \le T_D$ then

$$T_D\left(Z^S\right) + S = T_D(Z)$$

since the first exit time from D for Z_S can be completely understood in terms of the first exit time for Z. Suppose $f(Z_{T_D})$ is integrable, and h is bounded and $\mathfrak{F}_S$ measurable then

$$\begin{aligned}
\mathbb{E}_z\left[hf\left(Z_{T_D(\omega)}(\omega)\right)\right] &= \mathbb{E}_z\left[\mathbb{E}_z\left[hf(Z_{T_D})\,|\mathfrak{F}_S\right]\right] \\
&= \mathbb{E}_z\left[h\mathbb{E}_z\left[f\left(Z_{T_D(Z^S)+S}\right)|\mathfrak{F}_S\right]\right] \\
&= \mathbb{E}_z\left[h\mathbb{E}_{Z_S}\left[f(Z_{T_D})\right]\right] \\
&= \mathbb{E}_z\left[hf(Z_S)\right]
\end{aligned}$$

As h is arbitrary $\mathfrak{F}_S$ measurable one observes from the definition of conditional expectation that

$$\mathbb{E}_z\left[f\left(Z_{T_D}\right)|\mathfrak{F}_S\right]=f\left(Z_S\right)$$

and in particular $f\left(Z_{t\wedge T_D}\right)$ is a uniformly integrable and L^1 bounded martingale.

Since

$$f\left(z\right)=\frac{1}{2\pi}\int_{\mathbb{T}}\frac{1-\left|z\right|^2}{\left|z-e^{i\theta}\right|^2}f\left(e^{i\theta}\right)d\theta$$

we see from the lemma 96 that $f\left(z_n\right)\to f\left(z\right)$ whenever $z_n\to z\in D$,so f is continuous. A similar argument applied to the function

$$\theta\to\partial^j\bar{\partial}^k\frac{1-\left|z\right|^2}{\left|z-e^{i\theta}\right|^2}$$

yields the continuity of all derivatives of f .

Since $f\left(Z_{t\wedge T_D}\right)$ is a martingale we can conclude from the Itô formula that $\partial\bar{\partial}f$ is zero - at least almost everywhere, and as it is itself continuous it must be identically zero. So f is harmonic.

We must now prove that the function defined in this way is continuous up to the boundary of D.

Recall (you have seen examples such as this one) that if we have two concentric balls, one of radius r and the other of radius $R>2r$ then and if Brownian motion is started at a distance r from the smaller ball then the probability that it hits the smaller ball before the larger one is between them, then the probability that the process exits through the larger ball is

$$\frac{\log 2r-\log r}{\log R-\log r}=\frac{\log 2}{\log R-\log r}$$

and there is a function (excercise - compute it) $\delta\left(\varepsilon\right)$ so that if $r/R<\delta\left(\varepsilon\right)$ this probability is less than ε. This probability only depends on r/R.

Fix a continuous function f on $\mathbb{T}$ and $x\in\mathbb{T}$. The set $\mathbb{T}$ is compact, and thus there is an M so that for all $x'\in\mathbb{T}$

$$\left|f\left(x'\right)-f\left(x\right)\right|<M.$$

Now choose R so that if $\left|x'-x\right|\leq 2R$ then $\left|f\left(x'\right)-f\left(x\right)\right|\leq\varepsilon$. Suppose z is a distance at most $r=\delta\left(\varepsilon/M\right)$ from x then by considering the case of a ball of radius r touching the disk at x and just outside it, we see that Brownian motion must hit the smaller disk before leaving the larger one with probability at least $1-\frac{\varepsilon}{M}$, and in doing so must also have left D. Therefore we have

$$\mathbb{P}_z\left[\left|Z_{T_D}-x\right|<2R\right]>1-\frac{\varepsilon}{M}.$$

And so

$$\begin{aligned}\left|f\left(z\right)-f\left(x\right)\right| &= \left|\mathbb{E}_z\left[f\left(Z_{T_D}\right)\right]-f\left(x\right)\right|\\ &< \mathbb{E}\left[\left|f\left(Z_{T_D}\right)-f\left(x\right)\right|\right]\\ &< \varepsilon+M\frac{\varepsilon}{M}\\ &< 2\varepsilon\end{aligned}$$

■

This approach quickly gives a probabilistic proof of the maximum principle.

Corollary 95 *If D is any domain then $h(z) = \mathbb{E}_z(f(Z_{T_D}))$, $z \in D$ is harmonic provided $y \longmapsto \mathbb{E}_y[f(Z_{T_D})]$ is continuous on $\partial B(x,R)$ for any $\overline{B(x,R)} \subset D$.*

Proof. Suppose $z \in D$ then there exists $x \in D$ and $R > 0$ such that $z \in B(x,R)$ and $\overline{B(x,R)} \subset D$. Wlog we may suppose that $x = 0$, then the strong Markov property gives

$$\begin{aligned} \mathbb{E}_z[f(Z_{T_D})] &= \mathbb{E}_z\left[\mathbb{E}_{Z_{T_{B(0,R)}}}[f(Z_{T_D})]\right] \\ &= \int_0^{2\pi} \mathbb{E}_{Re^{i\theta}}[f(Z_{T_D})]\, P(z,\theta)\, d\theta. \end{aligned}$$

Since the function $y \longmapsto \mathbb{E}_y[f(Z_{T_D})]$ is continuous on $\partial B(x,R)$ we have that $h(z) = \mathbb{E}_z(f(Z_{T_D}))$ is harmonic by Theorem **??**. ■

One notices from this formula that

Lemma 96 *If $z_n \in D$ for $z_n \in \mathbb{N}$ and $z_n \to z_0$ then the functions $\theta \to \frac{1}{2\pi}\frac{1-|z_n|^2}{|z_n - e^{i\theta}|^2}$ are continuous on $\mathbb{T}$ and converge uniformly on $\mathbb{T}$.*

Natural Banach spaces of functions on $\mathbb{T}$ include the $L^p(\mathbb{T},\mu)$ spaces with $p \geq 1$ and norm[1]

$$\begin{aligned} \|f\|_p &= \left(\int_{\mathbb{T}} \left|f\left(e^{i\theta}\right)\right|^p \mu(d\theta)\right)^{1/p} \\ &= \left(\mathbb{E}_0\left[|f(Z_{T_D})|^p\right]\right)^{1/p} \end{aligned}$$

and the continuous functions $C(\mathbb{T})$with the uniform norm

$$\|f\|_\infty = \sup_{z\in\mathbb{T}} |f(z)|.$$

It is a well known, and not quite trivial fact that

Lemma 97 *The space $C(\mathbb{T})$ is dense in $L^p(\mathbb{T},\mu)$.*

This is standard analysis and has nothing to do with $\mathbb{T}$ being the circle. The proof is closely related to standard fact that for any Borel measure ν on a separable and completely metrizable space, any measurable subset A, and any $\varepsilon > 0$ one can always find a compact subset K of A so that $\nu(A\backslash K) < \varepsilon$.

As a consequence it is usually enough to prove estimates for $L^p(\mathbb{T},\mu)$ if to assume that the functions are continuous functions if that simplifies matters.

Special to $\mathbb{T}$ one could also consider the classes of f that have well behaved harmonic or analytic extension to D.

[1]Where Z is complex Brownian motion, D is the unit disk, and T_D is the first exit time from D.

Lemma 98 *Every element $f \in L^p(\mathbb{T}, \mu)$ extends in a unique way to a function harmonic on D so that*

$$f(Z_{t\wedge T_D}) = \mathbb{E}_0[f(Z_{T_D})|\mathfrak{F}_t]$$

and $f(Z_{t\wedge T_D})$ is an L^p bounded and uniformly integrable martingale.

Proof. As before, we define

$$\begin{aligned} f(z) &= \mathbb{E}_z[f(Z_{T_D})] \\ &= \mu_z(f) \end{aligned}$$

and this makes sense for any function $f \in L^p(\mathbb{T}, \mu_0)$ .Since the function $\theta \to \frac{1-|z|^2}{|z-e^{i\theta}|^2}$ is continuous into $C(\mathbb{T})$ with the uniform norm and $\int g_n f \to \int gf$ if the g_n converge uniformly to g and f is integrable, the existence, continuity and smoothness of f follow as before. The strong Markov argument again allows one to prove that $f(Z_{t\wedge T_D})$ is a local martingale.

We need to prove that it is a martingale and that the identity $f(Z_{t\wedge T_D}) = \mathbb{E}_0[f(Z_{T_D})|\mathfrak{F}_t]$ holds true.

Now

$$t \to \mathbb{E}_0[f(Z_{T_D})|\mathfrak{F}_t]$$

is a uniformly integrable martingale in L^p by definition. Choose f_n continuous with $f_n \to f$ in $L^p(\mathbb{T}, \mu)$. Using the extensions to D defined in the previous paragraph, we see that

$$\begin{aligned} f_n(z) &= \frac{1}{2\pi}\int_{\mathbb{T}} \frac{1-|z|^2}{|z-e^{i\theta}|^2} f_n(\theta)\, d\theta \\ &\to \\ f(z) &= \frac{1}{2\pi}\int_{\mathbb{T}} \frac{1-|z|^2}{|z-e^{i\theta}|^2} f(\theta)\, d\theta \end{aligned}$$

since $f_n \to f$ in $L^1(\mathbb{T}, \mu_0)$. On the other hand if X is $L^p(\Omega, \mathcal{F}, \mathbb{P})$ then Jensen's inequality tells us that

$$\sup_t \mathbb{E}[\mathbb{E}[|X||\mathfrak{F}_t]^p]^{1/p} \leq \mathbb{E}[|X|^p]^{1/p}$$

and so the map from functions to martingales

$$f_n \to (t \to \mathbb{E}_0[f_n(Z_{T_D})|\mathfrak{F}_t])$$

is continuous and the $t \to \mathbb{E}_0[f_n(Z_{T_D})|\mathfrak{F}_t]$ converge to the martingale $t \to \mathbb{E}_0[f(Z_{T_D})|\mathfrak{F}_t]$ in the L^p sense. On the other hand $t \to \mathbb{E}_0[f_n(Z_{T_D})|\mathfrak{F}_t]$ has been identified as $t \to f_n(Z_{t\wedge T_D})$ and this converges pointwise to $f(Z_{t\wedge T_D})$ at least on $[0, S]$ where $S < T_D$. If a martingale converges in L^p and pathwise to some process, then that pathwise limit has to coincide almost surely with the L^p limit (choose a subsequence to make the L^p convergence pointwise as well). So we conclude that the martingale coincides with $f(Z_{t\wedge T_D})$ on each $[0, S]$ and hence on $[0, T_D]$ since agreement at T_D was an assumption.

$$f(Z_{t\wedge T_D}) = \mathbb{E}_0[f(Z_{T_D})|\mathfrak{F}_t].$$

■

Corollary 99 *If $f \in L^p(\mathbb{T}, \mu_0)$ then the harmonic function extension of f to D has the property that*

$$\lim_{t \to T_D, t < T_D} f(Z_t) = f(Z_{T_D})$$

almost surely.

Proof. Apply the martingale convergence theorem. ■

Theorem 100 *If f is a harmonic function on D and $f(Z_t)$ is a uniformly integrable L^1 (or L^p bounded martingale) on $t < T_D$ then there is a function in $L^1(\mathbb{T})$ (or $L^p(\mathbb{T})$) for which f is an extension. In particular $\lim_{t \to T_D} f(Z_t)$ is Z_{T_D} measurable.*

Proving the last part of the result is the key to the whole proof but we cannot give it here.

Almost nothing that we have said above really uses the complex analysis and for example it is true without too many changes for the ball in d dimensions. Let us change notation a bit. Let u be a real valued function and integrable function on T. Then we can see that it has an extension and that this in turn has a unique harmonic conjugate v that is zero at zero. If u is bounded, then it is certainly in L^2 and the following argument, together with the above theorem allows one to show that the conjugate function is also in L^2 and can be thought of as a transformation of functions on the circle to functions on the circle. We can use our earlier work to get an array of impressive results.

Definition 101 *Suppose that $A(f)$ and $B(f)$ are two positive scalar quantities associated to f and that there is a strictly positive constants C so that*

$$\begin{aligned} A(f) &\leq cB(f) \\ B(f) &\leq CA(f) \end{aligned}$$

independently of f then we say $A(f) \approx B(f)$.

Recall a crucial fact about analytic functions. If $f(z) = u(z) + iv(z)$ then[2]

$$|\partial f(z)|^2 = |\nabla u(z)|^2 = |\nabla v(z)|^2$$

Theorem 102 *If $f(z) = u(z) + iv(z)$ is analytic on the disk $D = \{|z| < 1\}$ and u is continuous on the closed disk $\bar{D} = \{|z| \leq 1\}$ with $f(0) = 0$ then*

$$\begin{aligned} & \mathbb{E}_0 \left(\left| u(Z_{\cdot \wedge T_D})^*_{T_D} \right|^p \right)^{1/p} \\ \approx \quad & \mathbb{E}_0 \left(\left(\int_0^{T_D} |\nabla u(Z_t)|^2 \, dt \right)^{p/2} \right)^{1/p} \\ = \quad & \mathbb{E}_0 \left(\left(\int_0^{T_D} |\nabla v(Z_t)|^2 \, dt \right)^{p/2} \right)^{1/p} \\ \approx \quad & \mathbb{E}_0 \left(\left| v(Z_{\cdot \wedge T_D})^*_{T_D} \right|^p \right)^{1/p} \end{aligned}$$

[2]Check the powers of 2 here as it may not be quite correct

Definition 103 *We can introduce the real (Hardy Space) norm(s)*

$$\mathbb{E}_0\left(\left|u\left(Z_{.\wedge T_D}\right)^*_{T_D}\right|\right)$$
$$\mathbb{E}_0\left(\left(\int_0^{T_D}\left|\nabla u\left(Z_t\right)\right|^2 dt\right)^{1/2}\right)^{1/2}$$

Theorem 104 *The following are true.*

1. *Every function U in L^1 has a harmonic extension u and a unique conjugate function v with $v(0)=0$*
2. *v has boundary data V in L^p for all $p<1$*
3. *If U is in H^1 then V is in H^1*
4. *If U is in H^1 and V is in H^1 then U is in H^1.*
5. *U is in H^1 if and only if the nontangential maximal function of U is in L^1*

The last part of this theorem is not now a surprise as we can understand it as analogous to our result.

5.0.2 The Burkolder Gundy Silverstein Theorem

Definition 105 (H^p spaces) *Let $0<p<\infty$ then if f is a function defined on D then $f\in H^p$ if*

$$||f||_{H^p}=\sup_{0\leq r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^p d\theta\right)^{1/p}<\infty.$$

Let $0<r<1$, $D_r=\{z\in\mathbb{C}:|z|<r\}$ and denote its boundary by $\mathbb{T}_r$. Now suppose that u is a harmonic function on D and let v denote its unique harmonic conjugate with $v(0)=0$ and such that $F=u+iv$ is analytic. Our objective is to prove the following result of Burkholder, Gundy and Silverstein :

Theorem 106 *For every $0<p<\infty$ there exist constants c_p and C_p such that*

$$c_pE_0\left[u^*\left(Z_{T_D}\right)^p\right]^{1/p}\leq||F||_{H^p}\leq C_pE_0\left[u^*\left(Z_{T_D}\right)^p\right]^{1/p}.$$

(where recall that the maximal function is given by $u^*(Z_t)=\sup_{0\leq s\leq t}|u(Z_s)|$).

In other words, $F\in H^p$ if and only if $u^*\left(Z_{T_D}\right)\in L^p(\Omega,\mathcal{F},P)$. In particular, one may check if a given function $U\in L^p(\mathbb{T},\mu)$ is such that its harmonic extension u the real part of a H^p function by checking whether its maximal function is in $L^p(\Omega,\mathcal{F},P)$. An essential ingedient is the following inequality which relates the L^p norms for the maximal functions of a harmonic function to its conjugate.

Theorem 107 (Conjugate function inequality) *For each $0 < p < \infty$ there exists a constant c_p such that for any harmonic function u on D with conjugate v such that $v(0) = 0$ there is a constant c_p such that*

$$E_0 \left[v^* \left(Z_{T_D} \right)^p \right]^{1/p} \leq c_p E_0 \left[u^* \left(Z_{T_D} \right)^p \right]^{1/p}.$$

Let $0 < r < 1$, $D_r = \{z \in \mathbb{C} : |z| < r\}$ and denote its boundary by $\mathbb{T}_r$ and let T_{D_r} represent the exit time of Brownian motion from D_r. Introduce the maximal functions

$$\begin{aligned} F_r^* &= F^* \left(Z_{T_{D_r}} \right) = \sup_{0 \leq t < \infty} \left| F \left(Z_{t \wedge T_{D_r}} \right) \right| \\ u_r^* &= u^* \left(Z_{T_{D_r}} \right) = \sup_{0 \leq t < \infty} \left| u \left(Z_{t \wedge T_{D_r}} \right) \right|, \end{aligned}$$

then to prove the conjugate function inequality it suffices to prove for each $0 < r < 1$ that $E_0 \left[v^* \left(Z_{T_{D_r}} \right)^p \right]^{1/p} \leq c_p E_0 \left[u^* \left(Z_{T_{D_r}} \right)^p \right]^{1/p}$ for some constant c_p which does not depend on r. We do this using the following sequence of results.

Lemma 108 *Let $(\mathcal{F}_t)_{t \geq 0}$ be the filtration generated by $Z_{t \wedge T_D}$ and suppose S and T are $\mathcal{F}_t-$ stopping times such that $S \leq T$ a.s.. Then*

$$E_0 \left[\left| u \left(Z_{T \wedge T_{D_r}} \right) - u \left(Z_{S \wedge T_{D_r}} \right) \right|^2 \right] = E_0 \left[\left| v \left(Z_{T \wedge T_{D_r}} \right) - v \left(Z_{S \wedge T_{D_r}} \right) \right|^2 \right].$$

Proof. Without loss of generality we can (and will) assume $F(0) = 0$ throughout this and subsequent proofs. Since F^2 is analytic we know $F^2(Z_t)$ is a local martingale. Moreover, since F is bounded on D it follows that $F^2 \left(Z_{t \wedge T_{D_r}} \right)$ is a bounded martingale and then $F^2 \left(Z_{t \wedge S \wedge T_{D_r}} \right)$, $F^2 \left(Z_{t \wedge T \wedge T_{D_r}} \right)$ are bounded martingales as well. Optional stopping followed by martingale convergence gives (using $T_{D_r} < \infty$):

$$\begin{aligned} E_0 \left[F^2 \left(Z_{t \wedge S \wedge T_{D_r}} \right) \right] &= E_0 \left[F^2 \left(Z_{0 \wedge S \wedge T_{D_r}} \right) \right] = 0, \\ \lim_{t \to \infty} E_0 \left[F^2 \left(Z_{t \wedge S \wedge T_{D_r}} \right) \right] &= E_0 \left[F^2 \left(Z_{S \wedge T_{D_r}} \right) \right] = 0. \end{aligned}$$

Consequently,

$$0 = E_0 \left[u \left(Z_{S \wedge T_{D_r}} \right)^2 - v \left(Z_{S \wedge T_{D_r}} \right)^2 \right] + 2i E_0 \left[u \left(Z_{S \wedge T_{D_r}} \right) v \left(Z_{S \wedge T_{D_r}} \right) \right]$$

so $E_0 \left[u \left(Z_{S \wedge T_{D_r}} \right)^2 \right] = E_0 \left[v \left(Z_{S \wedge T_{D_r}} \right)^2 \right]$ and a similar calculation gives $E_0 \left[u \left(Z_{T \wedge T_{D_r}} \right)^2 \right] = E_0 \left[v \left(Z_{T \wedge T_{D_r}} \right)^2 \right]$. Now $s \to u \left(Z_{t \wedge S \wedge T_{D_r}} \right)$ is a one dimensional real martingale which is bounded (and so UI) so we have

$$E_0 \left[u \left(Z_{t \wedge T \wedge T_{D_r}} \right) | \mathcal{F}_S \right] = u \left(Z_{t \wedge S \wedge T_{D_r}} \right),$$

and then

$$\begin{aligned} E_0 \left[u \left(Z_{t \wedge T \wedge T_{D_r}} \right) u \left(Z_{t \wedge S \wedge T_{D_r}} \right) \right] &= E_0 \left[E_0 \left[u \left(Z_{t \wedge T \wedge T_{D_r}} \right) | \mathcal{F}_S \right] u \left(Z_{t \wedge S \wedge T_{D_r}} \right) \right] \\ &= E_0 \left[u \left(Z_{t \wedge S \wedge T_{D_r}} \right)^2 \right]. \end{aligned}$$

The same argument also gives this result for the conjugate function v therefore

$$\begin{aligned}
& E_0\left[\left|u\left(Z_{T\wedge T_{D_r}}\right)-u\left(Z_{S\wedge T_{D_r}}\right)\right|^2\right] \\
= \; & E_0\left[u\left(Z_{T\wedge T_{D_r}}\right)^2-2u\left(Z_{T\wedge T_{D_r}}\right)u\left(Z_{S\wedge T_{D_r}}\right)+u\left(Z_{S\wedge T_{D_r}}\right)^2\right] \\
= \; & E_0\left[u\left(Z_{T\wedge T_{D_r}}\right)^2-u\left(Z_{S\wedge T_{D_r}}\right)^2\right] \\
= \; & E_0\left[v\left(Z_{T\wedge T_{D_r}}\right)^2-v\left(Z_{S\wedge T_{D_r}}\right)^2\right] \\
= \; & E_0\left[\left|v\left(Z_{T\wedge T_{D_r}}\right)-v\left(Z_{S\wedge T_{D_r}}\right)\right|^2\right].
\end{aligned}$$

■

Lemma 109 *Suppose g is a measurable function on a measure space $(X,\mathcal{B},\mu)$ and $E\subset X$ is a measurable set with $0<\mu(E)<\infty$ and $g\geq 0$ on E. Suppose further that A and B are constants such that :*

1. $\frac{1}{\mu(E)}\int_E g d\mu \geq A>0$ *and*
2. $\frac{1}{\mu(E)}\int_E g^2 d\mu \leq B$.

Then for any $\delta\in(0,1)$,

$$\mu\{x : g(x)\geq \delta A\}\geq \mu(E)(1-\delta)^2\frac{A^2}{B}.$$

Proof. Let $E_\delta=\{x\in E : g(x)\geq \delta A\}$, then

$$\int_{E\setminus E_\delta} g d\mu \leq \delta A\mu(E)$$

so 1 gives

$$A\mu(E)\leq \int_E g d\mu=\int_{E\setminus E_\delta} g d\mu+\int_{E_\delta} g d\mu\leq \delta A\mu(E)+\int_{E_\delta} g d\mu$$

and hence

$$\int_{E_\delta} g d\mu\geq (1-\delta)A\mu(E).$$

But Cauchy-Schwarz and 2 give that

$$\int_{E_\delta} g d\mu\leq\left(\int_{E_\delta} g^2 d\mu\right)^{1/2}\mu(E_\delta)^{1/2},$$

so that

$$\mu(E_\delta)\geq\frac{A^2}{B}\left(1-\delta^2\right)\mu(E).$$

■

Lemma 110 *Let $a \geq 1$ and $b > 1$. There are constants $c_{a,b}$ and $d_{a,b}$ such that if $\lambda > 0$ and*

$$P\left(F_r^* > \lambda\right) \leq aP\left(F_r^* > b\lambda\right),$$

then

$$P\left(F_r^* > \lambda\right) \leq c_{a,b} P\left(u_r^* > b\lambda\right).$$

Proof. Define the stopping times $S = \inf\left\{t \geq 0 : \left|F\left(Z_{t\wedge T_{D_r}}\right)\right| > \lambda\right\}$ and $T =$

$\inf\left\{t \geq 0 : \left|F\left(Z_{t\wedge T_{D_r}}\right)\right| > b\lambda\right\}$, so that $S \leq T$ a.s.. On the set $\{F_r^* > b\lambda\} = \{T < \infty\}$ we have that $\left|F\left(Z_{S\wedge T_{D_r}}\right)\right| = \lambda$ and $\left|F\left(Z_{T\wedge T_{D_r}}\right)\right| = b\lambda$. Also, $\{F_r^* \leq \lambda\} = \{S = T = \infty\}$ and so on this set we must have $Z_{S\wedge T_{D_r}} = Z_{T\wedge T_{D_r}} = Z_{T_{D_r}}$. From lemma 109 we deduce that

$$\begin{aligned}
& E_0\left[\left|u\left(Z_{T\wedge T_{D_r}}\right) - u\left(Z_{S\wedge T_{D_r}}\right)\right|^2 1_{\{F_r^* > \lambda\}}\right] \\
= \; & E_0\left[\left|u\left(Z_{T\wedge T_{D_r}}\right) - u\left(Z_{S\wedge T_{D_r}}\right)\right|^2\right] \\
= \; & \frac{1}{2} E_0\left[\left|F\left(Z_{T\wedge T_{D_r}}\right) - F\left(Z_{S\wedge T_{D_r}}\right)\right|^2\right] \\
\geq \; & \frac{1}{2} E_0\left[\left|F\left(Z_{T\wedge T_{D_r}}\right) - F\left(Z_{S\wedge T_{D_r}}\right)\right|^2 1_{\{F_r^* > b\lambda\}}\right] \\
= \; & \frac{1}{2}\lambda^2 (b-1)^2 P\left(F_r^* > b\lambda\right) \geq \frac{\lambda^2 (b-1)^2}{2a} P\left(F_r^* > \lambda\right).
\end{aligned}$$

Also,

$$\begin{aligned}
& E_0\left[\left|u\left(Z_{T\wedge T_{D_r}}\right) - u\left(Z_{S\wedge T_{D_r}}\right)\right|^4 1_{\{F_r^* > \lambda\}}\right] \\
\leq \; & E_0\left[\left|F\left(Z_{T\wedge T_{D_r}}\right) - F\left(Z_{S\wedge T_{D_r}}\right)\right|^4 1_{\{F_r^* > \lambda\}}\right] \\
= \; & E_0\left[\left|F\left(Z_{T\wedge T_{D_r}}\right) - F\left(Z_{S\wedge T_{D_r}}\right)\right|^4 1_{\{F_r^* > b\lambda\}}\right] \\
& + E_0\left[\left|F\left(Z_{T\wedge T_{D_r}}\right) - F\left(Z_{S\wedge T_{D_r}}\right)\right|^4 1_{\{\lambda < F_r^* \leq b\lambda\}}\right] \\
\leq \; & \lambda^4 (b+1)^4 P\left(F_r^* > b\lambda\right) + 16 b^4 \lambda^4 P\left(\lambda < F_r^* \leq b\lambda\right) \\
\leq \; & \left((b+1)^4 + 16 b^4\right) \lambda^4 P\left(F_r^* > \lambda\right).
\end{aligned}$$

Applying lemma 109 for the choice

$$\begin{aligned}
g &= \left|u\left(Z_{T\wedge T_{D_r}}\right) - u\left(Z_{S\wedge T_{D_r}}\right)\right|^2, \; E = \left\{F_r^* > \lambda\right\}, \\
A &= \frac{\lambda^2 (b-1)^2}{2a} \text{ and } B = \left((b+1)^4 + 16 b^4\right) \lambda^4,
\end{aligned}$$

we see that for any $\delta \in (0,1)$

$$P\left(F_r^* > \lambda\right) \leq \frac{B}{A^2\left(1-\delta^2\right)} P\left(\left|u\left(Z_{T\wedge T_{D_r}}\right) - u\left(Z_{S\wedge T_{D_r}}\right)\right| \geq \left(\frac{\delta (b-1)^2}{2a}\right)^{1/2} \lambda\right).$$

Fixing a value of δ and observing that $\left|u\left(Z_{T\wedge T_{D_r}}\right) - u\left(Z_{S\wedge T_{D_r}}\right)\right| \leq 2u_r^*$ finishes the result. ■

We can now present :

Proof. (of lemma 107) Fix $p \in (0,\infty)$. For $x \geq 0$ let $G(x) = P(F_r^* > x)$ then

$$E\left[|F_r^*|^p\right] = -\int_0^\infty \lambda^p G(d\lambda).$$

Since F is bounded we have $G(x) = 0$ for all x large enough and so integration by parts gives

$$E\left[|F_r^*|^p\right] = \int_0^\infty G(\lambda)\, p\lambda^{p-1} d\lambda = \int_0^\infty p\lambda^{p-1} P(F_r^* > \lambda)\, d\lambda.$$

Let $a = 2^{p+1}$ and $b = 2$. If B denotes the set of all $\lambda > 0$ for which

$$P(F_r^* > \lambda) \leq aP(F_r^* > b\lambda)$$

then

$$\begin{aligned}
&\int_0^\infty p\lambda^{p-1} P(F_r^* > \lambda)\, d\lambda = \int_0^\infty p(bx)^{p-1} P(F_r^* > bx)\, bdx \\
= \;& b^p \left(\int_B px^{p-1} P(F_r^* > bx)\, dx + \int_{[0,\infty)\setminus B} px^{p-1} P(F_r^* > bx)\, dx \right) \\
\leq \;& b^p \left(\int_B px^{p-1} P(F_r^* > bx)\, dx + \frac{1}{a} \int_{[0,\infty)\setminus B} px^{p-1} P(F_r^* > x)\, dx \right) \\
= \;& b^p \int_B px^{p-1} P(F_r^* > bx)\, dx + \frac{b^p}{a} \int_0^\infty px^{p-1} P(F_r^* > x)\, dx \\
& - \frac{b^p}{a} \int_B px^{p-1} P(F_r^* > x)\, dx,
\end{aligned}$$

so that

$$\left(1 - \frac{b^p}{a}\right) \int_0^\infty px^{p-1} P(F_r^* > x)\, dx \leq b^p \left(1 - \frac{1}{a}\right) \int_B px^{p-1} P(F_r^* > x)\, dx$$

and

$$\begin{aligned}
\int_0^\infty px^{p-1} P(F_r^* > x)\, dx &\leq \frac{b^p\left(1 - \frac{1}{a}\right)}{\left(1 - \frac{b^p}{a}\right)} \int_B px^{p-1} P(F_r^* > x)\, dx \\
&\leq a \int_B px^{p-1} P(F_r^* > x)\, dx.
\end{aligned}$$

Thus an application of lemma 110 gives

$$\begin{aligned}
E\left[|F_r^*|^p\right] &\leq a \int_B px^{p-1} P(F_r^* > x)\, dx \\
&\leq a \int_B px^{p-1} c_{a,b} P(u_r^* > d_{a,b} x)\, dx \\
&\leq \frac{ac_{a,b}}{d_{a,b}} \int_0^\infty px^{p-1} P(u_r^* > x)\, dx \\
&= c_p E\left[|u_r^*|^p\right] \leq c_p E\left[|u^*|^p\right].
\end{aligned}$$

■

Remark 111 *In the course of the proof we have actually shown the stronger result that* $E\left[|v_r^*|^p\right] \leq c_p E\left[|u_r^*|^p\right]$ *for all* $r \in (0,1)$.

As a corollary of this result we can obtain Riesz's inequalities relating the H^p-norms of the harmonic function with its conjugate.

Corollary 112 (Riesz's Inequality) *For each* $p \in (1,\infty)$ *there is a constant* c_p, *which does not depend on* r, *such that whenever* u *is harmonic on* D *and* v *is its harmonic conjugate with* $v(0) = 0$ *and* $r \in (0,1)$,

$$\frac{1}{2\pi}\int_0^{2\pi} \left|v\left(re^{i\theta}\right)\right|^p d\theta \leq \frac{c_p}{2\pi}\int_0^{2\pi} \left|u\left(re^{i\theta}\right)\right|^p d\theta.$$

Proof. Again let $F = u + iv$. By conformal invariance $F\left(Z_{t\wedge T_{D_r}}\right)$ is a time changed Brownian motion and since F is bounded it is a martingale. It follows at once that $u(Z_{t\wedge T_{D_r}})$ is martingale and hence (conditional Hölder + Jensen with $x \longmapsto |x|^p$) $\left|u\left(Z_{t\wedge T_{D_r}}\right)\right|^p$ is a submartingale if $p \in (1,\infty)$. Doob's maximal inequality then gives

$$E\left[\sup_{0\leq t<\infty}\left|u\left(Z_{t\wedge T_{D_r}}\right)\right|^p\right] \leq \left(\frac{p}{p-1}\right)^p E\left[\left|u\left(Z_{T_{D_r}}\right)\right|^p\right].$$

Now using the conjugate function inequality and the above gives

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \left|v\left(re^{i\theta}\right)\right|^p d\theta &= rE\left[\left|v\left(Z_{T_{D_r}}\right)\right|^p\right] \\
&\leq rE\left[(v_r^*)^p\right] \\
&\leq rc_pE\left[(u_r^*)^p\right] \\
&\leq rc_p\left(\frac{p}{p-1}\right)^p E\left[\left|u\left(Z_{T_{D_r}}\right)\right|^p\right] \\
&= c_p\left(\frac{p}{p-1}\right)^p \frac{1}{2\pi}\int_0^{2\pi} \left|u\left(re^{i\theta}\right)\right|^p d\theta.
\end{aligned}$$

■

We may now prove the Burkholder Gundy Silverstein result :
Proof. (of theorem 106) We first notice that

$$\frac{1}{2\pi}\int_0^{2\pi} \left|v\left(re^{i\theta}\right)\right|^p d\theta = rE\left[\left|v\left(Z_{T_{D_r}}\right)\right|^p\right] \leq rE\left[(v_r^*)^p\right]$$

and similarly for v. The right hand inequality then follows almost immediately from the conjugate function inequality :

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \left|F\left(re^{i\theta}\right)\right|^p d\theta &\leq \frac{1}{2\pi}\int_0^{2\pi} 2^p\left(\max\left|u\left(re^{i\theta}\right)\right|, \left|v\left(re^{i\theta}\right)\right|\right)^p d\theta \\
&\leq \frac{2^p}{2\pi}\int_0^{2\pi} \left(\left|u\left(re^{i\theta}\right)\right| + \left|v\left(re^{i\theta}\right)\right|\right)^p d\theta \\
&\leq c_pE\left[(u_r^*)^p\right].
\end{aligned}$$

We now prove the left hand inequality. To do this suppose $F \in H^p$ then it follows from the lemma below that $\left(\left|F\left(Z_{t\wedge T_{D_r}}\right)\right|^{p/2}\right)_{t\geq 0}$ is a submartingale. Doob's maximal inequality then gives

$$\begin{aligned} E\left[(u_r^*)^p\right] &\leq E\left[\left(\sup_{0\leq t<\infty}\left|F\left(Z_{t\wedge T_{D_r}}\right)\right|^{p/2}\right)^2\right] \\ &\leq c_p E\left[\left|F\left(Z_{T_{D_r}}\right)\right|^p\right] \\ &= \frac{c_p}{2\pi r}\int_0^{2\pi}\left|F\left(re^{i\theta}\right)\right|^p d\theta, \end{aligned}$$

where the second line is established by the same reasoning as in Riesz's inequalities. Letting $r \to 1$ and using monotone convergence gives

$$\begin{aligned} E\left[(u^*)^p\right] &\leq c_p \lim_{r\to 1}\frac{1}{2\pi r}\int_0^{2\pi}\left|F\left(re^{i\theta}\right)\right|^p d\theta \\ &\leq c_p \sup_{0\leq r<1}\left(\int_0^{2\pi}\left|F\left(re^{i\theta}\right)\right|^p d\theta\right). \end{aligned}$$

■

To finish we need to show that $\left(\left|F\left(Z_{t\wedge T_{D_r}}\right)\right|^{p/2}\right)_{t\geq 0}$ is a submartingale. This is accompolished using a result of Doob, for which one needs the notion of a subharmonic function :

Definition 113 *A continuous function ϕ defined on a domain $U \subset \mathbb{C}$ is said to be subharmonic if for every $z \in U$ there is an R such that if $\{\zeta : |\zeta - z| \leq R\} \subset U$ and*

$$\phi(z) \leq \frac{1}{2\pi}\int_0^{2\pi}\phi\left(z + re^{i\theta}\right) d\theta \text{ for all } r \in (0, R).$$

Remark 114 *If F ia analytic on U and $c > 0$ then $|F|^c$ is subharmonic on U.*

The fact that $\left(\left|F\left(Z_{t\wedge T_{D_r}}\right)\right|^{p/2}\right)_{t\geq 0}$ is a submartingale is then seen by combining this remark with:

Lemma 115 *If ϕ is subharmonic on D_r and continuous on $\bar{D}_r$, then $\left(\phi\left(Z_{t\wedge T_{D_r}}\right)\right)_{t\geq 0}$ is a submartingale.*

Proof. See Petersen chapter 6. ■

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