Harmonic measure and SLE

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1 Introduction

The motivation for this paper is twofold: to study multifractal spectrum of the harmonic measure and to better describe the geometry of Schramm's SLE curves. The main result of this paper is the Theorem 2.2 in which we rigorously compute the average spectrum of domains bounded by SLE curves. Several results can be easily derived from this theorem: dimension estimates of the boundary of SLE hulls, Hölder continuity of SLE Riemann maps, Hölder continuity of SLE trace, and more. We also would like to point out that SLE seems to be *the only model* where the spectrum (even average) of harmonic measure is non-trivial and known explicitly.

1.1 Integral means spectrum

There are several equivalent definitions of harmonic measure that are useful in different contexts. For a domain Ω with a regular boundary we define the harmonic measure with a pole at $z \in \Omega$ as the exit distribution of the standard Brownian motion started at z. Namely, $\omega_z(A) = \mathbb{P}(B^z_\tau \in A)$, where $\tau = \inf\{t : B^z_t \notin \Omega\}$ is the first time the standard two-dimensional Brownian motion started at z leaves Ω.

Alternatively for a simply connected planar domain the harmonic measure is the image of the normalized length on the unit circle under the Riemann mapping that sends the origin to z.

It is easy to see that harmonic measure depends on z in a smooth (actually harmonic) way, thus the geometric properties do not depend on the choice of the pole. So we fix the pole to be the origin or infinity and eliminate it from notation.

Over the last twenty years it became clear that many extremal problems in the geometric function theory are related to the geometrical properties of harmonic measure and the proper language for these problems is *the multifractal analysis*.

Multifractal analysis operates with different spectra of measures and relations between them. In this paper we study the harmonic measure on simply connected domains, so we give the rigorous definition for this case only.

Let $\Omega = \mathbb{C} \setminus K$ where K is a connected compact set and let ϕ be a Riemann mapping from the complement of the unit disc $\mathbb{D}_-$ onto Ω such that $\phi(\infty) = \infty$. *The integral means spectrum* of ϕ (or Ω) is defined as

$$\beta_\phi(t) = \beta_\Omega(t) = \limsup_{r \to 1+} \frac{\log \int |\phi(re^{i\theta})|^t d\theta}{-\log(r-1)}.$$

The universal integral means spectrum is defined as

$$B(t) = \sup \beta_\Omega(t),$$

where supremum is taken over all simply connected domains with compact boundary.

On the basis of work of Brennan, Carleson, Clunie, Jones, Makarov, Pommerenke and computer experiments for quadratic Julia sets Kraetzer [14] in 1996 formulated the following universal conjecture:

$$\begin{aligned} B(t) = & \quad t^2/4, & |t| < 2, \\ B(t) = & \quad |t| - 1, & |t| \geq 2. \end{aligned}$$

It is known that many other conjectures follow from Kraetzer conjecture. In particular, Brennan's conjecture [4] about integrability of $|\psi'|$ where ψ is a conformal map from a domain to the unit disc is equivalent to $B(-2) = 1$, while Carleson-Jones conjecture [5] about the decay rate of coefficient of a univalent function and the growth rate of the length of the Green's lines is equivalent to $B(1) = 1/4$.

There are many partial results in both directions: estimates of $B(t)$ from above and below (see surveys [3, 12]). Upper bounds are more difficult and they still not that far from the trivial bound $B(1) \leq 1/2$. Currently the best upper bound is $B(1) \leq 0.46$ [11]. Until recently lower bounds were also quite far from the conjectured value.

The main problem in finding lower bounds is that it is almost impossible to compute the spectrum explicitly for any non-trivial domain. The origin of difficulties is easy to see: only fractal domains have interesting spectrum, but for them the boundary behavior of $|\phi'(re^{i\theta})|^t$ depends on θ in a very non smooth way, making it hard to find the average growth rate.

We claim that in order to overcome these problems one should work with regular random fractals instead of deterministic ones. For random fractals it is natural to study *the average integral means spectrum* which is defined as

$$\bar{\beta}(t) = \limsup_{r \to 1} \frac{\log \int \mathbb{E}\left[|\phi'(re^{i\theta})|^t\right] d\theta}{-\log|r-1|},$$

The advantage of this approach it that for many random fractals the average boundary behavior of $|\phi'|$ is a very smooth function of θ. Therefore it is sufficient to study average behavior along any particular radius. Regular (random) fractals are invariant under some (random) transformation, making $\mathbb{E}|\phi'|^t$

a solution of a specific equation. Solving this equation one can find the average spectrum.

Note that $\bar{\beta}(t)$ and $\beta(t)$ do not necessarily coincide. It can even happen (and in this paper we consider exactly this case) that $\bar{\beta}(t)$ is not a spectrum of any particular domain. But Makarov's fractal approximation [17] implies that $\bar{\beta}(t) \leq B(t)$, so any function $\bar{\beta}$ gives a lower bound on B.

Another important notion is the *dimension* or *multifractal spectrum* of harmonic measure which can be non-rigorously defined as

$$f(\alpha) = dim\{z : \omega(B(z,r)) \approx r^{\alpha}\},$$

where $\omega(B(z,r))$ is the harmonic measure of the disc of radius r centred at z. There are several ways to make this definition rigorous, leading to slightly different notions of spectrum. But it is known [17] that the universal spectrum $F(\alpha) = \sup f(\alpha)$ is the same for all definitions of $f(\alpha)$.

There is no simple relation between the integral means spectrum and the dimension spectrum for a given domain. But for regular (in some sense) fractals they are related by a Legendre type transform. It is also known [17] that the universal spectra are related by a Legendre type transform:

$$\begin{aligned} F(\alpha) &= \inf_t (t + \alpha(B(t) + 1 - t)), \\ B(t) &= \sup_{\alpha > 0} \frac{F(\alpha) - t}{\alpha} + t - 1. \end{aligned}$$

1.2 Schramm-Loewner Evolution

It is a common belief that planar lattice models at criticality have a conformally invariant scaling limits as mesh of the lattice tends to zero. Schramm [23] introduced a one parametric family of random curves which are called SLE_κ (SLE stands for Stochastic Loewner Evolution or Schramm-Loewner Evolution) that are the only possible limits of cluster perimeters for critical lattice models. It turned out to be also a very useful tool in many related problems.

In this section we give the definition of SLE and the necessary background. The discussion of various versions of SLE and relations between them can be found in Lawler's book [15].

To define SLE we need a classical tool from complex analysis: the Loewner evolution. In general this is a method to describe by an ODE the evolution of the Riemann map from a growing (shrinking) domain to a uniformization domain. In this paper we use the radial Loewner evolution (where uniformization domain is the complement of the unit disc) and its modifications.

Definition 1. The radial Loewner evolution in the complement of the unit disc with driving function $\xi(t) : \mathbb{R}_+ \to \mathbb{T}$ is the solution of the following ODE

$$\partial_t g_t(z) = g_t(z) \frac{\xi(t) + g_t(z)}{\xi(t) - g_t(z)}, \qquad g_0(z) = z. \tag{1}$$

It is a classical fact [15] that for any driving function ξ g_t is a conformal map from $\Omega_t \to \mathbb{D}_-$ where $\mathbb{D}_-$ is the complement of the unit disc and $\Omega_t = \mathbb{D}_- \setminus K_t$ is the set of all points where solution of (1) exists up to the time t.

The *Schramm-Loewner Evolution* SLE_κ is defined as a Loewner evolution driven by the Brownian motion with speed $\sqrt{\kappa}$ on the unit circle, namely $\xi(t) = e^{i\sqrt{\kappa}B_t}$ where B_t is the standard Brownian motion and κ is a positive parameter. Since ξ is random, we obtain a family of random sets. The corresponding family of compacts K_t is also called SLE (or the *hull of SLE*).

A number of theorems was already established about SLE curves. Rohde and Schramm [21] proved that SLE is a.s. generated by a curve. Namely, almost surely there is a random curve γ (called *trace*) such that Ω_t is the unbounded component of $\mathbb{D}_- \setminus \gamma_t$, where $\gamma_t = \gamma([0,t])$. The trace is almost surely a simple curve when $\kappa \le 4$. In this case the hull K_t is the same as the curve γ_t. For $\kappa \ge 8$ the trace γ_t is a space-filling curve. In the same paper they also proved that almost surely the Minkowski (and hence the Hausdorff) dimension of the SLE_κ trace is no more than $1 + \kappa/8$ for $\kappa \le 8$. Beffara [2] proved that this estimate is sharp for $\kappa = 6$, later expanding the result to all $\kappa \le 8$. Lind [16] proved that the trace is Hölder continuous.

Another natural object is the *boundary* of SLE hull, namely the boundary of K_t. For $\kappa \le 4$ the boundary of SLE is the same as SLE trace (since the trace is a simple curve). For $\kappa > 4$ the boundary is the subset of the trace. Rohde and Schramm [21] proved that for $\kappa > 4$ the dimension of the boundary is no more than $1 + 2/\kappa$.

In [6, 7] physicist Duplantier using conformal field theory and quantum gravity methods predicted the average multifractal spectrum of SLE. He also conjectured that the boundary of SLE_κ for $\kappa > 4$ is in the same measure class as the trace of $SLE_{16/\kappa}$ (this is the so-called *duality* of SLE).

In this paper we rigorously compute the average integral means spectrum of SLE and show that it coincides with the Duplantier's prediction. This gives new proofs that dimension of the boundary is no more than $1 + 2/\kappa$ for $\kappa > 4$ and SLE maps are Hölder continuous, and provides more evidence which supports the duality conjecture.

Since $\bar{\beta}$ is defined in the terms of a Riemann mapping, it is more convenient to work with $f_t = g_t^{-1}$. From the equation (1) one can derive an equation on f_t. Unfortunately this equation involves f_t' as well as $\partial_t f_t$, so we have a PDE instead of ODE.

There is another approach which leads to a nice equation. Changing the direction of the flow defined by the equation (1) we get the equation for "inverse" function g_{-t}. For a given driving function ξ, maps g_t^{-1} and g_{-t} are different, but in the case of Brownian motion they have the same distribution.The precise meaning is given by the following lemma (which is an analog of the Lemma 3.1 from [21]):

Lemma 1.1. *Let g_t be a radial SLE, then for all $t \in \mathbb{R}$ the map $z \mapsto g_{-t}(z)$ has the same distribution as the map $z \mapsto \hat{f}_t(z)/\xi_t$, where $\hat{f}_t(z) = g_t^{-1}(z\xi_t)$.*

Proof. Fix $s \in \mathbb{R}$. Let $\hat{\xi}(t) = \xi(s+t)/\xi(s)$. Then $\hat{\xi}$ has the same distribution as ξ. Let

$$\hat{g}_t(z) = g_{s+t}(g_s^{-1}(z\xi(s)))/\xi(s).$$

It is easy to check that $\hat{g}_0(z) = z$ and

$$\hat{g}_{-s}(z) = g_0(g_s^{-1}(z\xi(s)))/\xi(s) = \hat{f}_s(z)/\xi(s).$$

Differentiating $\hat{g}_t(z)$ with respect to t we obtain

$$\partial_t \hat{g}_t(z) = \hat{g}_t(z)\frac{\hat{\xi}(t) + \hat{g}_t(z)}{\hat{\xi}(t) - \hat{g}_t(z)},$$

hence $\hat{g}_t$ has the same distribution as SLE. □

This lemma proves that the solution of the equation

$$\partial_t f_t(z) = f_t(z)\frac{f_t(z) + \xi(t)}{f_t(z) - \xi(t)}, \qquad f_0(z) = z, \tag{2}$$

where $\xi(t) = e^{i\sqrt{\kappa}B_t}$ has the same distribution as g_t^{-1}. Abusing notations we call it also SLE_κ.

One of the most important properties of SLE is Markov property, roughly speaking it means that the composition of two independent copies of SLE is an SLE. The rigorous formulation is given by the following lemma.

Lemma 1.2. *Let $f_\tau^{(1)}$ be an SLE_κ driven by $\xi^{(1)}(\tau)$, $0 < \tau < t$ and $f_\tau^{(2)}$ be an SLE_κ driven by $\xi^{(2)}(\tau)$, $0 < \tau < s$, where $\xi^{(1)}$ and $\xi^{(2)}$ are two independent Brownian motions on the circle. Then $f_{s+t}(z) = f_t^{(2)}(f_t(z)/\xi^{(1)}(t))\xi^{(1)}(t)$ is SLE_κ at time $t+s$.*

Proof. This composition is the solution of Loewner Evolution driven by $\xi(\tau)$, where

$$\xi(\tau) = \begin{cases} \xi^{(1)}(\tau), & 0 < \tau \le t, \\ \xi^{(2)}(\tau - t)\xi^{(1)}(t), & t < \tau \le t+s. \end{cases}$$

It is easy to see that $\xi(\tau)$ is also a Brownian motion on the circle with the same speed $\sqrt{\kappa}$, hence f_{t+s} is also SLE_κ. □

We will need yet another modification of SLE which is in fact a manifestation of stationarity of radial SLE.

Definition 2. Let $\xi(t) = \exp(i\sqrt{\kappa}B_t)$ be a two-sided Brownian motion on the unit circle. The whole plane SLE_κ is the family of conformal maps g_t satisfying

$$\partial_t g_t(z) = g_t(z)\frac{\xi(t) + g_t(z)}{\xi(t) - g_t(z)},$$

with initial condition

$$\lim_{t\to-\infty} e^t g_t(z) = z, \qquad z \in \mathbb{C} \setminus \{0\}.$$

The whole-plane SLE satisfies the same differential equation as the radial SLE, the difference is in the initial conditions. One can think about the whole-plane SLE as about the radial SLE started at $t = -\infty$. And this is the way to construct the whole-plane SLE and prove the existence. Proposition 4.21 in [15] proves that the whole-plane Loewner Evolution g_t with the driving function $\xi(t)$ is the limit as $s \to -\infty$ of the following maps: $g_t^{(s)}(z) = e^{-t}z$ if $t \le s$, $g_t^{(s)}(z)$ is the solution to (1) with initial condition $g_s^{(s)}(x) = e^{-s}z$. The same is also true for inverse maps.

We use this argument to prove that there is a limit of $e^{-t}f_t$ as $t \to \infty$.

Lemma 1.3. *Let f_t be a radial SLE_κ then there is a limit in the sense of distribution of $e^{-t}f_t(z)$ as $t \to \infty$.*

Proof. The function $e^{-t}f_t$ is exactly the function which is used to define the whole-plane SLE. Multiplication by the exponent corresponds to the shift in time in the driving function. The function $e^{-t}f_t(z)$ has the same distribution as the inverse of $g_0^{(-t)}(z)$, hence it converges to F_0, where $F_\tau = g_\tau^{-1}$ and g_τ is a whole-plane SLE. □

1.3 Results, conjectures, and organization of the paper

It is easy to see that the geometry near "the tip" of SLE (the point of growth) is different from the geometry near "generic" points. This means that for some problems it is more convenient to work with the so-called *bulk* of SLE i.e. the part of the SLE hull which is away from the tip. In the following theorem we compute the average spectrum of SLE hull and SLE bulk.

Theorem 2.3 *The average integral means spectrum $\bar\beta(t)$ of SLE is equal to*

$$\begin{aligned} -t+\kappa\frac{4+\kappa-\sqrt{(4+\kappa)^2-8t\kappa}}{4\kappa} \qquad & t\le -1-\frac{3\kappa}{8}, \\ -t+\frac{(4+\kappa)(4+\kappa-\sqrt{(4+\kappa)^2-8t\kappa}\ )}{4\kappa} \qquad & -1-\frac{3\kappa}{8}\le t\le\frac{3(4+\kappa)^2}{32\kappa}, \\ t-\frac{(4+\kappa)^2}{16\kappa} \qquad & t\ge\frac{3(4+\kappa)^2}{32\kappa}. \end{aligned}$$

The average integral means spectrum $\bar\beta(t)$ of the bulk of SLE is equal to

$$\begin{aligned} 5-t+\frac{(4+\kappa)(4+\kappa-\sqrt{(4+\kappa)^2-8t\kappa}\ )}{4\kappa} \qquad & t\le\frac{3(4+\kappa)^2}{32\kappa}, \\ t-\frac{(4+\kappa)^2}{16\kappa}, \qquad & t\ge\frac{3(4+\kappa)^2}{32\kappa}. \end{aligned}$$

Remark 1. The local structure of the SLE bulk is the same for all versions of SLE which means that they all have the same average spectrum.

Remark 2. To prove this theorem we show that

$$\mathbb{E}|f'(re^{i\theta})|^t \asymp (r-1)^\beta((r-1)^2+\theta^2)^\gamma,$$

where β and γ are given by (12) and (11). We would like to point out that β and γ are local exponents so they are the same for different versions of SLE.

There are several corollaries that one can easily derive from Theorem 2.3

Corollary 1.4. *SLE map f is Hölder continuous with any exponent less than*

$$\alpha_\kappa = 1 - \frac{1}{\mu} - \sqrt{\frac{1}{\mu^2}+\frac{2}{\mu}},$$

where $\mu = (4+\kappa)^2/4\kappa$.

Corollary 1.5. *The Hausdorff dimension of the boundary of the SLE hull for $\kappa \geq 4$ is at most $1+2/\kappa$.*

Corollary 1.6. *SLE trace with natural parametrization is Hölder continuous.*

The first two results a conjectured to be sharp. They both have been previously published in [13] and [21] correspondingly. Both results can be easily derived from the properties of the spectrum (see [17]) and Theorem 2.3.

The third corollary first appeared in a paper by Lind [16] where she uses derivatives estimates by Rohde and Schramm. One can use Theorem 2.3 to prove this result.

The Theorem 2.3 gives the average spectrum of SLE. The question about spectra of individual realizations of SLE remains open. We believe that with probability one they all have the same spectrum $\beta(t)$ which we call the a.s. spectrum.

It is immediate that the tangent line at $t = 3(4+\kappa)^2/32\kappa$ intersects y-axis at $-(4+\kappa)^2/16\kappa < -1$. This contradicts Makarov's characterization of possible spectra [17] which means that $\bar{\beta}$ can not be a spectrum of any given domain. In particular $\bar{\beta}$ is not the a.s. spectrum of SLE. On the other hand it suggests that the following conjecture is true.

Conjecture 1. *Let t_{min} and t_{max} be the two points such that the tangent to $\bar{\beta}(t)$ intersects the y-axis at -1. The almost sure value of the spectrum is equal to $\bar{\beta}(t)$ for $t_{min} \leq t \leq t_{max}$ and continues as the tangents for $t < t_{min}$ and $t > t_{max}$. Explicit formulas for t_{min}, t_{max}, and tangent lines are given in* (4) *and* (5)*. See Figure 1 for plots of β and $\bar{\beta}$.*

The rest of the paper is organized in the following way. In the first part of the Section 2 we discuss Duplantier's prediction and the Conjecture 1. In the second part we compute the moments of $|f'|$ and prove Theorem 2.3. In the Section 3 me make some remarks about possible generalizations of SLE. In the last Section 4 we explain a possible approach to the Conjecture 1.

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2 Integral means spectrum of SLE

2.1 Duplantier's prediction for the spectrum of the bulk

In 2000 physicist Duplantier predicted [6, 7] by the means of quantum gravity that the Hausdorff dimension spectrum of the bulk of SLE is

$$f(\alpha) = \alpha - \frac{(25-c)(\alpha-1)^2}{12(2\alpha-1)},$$

where c is the central charge which is related to κ by

$$c = \frac{(6-\kappa)(6-16/\kappa)}{4}.$$

The negative values of f do not have a simple geometric interpretation, they correspond to negative dimensions (see papers by Mandelbrot [18, 19]) which appear only in the random setting. They correspond to the events that have zero probability in the limit, but appear on finite scales as exceptional events. There is another interpretation in terms of β beta spectrum which we explain below.

Since negative values of f correspond to zero robability events, it makes sense to introduce the positive part of the spectrum: $f^+ = \max\{f, 0\}$. We believe that f^+ is the almost sure value of the dimension spectrum. This is the dimension spectrum counterpart of Conjecture 1. The function f^+ is equal to f for $\alpha \in [\alpha_{min}, \alpha_{max}]$, where

$$\begin{aligned}
\alpha_{min} &= \frac{16+4\kappa+\kappa^2-2\sqrt{2}\sqrt{16\kappa+10\kappa^2+\kappa^3}}{(4-\kappa)^2}, \quad && \kappa \neq 4, \\
\alpha_{min} &= \frac{16+4\kappa+\kappa^2+2\sqrt{2}\sqrt{16\kappa+10\kappa^2+\kappa^3}}{(4-\kappa)^2}, \quad && \kappa \neq 4, \\
\alpha_{min} &= \frac{2}{3}, && \kappa = 4, \\
\alpha_{max} &= \infty, && \kappa = 4.
\end{aligned}$$

It is known (see [17]) that for regular fractals $\beta(t)$ spectrum is related to $f(\alpha)$ spectrum by the Legendre transform. We believe those relations to hold for SLE as well:

$$\begin{aligned}
\beta(t) - t + 1 &= \sup_{\alpha>0} (f(\alpha) - t)/\alpha, \\
f(\alpha) &= \inf_t (t + \alpha(\beta(t) - t + 1)).
\end{aligned}$$

The Legendre transform of f^+ is supposed to be equal to the almost sure value of the integral means spectrum $\beta(t)$, while the Legendre transform of f is believed to be equal to the average integral means spectrum $\bar{\beta}(t)$.

The Legendre transform of f^+ has two phase transitions: one for negative t and one for positive. The Legendre transform of f^+ is equal to

$$\begin{aligned}
\beta(t) &= t\left(1-\frac{1}{\alpha_{min}}\right)-1, && t \le t_{min},\\
\beta(t) &= -t+\frac{(4+\kappa)\left(4+\kappa-\sqrt{(4+\kappa)^2-8t\kappa}\right)}{4\kappa}, && t_{min}<t<t_{max},\\
\beta(t) &= t\left(1-\frac{1}{\alpha_{max}}\right)-1, && t \ge t_{max},
\end{aligned} \tag{3}$$

where

$$\begin{aligned}
t_{min} &= -f'(\alpha_{min})\alpha_{min}, && \kappa > 0,\\
t_{max} &= -f'(\alpha_{max})\alpha_{max}, && \kappa \neq 4,\\
t_{max} &= 3/2, && \kappa = 4.
\end{aligned}$$

We can also express t_{min} and t_{max} in terms of $\mu = 4/\kappa+2+\kappa/4 = (4+\kappa)^2/4\kappa$:

$$\begin{aligned}
t_{min} &= \frac{-1-2\mu-(1+\mu)\sqrt{1+2\mu}}{\mu},\\
t_{max} &= \frac{-1-2\mu+(1+\mu)\sqrt{1+2\mu}}{\mu}.
\end{aligned} \tag{4}$$

And the linear functions in (3) can be written as

$$\begin{aligned}
&t\left(\frac{1}{\sqrt{1-2t_{min}/\mu}}-1\right)-1\\
&t\left(\frac{1}{\sqrt{1-2t_{max}/\mu}}-1\right)-1.
\end{aligned} \tag{5}$$

For convenience we introduce

$$\tilde{\beta}(t) = -t+\frac{(4+\kappa)\left(4+\kappa-\sqrt{(4+\kappa)^2-8t\kappa}\right)}{4\kappa},$$

which is the analytic part of the spectrum and defined for all $t < (4+\kappa)^2/8\kappa$. This function is the analytic part of the Legendre transform of f. The critical points t_{max} and t_{min} are the points where the tangent line to the graph of $\bar{\beta}(t)$ intersects the y-axis at -1. The Legendre transform of f^+ is equal to $\tilde{\beta}(t)$ between these two critical points and than continues as a linear function.

Note that Makarov's theorem [17] states that all possible integral means spectra satisfy the following conditions: they are non-negative convex functions bounded by the universal spectrum such that tangent line at any point intersects y-axis between 0 and -1. So there is another way to describe the Legendre transform of f^+: it coincides with $\tilde{\beta}$ as long as this does not contradict Makarov's criteria and than continues in the only possible way.

Figure 1: Plots of β and $\bar{\beta}$ spectra.

If we do not cut off the negative part of f, then the picture is a bit different. There is no phase transition for negative t. For positive t phase transition occurs later, and it happens because the derivative of $f(\alpha)$ is bounded at infinity. For large α

$$f(\alpha) = \alpha \left(1 - \frac{(4+\kappa)^2}{16\kappa}\right) + \frac{3(4+\kappa)^2}{32\kappa} + O\left(\frac{1}{\alpha}\right),$$

hence

$$\begin{aligned} \bar{\beta}(t) &= -t + \frac{(4+\kappa)\left(4+\kappa-\sqrt{(4+\kappa)^2 - 8t\kappa}\right)}{4\kappa}, & t \le \frac{3(4+\kappa)^2}{32\kappa}, \\ \bar{\beta}(t) &= 1 - \frac{(4+\kappa)^2}{16\kappa} + t - 1 = t - \frac{(4+\kappa)^2}{16\kappa}, & t > \frac{3(4+\kappa)^2}{32\kappa}. \end{aligned}$$

The explanation of this phase transition is rather simple. It is obvious that $\bar{\beta}(t)$ is a convex function, and it follows from Makarov's fractal approximation that the average spectrum is bounded by the universal spectrum. It is known that for the large values of $|t|$ the universal spectrum is equal to $|t|-1$. Altogether it implies that $|\bar{\beta}'(t)| \le 1$ and if it is equal to 1 at some point then $\bar{\beta}$ should be linear after this point. And $\bar{\beta}' = 1$ exactly at $t = 3(4+\kappa)^2/32\kappa$.

The left part of the Figure 1 shows plots of β (thick line) and $\bar{\beta}$ for $\kappa = 0.2$. On the right part one can see the plot of the $\tilde{\beta}$ (thick) and three tangent lines, two of them are crossing the y-axis at -1 and define β after phase transition, the third one has a slope 1 and defines $\bar{\beta}$ after the phase transition.

2.2 Rigorous computation of the spectrum

In this section we compute the average integral means spectrum of SLE (and its bulk) and show that it coincides with the Legendre transform of the dimension spectrum predicted by Duplantier.

The average integral means spectrum is the growth rate of $\tilde{F}(z,\tau) = \mathbb{E}\left[|f'_\tau(z)|^t\right]$, where f_τ is a radial SLE_κ. Actually, this function depends also on t and κ, but they are fixed throughout the proof and we will not mention this dependence to simplify the notation.

Lemma 2.1. *The function $\tilde{F}(z,\tau)$ is a solution of*

$$\begin{aligned} t\frac{r^4+4r^2(1-r\cos\theta)-1}{(r^2-2r\cos\theta+1)^2}\tilde{F}+\frac{r(r^2-1)}{r^2-2r\cos\theta+1}\tilde{F}_r- \\ \frac{2r\sin\theta}{r^2-2r\cos\theta+1}\tilde{F}_\theta+\frac{\kappa}{2}\tilde{F}_{\theta,\theta}-\tilde{F}_\tau=0. \end{aligned} \tag{6}$$

Proof. The idea of the proof is to construct a martingale $\mathcal{M}_s$ (w.r.t filtration defining SLE) which involves $\tilde{F}$. The ds term in its Itô derivative should vanish. This will give us a partial differential equation on $\tilde{F}$. We set

$$\mathcal{M}_s=\mathbb{E}\left[|f'_\tau(z)|^t \mid \mathcal{F}_s\right].$$

By the Lemma 1.2

$$\begin{aligned} \mathbb{E}\left[|f'_\tau(z)|^t \mid \mathcal{F}_s\right] &= \mathbb{E}\left[|f'_s(z)|^t|f'_{\tau-s}(f_s(z)/\xi_s)|^t \mid \mathcal{F}_s\right] \\ &= |f'_s(z)|^t\tilde{F}(z_s,\tau-s), \end{aligned}$$

where $z_s=f_s(z)/\xi_s$.

We will need derivatives of z_s and $|f'_s|^t$

$$\begin{aligned} \partial_t\log|f'_s(z)| &= \operatorname{Re}\frac{\partial_z f_s\frac{f_s+\xi_s}{f_s-\xi_s}}{f'_s}=\operatorname{Re}\left[\frac{f_s+\xi_s}{f_s-\xi_s}-\frac{2\xi_s f_s}{(f_s-\xi_s)^2}\right] \\ &= \operatorname{Re}\frac{z_s^2-1-2z_s}{(z_s-1)^2}=\frac{r^4+4r^2(1-r\cos\theta)-1}{(r^2-2r\cos\theta+1)^2}, \end{aligned}$$

where $z_s=r\exp(i\theta)$. Next we have to find the derivative of $z_s=re^{i\theta}$

$$d\log z_s=d\log r+id\theta=d\log f_s-i\sqrt{\kappa}dB_s,$$

where

$$d\log f_s=\frac{df_s}{f_s}=\frac{z_s+1}{z_s-1}ds.$$

Writing everything in terms of r and θ we get

$$\begin{aligned} d\log r+id\theta=\frac{z_s+1}{z_s-1}ds-i\sqrt{\kappa}dB_s= \\ \frac{r^2-1}{r^2-2r\cos\theta+1}ds+i\left(-\frac{2r\sin\theta}{r^2-2r\cos\theta+1}ds-\sqrt{\kappa}dB_s\right). \end{aligned}$$

Summing it all up we obtain

$$\partial_t\log|f'_s(z)|=\frac{r^4+4r^2(1-r\cos\theta)-1}{(r^2-2r\cos\theta+1)^2}, \tag{7}$$

$$d\theta=-\frac{2r\sin\theta}{r^2-2r\cos\theta+1}ds-\sqrt{\kappa}dB_s, \tag{8}$$

$$dr=rd\log r=\frac{r(r^2-1)}{r^2-2r\cos\theta+1}ds. \tag{9}$$

Let us write $F(z,\tau)$ as $F(r,\theta,\tau)$. The ds term in Itô derivative of $\mathcal{M}$ is equal to

$$|f_s'(z)|^t \left(t\frac{r^4+4r^2(1-r\cos\theta)-1}{(r^2-2r\cos\theta+1)^2}\tilde{F} + \frac{r(r^2-1)}{r^2-2r\cos\theta+1}\tilde{F}_r - \frac{2r\sin\theta}{r^2-2r\cos\theta+1}\tilde{F}_\theta + \frac{\kappa}{2}\tilde{F}_{\theta,\theta} - \tilde{F}_\tau \right).$$

This derivative should be 0 and, since f_s is a univalent function and its derivative never vanishes, $\tilde{F}$ is a solution of (6). □

By the Lemma 1.3 there is a limit of $e^{-\tau} f_\tau$ as $\tau \to \infty$. Hence we can introduce

$$F(z) = \lim_{\tau\to\infty} e^{-\tau t} F(z,\tau).$$

Passing to the limit in (6) we can see that $F(z)$ is a solution of

$$t\left(\frac{r^4+4r^2(1-r\cos\theta)-1}{(r^2-2r\cos\theta+1)^2} - 1\right)F + \frac{r(r^2-1)}{r^2-2r\cos\theta+1}F_r - \frac{2r\sin\theta}{r^2-2r\cos\theta+1}F_\theta + \frac{\kappa}{2}F_{\theta,\theta} = 0. \tag{10}$$

Notation 1. We define two constants β and γ:

$$\gamma = \gamma(t,\kappa) = \frac{4+\kappa-\sqrt{(4+\kappa)^2-8t\kappa}}{2\kappa}, \tag{11}$$

$$\beta = \beta(t,\kappa) = t - \frac{(4+\kappa)\gamma}{2}. \tag{12}$$

It is easy to see that the second constant β is equal to $-\tilde{\beta}$.

Let us explain where these constants come from. Roughly speaking spectrum $\beta(t)$ is the growth rate of F as $r \to 1$. F is a solution of the parabolic equation (10) that has a singularities at $|z| = 1$ and $z = 1$. Let us assume that F has a power series expansion near 1. Then we can write power series expansion of coefficients of (10) and assuming that the leading term is $(r-1)^\beta((r-1)^2+\theta^2)^\gamma$ we get an equation on β and γ. Constants γ and β are solution of these equation. Now let us explain why it makes sense to consider this expansion. Instead of radial/whole-plane SLE we can write the same martingale for the chordal SLE. In this case (if we forget dependence on τ) the equation will be

$$2t\frac{x^2-y^2}{(x^2+y^2)^2}F - \frac{2x}{x^2+y^2}F_x + \frac{2y}{x^2+y^2}F_y + \frac{\kappa}{2}F_{xx} = 0. \tag{13}$$

This equation is "tangent" to (10) at $r = 1$ and $\theta = 0$.

This equation has a solution of the form $y^\beta(x^2+y^2)^\gamma$, where β and γ as above. Actually this is the way how we found this exponents. This approach seems to be easier, but there are two major problems. First it is not easy to argue that we can neglect the derivative with respect to τ. Another problem is that $y^\beta(x^2+y^2)^\gamma$ can not be equal to F since it blows up at infinity and we have to show that the local behavior does not depend on the boundary conditions at infinity.

Similar equation appeared in [21] and when this work was finished we learned from I. Gruzberg that equation (13) appeared several years ago in the paper by Hastings [10].

Theorem 2.2. *Let*

$$t \le \frac{3(4+\kappa)^2}{32\kappa}.$$

Then we have

$$\mathbb{E}\left[\int_{|z|=r} |F_0'(re^{i\theta})|^t d\theta\right] \asymp \left(\frac{1}{r-1}\right)^{\bar\beta(t)},$$

where the expectation is taken for a;; whole-plane SLE maps $F_0 = \lim e^{-\tau} f_\tau$ *and* $\bar\beta(t)$ *is equal to*

$$\begin{array}{rl} -\beta(t,\kappa), & t > -1 - \dfrac{3\kappa}{8}, \\ -\beta(t,\kappa) - 2\gamma(t,\kappa) - 1, & t \le -1 - \dfrac{3\kappa}{8}. \end{array} \tag{14}$$

Proof. Suppose that we can find positive functions bounded away from the unit circle ϕ_+ and ϕ_- such that $\Lambda\phi_- < 0$ and $\Lambda\phi_+ > 0$ then by the maximum principle any positive solution of (10) is between $c_+\phi_+$ and $c_-\phi_-$, where c_+ and c_- are positive constants.

In the Lemma 2.5 we will construct such functions ϕ_- and ϕ_+. They are of the form

$$\phi_\pm = (r-1)^\beta (r^2 - 2r\cos\theta + 1)^\gamma (-\log(r-1))^{\mp 1} g(r^2 - 2r\cos\theta + 1),$$

where $g > 0$ for $r = 1$. Both functions have the same polynomial growth rate as $r \to 1$, thus F has also the same growth rate. By Tonelli theorem

$$\mathbb{E}\left[\int |f_\tau'|^t\right] = \int \mathbb{E}\left[|f_\tau'(r,\theta)|^t\right] d\theta \approx \int (r-1)^\beta (r^2 - 2r\cos\theta + 1)^\gamma d\theta,$$

where $\approx$ means that functions have the same polynomial growth rate. For $\gamma > -1/2$ the weight $(r^2 - 2r\cos\theta + 1)^\gamma$ is integrable up to the boundary and we immediately get

$$\mathbb{E}\left[\int_{|z|=r} |f_\tau'|^t\right] \approx \left(\frac{1}{r-1}\right)^{-\beta}.$$

For $\gamma \le -1/2$ the situation is a bit different. In this case the integral of the weight blows up as $(r-1)^{2\gamma+1}$. Which gives us $\mathbb{E}\left[\int |f_\tau'|^t d\theta\right] \approx (r-1)^{\beta+2\gamma+1}$. It is easy to check that $\gamma \le -1/2$ if and only if $t \le -1 - 3\kappa/8$. □

Remark 3. The growth rate of $\mathbb{E}\left[\int |f'|^t\right]$ is similar to $\bar\beta(t)$ predicted by Duplantier. The phase transition at $t = -1 - 3\kappa/8$ is due to the exceptional behavior of SLE at the tip. If we integrate over values of θ bounded away from 0 then the weight $|z-1|^{2\gamma}$ does not blow up and we have no phase transition at $t = -1 - 3\kappa/8$ any more. This gives us the spectrum of the bulk of SLE.

We can also state this theorem in terms of average integral means spectrum defined in the introduction. This theorem proves that Duplantier's prediction for $\bar\beta(t)$ is correct.

Theorem 2.3. *The average integral means spectrum $\bar\beta(t)$ of SLE is equal to*

$$\begin{aligned} -t+\kappa\frac{4+\kappa-\sqrt{(4+\kappa)^2-8t\kappa}}{4\kappa} \qquad & t\le -1-\frac{3\kappa}{8},\\ -t+\frac{(4+\kappa)(4+\kappa-\sqrt{(4+\kappa)^2-8t\kappa}\,)}{4\kappa} \qquad & -1-\frac{3\kappa}{8}\le t\le \frac{3(4+\kappa)^2}{32\kappa},\\ t-\frac{(4+\kappa)^2}{16\kappa} \qquad & t\ge \frac{3(4+\kappa)^2}{32\kappa}. \end{aligned}$$

The average integral means spectrum $\bar\beta(t)$ of the bulk of SLE is equal to

$$\begin{aligned} -t+\frac{(4+\kappa)(4+\kappa-\sqrt{(4+\kappa)^2-8t\kappa}\,)}{4\kappa} \qquad & t\le \frac{3(4+\kappa)^2}{32\kappa},\\ t-\frac{(4+\kappa)^2}{16\kappa}, \qquad & t\ge \frac{3(4+\kappa)^2}{32\kappa}. \end{aligned}$$

Proof. The Theorem 2.2 gives us the value of $\bar\beta(t)$ for $t \le 3(4+\kappa)^2/32\kappa$. Direct computations show that derivative of $-\beta(t,\kappa)$ at $t = 3(4+\kappa)^2/32\kappa$ is equal to one. As we mentioned before, the $\bar\beta$ spectrum is a convex function bounded by the universal spectrum, and the universal spectrum is equal to $|t|-1$ for the large values of $|t|$. This means that if $\bar\beta' = 1$ at some point then it should continue as a linear function with slope one. Hence $\bar\beta$ should continue as $t-(4+\kappa)^2/16\kappa$ for $t > 3(4+\kappa)^2/32\kappa$. Plugging in the values of β and γ we finish the proof of the theorem. □

To complete the proof of the Theorem 2.2 we have to construct functions ϕ_- and ϕ_+. We do it in three steps, first we write the restriction of the equation (10) to the unit circle, then we find a positive solution g of the resulting equation. Finally we construct ϕ_- and ϕ_+ out of g.

We look for a solution in the following form:

$$f(r,\theta) = (r-1)^\beta (r^2-2r\cos\theta+1)^\gamma g(r^2-2r\cos\theta+1).$$

Plugging f into (10), factoring $(r-1)^\beta(r^2-2r\cos\theta+1)^\gamma$ out, and taking $r=1$ we obtain a differential equation on $g(2-2\cos\theta)$

$$\begin{aligned} (-2t+4\beta-2\gamma-2\gamma\kappa+\gamma^2\kappa+(4t-4\beta+2\kappa\gamma)\cos\theta\\ -(2t+\gamma(\gamma\kappa-2)\cos(2\theta))g(2-2\cos\theta)\\ +(2-2\cos\theta)(-2-\kappa+2\gamma\kappa+2\kappa\cos\theta-(\kappa+2\gamma\kappa-2)\cos(2\theta))\\ \times g'(2-2\cos\theta)+2\kappa(2-2\cos\theta)(\sin\theta)^2 g''(2-2\cos\theta)=0. \end{aligned} \tag{15}$$

Lemma 2.4. *The equation (15) has a smooth (with possible exception at $\theta = 0$) positive bounded solution on the circle if and only if*

$$t \leq \frac{3(4+\kappa)^2}{32\kappa}. \tag{16}$$

Proof. Changing the variable to $x = 2 - 2\cos\theta$ we rewrite (15) as a hypergeometric equation

$$\begin{aligned} \gamma(2+\kappa)g(x) + (8 - 2x + \kappa(x-2) + 2\gamma\kappa(x-4))g'(x) + \\ \kappa(x-4)xg''(x) = 0, \end{aligned} \tag{17}$$

which has two independent solutions

$$g_1(x) = {}_2F_1(a, b, \frac{1}{2} + a + b, \frac{x}{4})$$

and

$$g_2(x) = x^{1/2-a-b} {}_2F_1(\frac{1}{2} - a, \frac{1}{2} - b, \frac{3}{2} - a - b, \frac{x}{4}),$$

where

$$a = \gamma - \frac{1}{\kappa} - \frac{\sqrt{1-2t\kappa}}{\kappa},$$
$$b = \gamma - \frac{1}{\kappa} + \frac{\sqrt{1-2t\kappa}}{\kappa}.$$

Function $g(2 - 2\cos\theta)$ should have a second derivative everywhere on the unit circle except at the point $\theta = 0$. This means that $g(x)$ should have expansion $c + O(4-x)$ at the endpoint 4.

Any solution of (15) is a linear combination of g_1 and g_2: $g = c_1 g_1 + c_2 g_2$. We want to find coefficients c_1 and c_2 such that this sum is bounded and has a correct expansion at $x = 4$.

Expansions of g_1 and g_2 at 4 are

$$\begin{aligned} g_1(x) &= \frac{\sqrt{\pi}\Gamma(1/2+a+b)}{\Gamma(1/2+a)\Gamma(1/2+b)} - \\ &\quad \frac{\sqrt{\pi}\Gamma(1/2+a+b)}{\Gamma(a)\Gamma(b)}\sqrt{4-x} + O(4-x), \end{aligned}$$

and

$$\begin{aligned} g_2(x) &= \frac{2^{1-2a-2b}\sqrt{\pi}\Gamma(3/2-a-b)}{\Gamma(1-a)\Gamma(1-b)} - \\ &\quad \frac{2^{1-2a-2b}\sqrt{\pi}\Gamma(3/2-a-b)}{\Gamma(1/2-a)\Gamma(1/2-b)}\sqrt{4-x} + O(4-x). \end{aligned}$$

If $c_2 \neq 0$ then $1/2 - a - b$ should be nonnegative, otherwise g is not bounded at 0. Note that

$$\frac{1}{2} - a - b = \frac{4 + \kappa - 4\gamma\kappa}{2\kappa}$$

which is nonnegative if and only if

$$t \leq \frac{3(4+\kappa)^2}{32\kappa}$$

which is exactly the restriction from the statement of the lemma. If $t > 3(4+\kappa)^2/32\kappa$ then $c_2 = 0$. In this case g has a correct expansion at 4 if and only if $\Gamma(a) = 0$ or $\Gamma(b) = 0$, but $1 - 2t\kappa < 0$ so both a and b are not real number and gamma function has only real roots.

We can introduce

$$C = \frac{\Gamma(1/2+a+b)\Gamma(1/2-a)\Gamma(1/2-b)}{2^{1-2a-2b}\Gamma(a)\Gamma(b)\Gamma(3/2-a-b)},$$

and

$$g_3(x) = g_1(x) - Cg_2(x).$$

By construction $g_3(x) = \text{const} + O(4-x)$ near 4. Finally we have to prove that g_3 is a positive function. Note that in (17) g and g'' have coefficients of different signs. Obviously, $g_3(0) = 1$. Suppose that g_3 has a local minimum inside the interval $(0,4)$, then $g_3' = 0$ and $g_3'' \geq 0$ at this point, hence g_3 is also positive. Thus it is sufficient to check that $g_3(4) > 0$. The value of $g_3(4)$ is easy to evaluate

$$\begin{aligned} g_3(4) &= \sqrt{\pi}\Gamma(1/2+a+b) \\ &\quad \times \left(\frac{1}{\Gamma(1/2+a)\Gamma(1/2+b)} - \frac{\Gamma(1/2-a)\Gamma(1/2-b)}{\Gamma(a)\Gamma(b)\Gamma(1-a)\Gamma(1-b)} \right) \\ &= \frac{\sqrt{\pi}\Gamma(1/2+a+b)\cos(\pi(a+b))}{\Gamma(1/2+a)\Gamma(1/2+b)\cos(\pi a)\cos(\pi b)} \\ &= \pi^{-3/2}\Gamma(1/2+a+b)\cos(\pi(a+b))\Gamma(1/2-a)\Gamma(1/2-b). \end{aligned}$$

By (16), $a+b < 1/2$, hence $\Gamma(1/2+a+b)\cos(\pi(a+b)) > 0$. Finally we have to show that $\Gamma(1/2-a)\Gamma(1/2-b) > 0$. We consider two different cases: when $t \leq 1/2\kappa$ and $t > 1/2\kappa$. In the second case a and b are conjugated and $\Gamma(1/2-a)\Gamma(1/2-b) = |\Gamma(1/2-a)|^2 > 0$. In the first case we will prove that $1/2 - a > 0$ and $1/2 - b > 0$. It is easy to see that $1/2 - b < 1/2 - a$, hence it is sufficient to prove that $1/2 - b > 0$. Recall that

$$\frac{1}{2} - b = \frac{1}{2} - \gamma + \frac{1}{\kappa} - \frac{\sqrt{1-2t\kappa}}{\kappa},$$

hence

$$\partial_t(1/2-b) = \frac{1}{\sqrt{1-2\kappa t}} - \frac{2}{\sqrt{(4+\kappa)^2 - 8t\kappa}} > 0.$$

This means that $1/2 - b$ has a minimum when $t = 0$, this minimum is

$$\frac{1}{2} - b(0) = \frac{1}{2} - \gamma(0) = \frac{1}{2} > 0.$$

This proves that $g_3(x) > 0$ on $[0,4]$. □

Lemma 2.5. *Let g be a positive bounded solution of (15) and*

$$\begin{aligned}F =& f(r,\theta)(-\log(r-1))^{\delta} \\ =& (r-1)^{\beta}(r^2-2r\cos\theta+1)^{\gamma} g(r^2-2r\cos\theta+1)(-\log(r-1))^{\delta}.\end{aligned}$$

Then

$$\begin{aligned}\Lambda F > 0, \quad \delta < 0, \\ \Lambda F < 0, \quad \delta > 0,\end{aligned}$$

for r sufficiently close to 1.

Proof. Applying Λ we find

$$\Lambda F = (-\log(r-1))^{\delta}\left(\Lambda f - f\frac{r(r+1)\delta}{(r^2-2r\cos\theta+1)(-\log(r-1))}\right).$$

By Lemma 2.4 $\Lambda f = (r-1)^{\beta}(r^2-2r\cos\theta+1)^{\gamma} O(r-1)$, hence

$$\begin{aligned}\Lambda F =& (-\log(r-1))^{\delta}(r-1)^{\beta}(r^2-2r\cos\theta+1)^{\gamma} \\ & \times\left(O(r-1) - \frac{r(r+1)\delta(g(2-2\cos\theta)+O(r-1))}{w(-\log(r-1))}\right).\end{aligned}$$

The sign of the main term is opposite to the sign of δ. This proves the claim. □

Remark 4. Note that we proved a stronger result than announced in Theorem 2.2: $\mathbb{E}\int |F'|^t$ has growth rate $(r-1)^{\beta}$ up to a factor $\log^{\delta}(r-1)$ for *arbitrary small* $|\delta|$.

3 Loewner Evolution driven by other processes

It is known that Loewner Evolution can be defined for a very large class of driving functions. In particular, they do not have to be continuous. In [3] we proposed to study Lévy-Loewner Evolution (LLE), which is the Loewner Evolution driven by a Lévy process (i.e. process with independent stationary increments). This defines a very rich class of random fractals. It seems that it is still possible to find the spectrum of harmonic measure for this class explicitly.

In the fundamental Lemma 2.1 we only use the fact that the Brownian motion is a Lévy process. So the same argument can be applied for LLE. As the result we get that $F = \mathbb{E}\left[|e^{-\tau} f'_{\tau}(z)|^t\right]$ is the solution of

$$\begin{aligned}t\left(\frac{r^4+4r^2(1-r\cos\theta)-1}{(r^2-2r\cos\theta+1)^2}-1\right)F + \frac{r(r^2-1)}{r^2-2r\cos\theta+1}F_r \\ -\frac{2r\sin\theta}{r^2-2r\cos\theta+1}F_{\theta} + \Lambda F = 0.\end{aligned}$$

where Λ is the generator of the driving Lévy process. Thus again finding spectrum boils down to the analysis of a parabolic type integro-differential equation.

We have a freedom to chose the driving process (and the generator Λ), so it seems possible to find such driving process that this equation could be solved and gives large spectrum.

This paper was in preparation for a long time. During this period there appeared several paper studying the most natural LLE where the driving force is a symmetric α stable process (or a sum of a Brownian motion and stable process). Unpublished computer experiments by Meyer [20] suggested that the spectrum for 1-stable process could be large (and possibly equal to the conjectured universal spectrum). Unfortunately later work by Gruzberg, Guan, Kadanoff, Oikonomou, Rohde, Rushkin, Winkel, and others [22, 9, 8] showed that this is wrong. But there is still a possibility that computer experiments exposed an existing phenomenon. It could be that the integral means grow fast for a few (relatively) large scales and when we approach the boundary their growth slows down. If this is true, one can use LLE as a building block in a snowflake (or any other construction which allows to replicate scales). In this way one can hope to construct a domain with large integral means on *all* scales.

4 Almost sure value of the spectrum

In this section we speculate about what should be done to prove that the almost sure value of the spectrum is given by (3).

Let us introduce random variables $X_k(n) = |f'((1+2^{-n})e^{2\pi ik/2^n})|^t$. The spectrum is the growth rate of $2^{-n}\sum_k X_k$. We know that

$$2^{-n}\sum_{k=1}^{2^n}\mathbb{E}X_k \asymp 2^{n\beta(t)}.$$

We want to show that the probability

$$\mathbb{P}\left\{2^{-n}|\sum X_k - \mathbb{E}X_k| > 2^{n(\beta(t)-\delta)}\right\} \tag{18}$$

is summable for some positive δ. This will clearly imply that spectrum of SLE is equal to $\beta(t)$ with probability one.

Conformal field theory considerations suggest that X_k and X_l are essentially independent if $|k-l| \gg 1$ (in other words the distance between points should be much larger than their distance to the boundary). In fact it is believed that derivatives are essentially independent if the distance between points is greater that any power (less than one) of the distance to the boundary. Let us exaggerate it a little bit more and assume that X_k and X_l are independent for any $k \neq l$.

Let us denote $X_k - \mathbb{E}X_k$ by Y_k. By Chebyshev inequality the probability (18) is less than

$$\frac{\mathbb{E}|\sum Y_k|^{1+\epsilon}}{2^{n(1+\epsilon)(\beta(t)+1-\delta)}}.$$

It is known (see [1]) that for independent random variables with zero mean $\mathbb{E}|\sum Y_k|^{1+\epsilon} \leq c\sum \mathbb{E}|Y_k|^{1+\epsilon}$, where c is an absolute constant which does not depend on the number of terms. Using this we can estimate the fraction above by

$$\frac{\sum \mathbb{E}|Y_k|^{1+\epsilon}}{2^{n(1+\epsilon)(\beta(t)+1-\delta)}} \leq c\frac{2^n 2^{n\beta(t+t\epsilon)}}{2^{n(1+\epsilon)(\beta(t)+1-\delta)}} = c2^{n(1+\beta(t+t\epsilon)-\beta(t)-1+\delta-\epsilon\beta(t)-\epsilon+\epsilon\delta)}. \tag{19}$$

For small $\epsilon < \epsilon_0(t)$ the exponent in the last formula is bounded by

$$n(\beta'(t)t\epsilon + \epsilon^{3/2} + \delta - \epsilon\beta(t) - \epsilon + \epsilon\delta) = n(\epsilon(\beta'(t)t - \beta(t) - 1) + \epsilon^{3/2} + \delta + \epsilon\delta).$$

If $\beta'(t)t - \beta(t) - 1 = c(t) < 0$, then we can find a small ϵ_t (depending on t only) such that $\epsilon_t(\beta'(t)t - \beta(t) - 1) + \epsilon_t^{3/2} < c(t)\epsilon_t/2$. Fix $\delta = -\epsilon_t c(t)/4$, then the exponent in (19) is negative. This implies that the probability in (18) is summable if $-1 < \beta(t) - t\beta'(t)$. The last inequality means that the tangent line to β at point t intersects the y axis above -1. This is exactly the condition which appeared in (3).

Thus, assuming the independence of derivatives, we can prove that the almost sure value of the spectrum is equal to $\bar{\beta}(t)$ for $t_{min} < t < t_{max}$. For other values of t Makarov's theorem implies that the spectrum should continue as a straight line tangent to $\bar{\beta}(t)$ at t_{min} and t_{max} correspondingly.

References

[1] B. v. Bahr and C.-G. Esseen. Inequalities for the rth absolute moment of a sum of random variables, $1 \leq r \leq 2$. *Ann. Math. Stat.*, 36:299–303, 1965.

[2] V. Beffara. Hausdorff dimensions for SLE_6. *Ann. Probab.*, 32(3B):2606–2629, 2004.

[3] D. Beliaev and S. Smirnov. Harmonic measure on fractal sets. In *European Congress of Mathematics*, pages 41–59. Eur. Math. Soc., Zürich, 2005.

[4] J. E. Brennan. The integrability of the derivative in conformal mapping. *J. London Math. Soc. (2)*, 18(2):261–272, 1978.

[5] L. Carleson and P. W. Jones. On coefficient problems for univalent functions and conformal dimension. *Duke Math. J.*, 66(2):169–206, 1992.

[6] B. Duplantier. Conformally invariant fractals and potential theory. *Phys. Rev. Lett.*, 84(7):1363–1367, 2000.

[7] B. Duplantier. Higher conformal multifractality. *J. Statist. Phys.*, 110(3-6):691–738, 2003.

[8] Q.-Y. Guan. Cadlag curves of *SLE* driven by Levy processes. `arXiv:0705.2321`, 2007.

[9] Q.-Y. Guan and M. Winkel. *SLE* and alpha-*SLE* driven by Levy processes. `arXiv:math/0606685`, 2006.

[10] M. B. Hastings. Exact multifractal spectra for arbitrary laplacian random walks. *Phys. Rev. Lett.*, 88(5):055506, Jan 2002.

[11] H. Hedenmalm and S. Shimorin. Weighted Bergman spaces and the integral means spectrum of conformal mappings. *Duke Math. J.*, 127(2):341–393, 2005.

[12] H. Hedenmalm and A. Sola. Spectral notions for conformal maps: a survey. *preprint*.

[13] N.-G. Kang. Boundary behavior of SLE. *J. Amer. Math. Soc.*, 20(1):185–210 (electronic), 2007.

[14] P. Kraetzer. Experimental bounds for the universal integral means spectrum of conformal maps. *Complex Variables Theory Appl.*, 31(4):305–309, 1996.

[15] G. Lawler. *Conformally Invariant Processes in the Plane*, volume 114 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005.

[16] J. Lind. Holder regularity for the SLE trace. *preprint*.

[17] N. G. Makarov. Fine structure of harmonic measure. *St. Petersburg Math. J.*, 10(2):217–268, 1999.

[18] B. Mandelbrot. Negative fractal dimensions and multifractals. *Phys. A*, 163(1):306–315, 1990. Statistical physics (Rio de Janeiro, 1989).

[19] B. Mandelbrot. Multifractal power law distributions: negative and critical dimen sions and other "anomalies," explained by a simple example. *J. Statist. Phys.*, 110(3-6):739–774, 2003. Special issue in honor of Michael E. Fisher's 70th birthday (Piscat away, NJ, 2001).

[20] D. Meyer. Private communications.

[21] S. Rohde and O. Schramm. Basic properties of SLE. *Ann. of Math. (2)*, 161(2):883–924, 2005.

[22] I. Rushkin, P. Oikonomou, L. P. Kadanoff, and I. A. Gruzberg. Stochastic Loewner evolution driven by Levy processes. `arXiv:cond-mat/0509187`, 2005.

[23] O. Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, 118:221–288, 2000.