INTEGRAL MEANS SPECTRUM OF RANDOM CONFORMAL SNOWFLAKES

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1. Introduction

It is known that extremal configurations in many important problems in classical complex analysis exhibit complicated fractal structure. This makes such problems extremely difficult. The classical example is the coefficient problem for the class of bounded univalent functions. Let $\phi(z) = z + a_2 z^2 + a_3 z^3 + \dots$ be a bounded univalent map in the unit disc. One can ask what are the maximal possible values of coefficients a_n, especially when n tends to infinity. We define γ as the best possible constant such that $|a_n|$ decays as $n^{\gamma-1}$. In [4] Carleson and Jones showed that this problem is related to another classical problem about the growth rate of the length of Greens' lines. In particular, they showed that the extremal configurations for both of this problems should be of a fractal nature.

During the last decade it became clear that the right language for these problems, as well as many other classical problems, is *the maltifractal analysis*. It turned out that all these problems could be reduced to the problem of finding the maximal value of *the integral means spectrum*.

In the recent paper [2] S. Smirnov and the author introduced and studied a new class of random fractals, the so-called *random conformal snowflakes*. In particular, they proved the *fractal approximation* for this class, which means that one can find conformal snowflakes with spectra arbitrary close to the maximal possible spectrum. In this paper we report on our search for snowflakes with large spectrum.

The paper organized as follows: in the introduction we give some basic information about integral means spectrum, define random conformal snowflakes, and state the main facts about spectrum of the snowflakes. In the Section 2 we give numerical estimates of the spectra of several snowflakes for different values of parameters. In the last Section we give rigorous lower bound for the spectrum at $t = 1$.

1.1. Integral means spectrum.

Here we briefly sketch the necessary definitions and the background. For the detailed discussion of the subject we recommend surveys [5, 8, 1] and books [6, 7].

Let $\Omega \in \hat{\mathbb{C}}$ be a simply connected domain in the complex sphere which contains infinity. By the Riemann uniformization theorem, there is a conformal map ϕ from the complement of the unit disc onto Ω such that $\phi(\infty) = \infty$. The *integral means spectrum* is defined as

$$\beta(t) = \beta_\Omega(t) = \beta_\phi(t) = \limsup_{r \to 1+} \frac{\log \int_0^{2\pi} |\phi'(re^{i\theta})|^t d\theta}{|\log(r-1)|}, \ t \in \mathbb{R} .$$

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The *universal integral means spectrum* is

$$B(t) = \sup \beta(t),$$

where supremum is over all simply connected domains Ω.

The universal spectrum plays the central role in the Geometric Function Theory. A lot of major problems in the field can be stated in terms of $B(t)$. In particular, Brennan's conjecture [3] asserting that $|\psi'|^{4-\epsilon}$ is integrable for any $\epsilon > 0$ and any conformal $\psi : \Omega \to \mathbb{D}$ is equivalent to $B(-2) = 1$. The Carleson-Jones conjecture, $\gamma = 1/4$, is equivalent to $B(1) = 1/4$. In 1996 Kraetzer formulated the ultimate conjecture on integral means:

$$\begin{aligned} B(t) &= t^2/4, \quad |t| \le 2, \\ B(t) &= |t| - 1, \quad |t| > 2. \end{aligned}$$

This conjecture is based on the above mentioned conjectures, computer experiments and the believe that the spectrum should be relatively simple function. Kraetzer performed computer experiments with quadratic Julia sets. The results were within 5 percents from $t^2/4$. We should point out that the experiments were completely non-rigorous, and, as far as we know, the error can not be estimated using available techniques.

In the present paper we give a computer assisted lower bound on the universal spectrum which are within $4-7$ percents from $t^2/4$ but they are "semi-rigorous". By this we mean that we use computer only for numerical integration. For numerical integration of an explicit function one can write estimates of the error term and this will give a rigorous estimate. We give an estimate of the error term only for $t = 1$ (this case is of special interest and estimates are a bit simpler). We prove that $B(1) > 0.23$ which is a significant improvement over previously known $B(1) > 0.17$.

Essentially the same argument as for the case $t = 1$ could be used to estimate the error for other values of t, although we are not doing it here.

1.2. **Conformal snowflake.** We denote by Σ' the class of univalent maps ϕ from the complement of the unit disc $\mathbb{D}_-$ into itself such that $\phi(\infty) = \infty$ and $\phi'(\infty) \in \mathbb{R}$. Let $K_n\phi$ be the Koebe transform of $\phi \in \Sigma'$:

$$(K_n\phi)(z) = (K_n\phi)(z) = \sqrt[n]{\phi(z^n)}.$$

We denote the conjugation by rotation by $\phi_\theta(z) = e^{i\theta}\phi(ze^{-i\theta})$.

To construct a random snowflake we need two components: a building block $\phi \in \Sigma'$ and an integer $k \ge 2$. The n-th approximation to the snowflake is

$$f_n(z) = f_{n-1}(K_{k^n}\phi_{\theta_n}(z)) = \phi_{\theta_0}(\phi_{\theta_1}^{1/k}(\dots\phi_{\theta_n}^{1/k}(z^{k^n})\dots),$$

where θ_j are independent variables uniformly distributed between 0 and 2π. The conformal snowflake f is the limit of f_n. For the proof that conformal snowflakes are well defined see [2]. $S = \mathbb{C} \setminus f(\mathbb{D}_-)$ and $g = f^{-1}$ are also referred to as snowflakes.

It turns out that the spectrum of a snowflake for a particular value of t is closely related to the spectrum of an integral operator P whose kernel is defined in terms of k, t, and ϕ. This operator is defined as

$$P\nu(r) := r^{1-\frac{(k-1)t}{k}} \int_0^{2\pi} \frac{\nu(\phi(r^{1/k}e^{i\theta}))}{|\phi(r^{1/k}e^{i\theta})|} |\phi'(r^{1/k}e^{i\theta})|^t \frac{d\theta}{2\pi}. \tag{1}$$

Let R be any radius such that $D_R \subset \psi^k(D_R)$ where $D_R = \{z : 1 < |z| < R\}$ and $\psi = \phi^{-1}$.

FIGURE 1. The image of a small boundary arc under f_3 with three Green's lines.

One can show (see [2] for the proof) that if λ is the maximal eigenvalue of P then

$$\beta(t) \geq \log \lambda / \log k, \tag{2}$$

in particular

$$\beta(t) \geq \min_{1 \leq r \leq R} \log \left(\frac{P\nu(r)}{\nu(r)} \right) / \log k \tag{3}$$

for any positive test function ν.

1.3. **Random fractals.** One of the main problems in the computation of the integral means spectrum (or other multifractal spectra) is the fact that the derivative of a Riemann map for a fractal domain depends on the argument in a very non regular way: ϕ' is a "fractal" object in itself. The solution to this problem is to study random fractals for which the boundary behavior of ϕ' is statistically independent of the argument. In this case it is natural to consider the *average integral means spectrum:*

$$\begin{aligned} \bar{\beta}(t) &= \sup \left\{ \beta : \int_1 (r-1)^{\beta-1} \int_0^{2\pi} \mathbb{E}\left[|f'(re^{i\theta})|^t \right] d\theta dr = \infty \right\} \\ &= \inf \left\{ \beta : \int_1 (r-1)^{\beta-1} \int_0^{2\pi} \mathbb{E}\left[|f'(re^{i\theta})|^t \right] d\theta dr < \infty \right\}. \end{aligned}$$

The average spectrum does not have to be related to the spectra of a particular realization. We want to point out that even if ϕ has the same spectrum a.s. it does not guarantee that $\bar{\beta}(t)$ is equal to the a.s. value of $\beta(t)$. Moreover, it can happen that $\bar{\beta}$ is not a spectrum of *any* particular domain. On the other hand, Makarov's fractal approximation ([5]) implies that $\bar{\beta}(t) \le B(t)$. Thus lower bound for any $\bar{\beta}$ is a lower bound for B. In the remainder of the paper we will work only with average integral means spectrum. We refer to it as "the spectrum" and denote it by β.

2. Numerical estimates of the spectrum

In this section we give a numerical support of Kraetzer's conjecture. To achieve this we estimate the spectrum of snowflakes with a straight slit building blocks (see Figure 1). We use the following scheme: first we compute the disctretized operator P and find its main eigenvalue and eigenvector. We approximate the main eigenvector by a nice function ν and use it as a test function in (3).

First we define the basic slit function

$$\phi(z,l) = \phi_l(z) = \mu_2\left(\frac{\sqrt{\mu_1^2(zs) + l^2/(4k+4)}}{\sqrt{1 + l^2/(4l+4)}}\right), \tag{4}$$

where s is a constant close to 1, μ_1 and μ_2 are the Möbius transformation that maps $\mathbb{D}_-$ onto the right half plane and its inverse:

$$\mu_1(z) = \frac{z-1}{z+1},$$
$$\mu_2(z) = \frac{z+1}{z-1}.$$

We also need the inverse function

$$\psi(z,l) = \psi_l(z) = \phi(z,l)^{-1}. \tag{5}$$

The image $\phi_l(\mathbb{D}_-)$ is $\mathbb{D}_-$ with a horizontal slit originating at 1. The length of the slit is l. The derivative of a slit map has singularities at points mapped to 1. But for $s > 1$ these singularities are not in $\mathbb{D}_-$.

Given k, l (and s) we can numerically find the critical radius R and define the operator P. For the first step, we use the following discrete operator which approximates P. Choose sufficiently large N and M. Let $r_n = 1 + (R-1)n/N$ and $\theta_m = 2\pi m/M$. P is approximated by an $N \times N$ matrix with elements

$$P_{n,n'} = \sum_m r_n^{1-t(k-1)/k} \frac{|\phi'(r_n^{1/k} e^{i\theta_m})|^t}{|\phi(r_n^{1/k} e^{i\theta_m})| M},$$

where summation is over all indexes m such that $r_{n'}$ is the closest point to $|\phi(r_n^{1/k} e^{i\theta_m})|$. This defines the discretized operator P_N. Let λ_N and V_N be the main eigenvalue and the corresponding eigenvector.

It is highly plausible that the operators P_N converge to P and the λ_N converge to the main eigenvalue of P, but it seems to be hard to prove. Moreover, the rate of convergence is difficult to estimate. But this crude approximation gives us the fast test on whether the pair ϕ and k defines a snowflake with large spectrum or not.

For each value of t we use this method to search for the optimal values of k and l. This gives us snowflakes that a supposed to have large spectrum. To estimate the spectrum we use the following trick. We approximate the eigenvector V_N by a simple function (or just linearly interpolate it). The result is used as a test function ν. Than we use numerical integration to calculate $P\nu$. By (3) it gives us the lower bound of β. For general values of t we do it without estimates of the error term, we just use Euler quadrature formula doubling number of nodes until the difference is

t	k	l	$\log_k \lambda$	$\beta(t)$	Kraetzer	$t^2/4$
-2.0	34	1	1.262	–	–	1.0
-1.8	34	1	1.068	–	–	0.81
-1.6	34	1	0.8761	–	–	0.64
-1.4	34	1	0.6879	–	0.476	0.49
-1.2	34	1	0.5059	–	0.340	0.36
-1.0	34	1	0.3354	–	0.231	0.25
-0.8	34	1	0.1865	–	0.149	0.16
-0.6	24	21	0.0848	0.0710	0.085	0.09
-0.4	20	25	0.0377	0.0352	0.037	0.04
-0.2	31	44	0.0093	0.0083	0.0095	0.01
0.2	5	7	0.0091	0.00897	0.0094	0.01
0.4	11	30	0.0376	0.03767	0.037	0.04
0.6	14	68	0.0851	0.08442	0.086	0.09
0.8	12	67	0.1514	0.1511	0.154	0.16
1.0	13	73	0.2362	0.2340	0.242	0.25
1.2	10	67	0.3425	0.3350	0.346	0.36
1.4	8	55	0.4680	0.4586	0.476	0.49
1.6	6	39	0.6137	0.6091	–	0.64
1.8	6	39	0.7790	0.7713	–	0.81
2.0	4	21	0.9548	0.9296	–	1.0

TABLE 1. Spectra of nearly optimal snowflakes for different values of t. Parameters k and l define the snowflake (s=1), λ is the main eigenvalue of P_N (computed with $N = 2000$ and $M = 1000$), $\beta(t)$ is the lower bound given by (3). In the last two columns we give lower bounds obtained by Kraetzer and values of the conjectured universal spectrum.

small. We also can argue that since functions seems to be quite smooth (there is no oscillation), the precision of numerical integration should be much better than what follows from the standard estimates of the error term. The results of our computations are given in the Table 1.

We want to make a few comments about negative values of t. The derivative of ϕ has a zero of the first order at $z = 1$, this means that for $\rho = 1$ the integral in the definition of P diverges at $r = 1$. This is the reason why we can not find a good test function and the estimated λ is probably far from the correct one for $t \leq -0.8$. The problem with λ could be solved by increasing N and M. For example if we set $N = 3000$ and $M = 2000$ then for $t = -0.6, \ldots, -1.2$ the logarithm of λ will be equal to 0.0847, 0.1579, 0.2926, and 0.4509 correspondingly. This means that the value for $t = -0.6$ is probably correct even for smaller values of N and M and the value for $t = -1.2$ is still far from its true value.

3. SPECTRUM AT $t = 1$

In this section we estimate the error term in the numerical integration of $P\nu$. This gives a rigorous proof that $B(1) > 0.23$ which is a significant improvement over previously known $B(1) > 0.17$. We also would like to point out that this computation is not specific for $t = 1$ (though it is a bit easier in this case) and can be done for other values as well.

For $t = 1$ we study the snowflake with $l = 73$ and $k = 13$. The Figure 1 shows the image of a small arc under f_3 and three Green's lines. This snowflake has a spectrum close to 0.234. Unfortunately there is a singularity in the kernel of P (it is integrable, so one can still estimate the error term). This singularity is the reason why we introduced the parameter s. If we choose s close to 1 then snowflake is close to the snowflake with $s = 1$, but derivative of ϕ is continuous up to the boundary. In our case we use $s = 1.002$.

First we have to find the critical radius R such that $D_R \subset \psi^k(D_R)$. By symmetry of ϕ, the critical radius is the only positive solution of

$$\psi^k(x) = x.$$

This equation can not be solved explicitly, but we can solve it numerically (we don't care about error term since we can use any upper bound for R). The approximate value of R is 76.1568. To be on the safe side we fix $R = 76.2$.

To find the test function we approximate P by P_N where $N = 1000$ and $M = 500$. The logarithm of the first eigenvalue is 0.2321 (it is 0.23492 if we take $s = 1$). We scale the coordinates of the main eigenvector of P_N from $[1, 1000]$ to the interval $[1, R]$ and approximate by a rational function ν which we will use as a test function:

$$\nu(x) = (7.1479 + 8.9280x - 0.07765x^2 + 1.733 \times 10^{-3}x^3 -$$
$$2.0598 \times 10^{-5}x^4 + 9.5353 \times 10^{-8}x^5)/(2.7154 + 13.2845x).$$

To estimate $\beta(1)$ we have to integrate $\nu(|\phi|)|\phi'|/|\phi|$. It is easy to see that the main contribution to the derivative is given by a factor $|\phi'|/|\phi|$. Assume for a while that $s = 1$. The fraction $|\phi'(z)/\phi(z)|$ can be written as

$$\frac{|z-1|}{|z|\sqrt{|(z-z_1)(z-z_2)}|}, \tag{6}$$

where

$$z_1 = \frac{-5033 - 292i\sqrt{74}}{5625} \approx -0.894756 - 0.446556i,$$
$$z_2 = \frac{-5033 + 292i\sqrt{74}}{5625} \approx -0.894756 + 0.446556i.$$

Singular points z_1 and z_2 are mapped to 1 and ϕ' has a square root type singularity at these points. They will play essential role in all further calculations. We introduce notation $z_1 = x + iy$ and $z_2 = x - iy$, for z we will use polar coordinates $z = re^{i\theta}$.

We compute the integral of $f = |\nu(\phi)\phi'/\phi|$ using the Euler quadrature formula based on the trapezoid quadrature formula

$$\int_0^{2\pi} f(x)dx \approx S^n_\epsilon(f) = S_\epsilon(f) - \sum_{k=1}^{n-1} \gamma_{2k}\epsilon^{2k}\left(f^{(2k-1)}(2\pi) - f^{(2k-1)}(0)\right),$$

where $S_\epsilon(f)$ is a trapezoid quadrature formula with step ϵ and $\gamma_k = B_k/k!$ were B_k is the Bernoulli number. The error term in the Euler formula is

$$-\gamma_{2n} \max f^{(2n)}\epsilon^{2n}2\pi. \tag{7}$$

In our case function f is periodic and terms with higher derivatives vanish. This means that we can use (7) for any n as an estimate of the error in the trapezoid quadrature formula.

Function ϕ has two singular points: z_1 and z_2. Derivative of ϕ blows up near these points. This is why we introduce scaling factor s. We can write a power series of ϕ near z_1 (near z_2 situation is the same by the symmetry)

$$\phi^{(k)} = c_{-k}(z-z_1)^{-k+1/2} + c_{-k+1}(z-z_1)^{-k+3/2} + \cdots + c_0 + \dots\ .$$

This means that for $s > 1$ derivative can be estimated by

$$|c_{-k}|(s-1)^{-k+1/2} + |c_{k+1}|(s-1)^{-k+3/2} + \dots\ .$$

The series converges in a disc of a fixed radius (radius is $|z_1+1|$), so its tail can be estimated by a sum of a geometric progression. Writing these power series explicitly we find (for $s = 1.002$)

$$\begin{aligned} &|\phi'| < 55, && |\phi''| < 11800, && |\phi^{(3)}| < 8.69 \times 10^6, \\ &|\phi^{(4)}| < 1.08 \times 10^{10}, && |\phi^{(5)}| < 1.90 \times 10^{13}, && |\phi^{(6)}| < 4.25 \times 10^{16}, \\ &|\phi^{(7)}| < 1.17 \times 10^{20}. \end{aligned}$$

The maximal values for first six derivatives of ν are

$$\begin{aligned} |\nu'| &< 0.28, & |\nu''| &< 0.45, & |\nu^{(3)}| &< 1.12, \\ |\nu^{(4)}| &< 3.69, & |\nu^{(5)}| &< 15.3, & |\nu^{(6)}| &< 76.2. \end{aligned}$$

The derivative $\partial_\theta |\phi|$ can be estimated by $r|\phi'|$. We can write sixth derivative of $\nu(|\phi|)|\phi'|/|\phi|$ as a rational function of partial derivatives of $|\phi|$, $|\phi'|$, and ν. Than we apply triangle inequality and plug in the above estimates. Finally we have

$$\left| \frac{\partial \left(\frac{\nu(|\phi|)|\phi'|}{|\phi|} \right)}{\partial \theta^6} \right| < 1.65 \times 10^{21}.$$

Plugging in the value $\epsilon = \pi/5000$ and the estimate on the sixth derivative into (7) we find that error term in this case is less than 0.0034.

Next we have to estimate modulus of continuity with respect to r. First we calculate

$$\partial_r |z-(a+bi)|^2 = 2r - 2(a\cos\theta + b\sin\theta).$$

Applying this formula several times we find

$$\begin{aligned} \partial_r \left(\frac{|\phi'|}{|\phi|} \right) &= \partial_r \left(\frac{|z-1|}{r\sqrt{|z-z_1||z-z_2|}} \right) \le \partial_r \left(\frac{|z-1|}{\sqrt{|z-z_1||z-z_2|}} \right) \\ &= \frac{r-\cos\theta}{|z-1|S} - \frac{r - x\cos\theta - y\sin\theta}{2|z-z_1|^2 S}|z-1| - \frac{r - x\cos\theta + y\sin\theta}{2|z-z_2|^2 S}|z-1|, \end{aligned}$$

where $S = \sqrt{|z-z_1||z-z_2|}$. Factoring out

$$\frac{1}{2|z-1| \cdot |z-z_1|^{5/2}|z-z_2|^{5/2}}$$

we get

$$\begin{aligned}2(r-\cos\theta)|z-z_1|^2|z-z_2|^2-(r-x\cos\theta-y\sin\theta)|z-1|^2|z-z_2|^2\\ -(r-x\cos\theta+y\sin\theta)|z-1|^2|z-z_1|^2\\ =-2(r^2-1)(2\cos^2\theta r(x-1)+\cos\theta(r^2+1)(x-1)+2ry^2).\end{aligned}$$

This is a quadratic function with respect to $\cos\theta$. Taking values of x and y into account we can write it as

$$\cos^2\theta+\cos\theta\left(r+\frac{1}{r}\right)\frac{1}{2}-\frac{592}{5625}\ .$$

This quadratic function has two real roots. Their average is $-(r+1/r)/2<-1$, hence one root is definitely less than -1. The product of roots is a small negative number, which means that the second root is positive and less than 1. Simple calculation shows that this root decreases as r grows. This means that the corresponding value of θ increases. Hence it attains its maximal value at $r=1.4$ and the maximal value is at most 1.48. This gives us that the radial derivative of $|\phi'|/|\phi|$ can be positive only on the arc $\theta\in[-1.48,1.48]$. By subharmonicity it attains the maximal on the boundary of $\{z\mid 1<r<1.4,\ -1.48<\theta<1.48\}$. It is not very difficult to check that maximum is at $z=1.4$ and it is equal to 0.36.

Let

$$I(r)=\int_{-\pi}^{\pi}\nu(|\phi(r^{1/k}e^{i\theta})|\left|\frac{\phi'(r^{1/k}e^{i\theta})}{\phi(r^{1/k}e^{i\theta})}\right|\frac{d\theta}{2\pi}.$$

The derivative is

$$I'(r)=\frac{1}{kr^{1-1/k}}\left(\int_{-\pi}^{\pi}\nu'(|\phi|)\partial_r|\phi|\frac{|\phi'|}{|\phi|}\frac{d\theta}{2\pi}+\int_{-\pi}^{\pi}\nu(|\phi|)\partial_r\left(\frac{|\phi'|}{|\phi|}\right)\frac{d\theta}{2\pi}\right).$$

By the symmetry the first integral is zero. In the second integral

$$\nu(|\phi|)\partial_r\left(\frac{|\phi'|}{|\phi|}\right)$$

can be positive only when $\theta\in[-1.48,1.48]$ and even in this case it is bounded by $0.36/(r^{1-1/k}k2\pi)$. Hence

$$I'(r)<2\cdot 1.48\cdot 0.36/(r^{1-1/k}k2\pi)<0.0131r^{1/k-1}.$$

If we compute values $I(r_1)$ and $I(r_2)$ (with precision 0.0034) then the minimum of $P(\nu)/\nu$ on $[r_1,r_2]$ is at least

$$r_1^{1/k}(\min\{I(r_1),I(r_2)\}-0.0034-0.0131(r_2-r_1)r^{1/k-1})/\nu(r_1). \tag{8}$$

We take 3000 equidistributed points on $[1,R]$ and compute $I(r)$ at these points. Applying the error estimate (8) we find a rigorous estimate from below of $P\nu/\nu$. The minimum of $P\nu/\nu$ is at least 1.8079 which means that

$$\beta(1)>0.2308.$$

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