Tautologial integrals on curvilinear Hilbert schemes

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- $X_p^{[k]} = \{\xi \in X^{[k]} : \mathrm{supp}(\xi) = p\} = \rho^{-1}(np)$ Punctual Hilbert scheme at p where $\rho : X^{[k]} \to S^k X$ $\xi \mapsto \sum_{p \in X} l(\mathcal{O}_{\xi,p})p$ is the Hilbert-Chow morphism.

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For $n > 2$ $\overline{CX_p^{[k]}}$ is a component of the punctual Hilbert scheme.

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$f_\xi(z) = zf'(0) + \frac{z^2}{2!}f''(0) + \ldots + \frac{z^k}{k!}f^{(k)}(0) \in J_k(1,n)$ and
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Then $f \circ \varphi(z) =$

$$= (f'(0)\alpha_1)z + (f'(0)\alpha_2 + \frac{f''(0)}{2!}\alpha_1^2)z^2 + \ldots + \left(\sum_{i_1+\ldots+i_l=k} \frac{f^{(l)}(0)}{l!}\alpha_{i_1}\ldots\alpha_{i_l}\right) z^k =$$

$$= (f'(0), \ldots, f^{(k)}(0)/k!) \cdot \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \ldots & 2\alpha_1\alpha_{k-1} + \ldots \\ 0 & 0 & \alpha_1^3 & \ldots & 3\alpha_1^2\alpha_{k-2} + \ldots \\ 0 & 0 & 0 & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \alpha_1^k \end{pmatrix}$$

where the polynomial in the (i,j) entry is

$$p_{i,j}(\bar{\alpha}) = \sum_{a_1+a_2+\ldots+a_i=j} \alpha_{a_1}\alpha_{a_2}\ldots\alpha_{a_i}.$$

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In coordinates: $f_\xi = f_1 z + \ldots + f_k z^k$ with $f_i \in \mathbb{C}^n$;
$g(v) = Av + Bv^2 + \ldots$ with $A \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C})$, $B \in \mathrm{Hom}(\mathrm{Sym}^2(\mathbb{C}^n), \mathbb{C}), \ldots$
then $g \circ f_\xi = 0$ has the form:

$$A(f_1) = 0,$$
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Theorem

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$$(f', \ldots, f^{(k)}) \mapsto \mathrm{Span}(f', f'' + (f')^2, \ldots, \sum_{a_1 + a_2 + \ldots + a_i = j} f^{(a_1)} f^{(a_2)} \ldots f^{(a_i)}, \ldots),$$

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2. For $k \leq n$ $\overline{\rho(J_k(1,n))} = \overline{\mathrm{GL}(n) \cdot \rho(e_1, \ldots, e_n)}$ where $\{e_1, \ldots, e_n\}$ is a basis of $\mathbb{C}^n$.

Tautological integrals

Ultimate goal: Determine the geometric and topological invariants of $X^{[n]}$.
For surfaces these were extensively studied such as **Betti numbers** (Ellingsrud, Stromme, Göttsche), **Hodge numbers** (Sörgel, Göttsche), **cohomology ring** (Nakajima, Grojnowski), **Chern numbers of tautological bundles** (Lehn, Rennemo, Marian-Oprea-Pandharipande).

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Following Rennemo (2014) $\mathcal{X} \subset X^{[k]}$ is called **geometric subset** when

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Thm (Rennemo, Tzeng 2013-14): $\int_{\overline{\mathcal{X}}} P(c_i(F^{[n]})) = \tilde{P}(c_i(F), c_i(X))$.

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Want: Closed formula for $\int_{\overline{CX}^{[k]}} P(c_i(F^{[n]}))$.
We have: Closed formula for $\int_{\widetilde{CX}^{[k]}} P(c_i(F^{[n]}))$ for some geometric partial resolution $\widetilde{CX}^{[k]} \to \overline{CX}^{[k]}$.

Step 1 $X^{[k_1,\ldots,k_t]}=\{(\xi_1\subset\xi_2\subset\ldots\subset\xi_t):\xi_i\in X^{[k_i]}\}\subset X^{[k_1]}\times\ldots\times X^{[k_t]}$
Define

$$\tilde{\rho}:CX_p^{[k+1]}\hookrightarrow X^{[2,\ldots,k+1]}$$

$$\xi\mapsto \text{ Unique flag } (\xi_1\subset\ldots\subset\xi_k) \text{ such that } \xi_i\in CX_p^{[i]}$$

$$f=f_\xi\mapsto(\mathrm{Ann}(f_{[1]})^*\subset\mathrm{Ann}(f_{[2]})^*\subset\ldots\subset\mathrm{Ann}(f_{[k]})^*)=$$

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Step 1 $X^{[k_1,\ldots,k_t]} = \{(\xi_1 \subset \xi_2 \subset \ldots \subset \xi_t) : \xi_i \in X^{[k_i]}\} \subset X^{[k_1]} \times \ldots \times X^{[k_t]}$
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Step 2 Assume $k\le n$. Let $\pi:J_k(1,n)\simeq\oplus_{i=1}^k\mathrm{Sym}^i\mathbb{C}^n\to\mathbb{C}^n$ be the projection.

$$\widetilde{CX}_p^{[k+1]}=\{(\tilde\xi_1\subset\ldots\subset\tilde\xi_k),(V_1\subset\ldots\subset V_k):\pi(\tilde\xi_i)\subset V_i\}\subset\widetilde{\widetilde{CX}}_p^{[k+1]}\times\mathrm{Flag}(k,J_k(1,n)$$

Equivalently:

$$\widetilde{CX}_p^{[k+1]}=\mathrm{GL}(n)\times_{B_n}\overline{B_n\cdot z_k}\to\overline{\mathrm{GL}(n)\cdot z_k}=\widetilde{\widetilde{CX}}_p^{[k+1]}$$

Equivariant localisation on $\widetilde{\widetilde{CX}}_p^{[k]}$

Set-up: $k \le n$. $\widetilde{\widetilde{CX}}^{[k+1]} \to CX^{[k+1]} \to X$ partial resolution. Fibre over $p \in X$

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Let $T = (\mathbb{C}^*)^n$ act on $T_p^* X_p$ with weights $\lambda_1, \ldots, \lambda_n$.

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where s_X is the total Segre class of X.

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$\mathcal{B}_k = \overline{B_n z_k} \subset \mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n)$. For $p \in \mathcal{B}_k^T$ replace

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Hard part: we don't know which fixed points of the ambient flag sit in $\mathcal{B}_k$.
Still, we can prove the following vanishing property

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Theorem (B-Szenes, B)

Let $n \leq k$. Only one fixed point contributes to the iterated residue. This distinguished fixed point corresponds to the sequence $\pi = (1, 2, \dots, k)$ which is the ideal $\xi = m_k^2 \subset m_n^2$

Let $Q_k(z_1, \dots, z_k) = \mathrm{mdeg}(\mathcal{Y}_{1,\dots,k}\mathcal{B}_k, T_\pi\mathrm{Flag})$. Then

Theorem

For any n, k (we can drop $k \le n$ condition!)

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$$\epsilon = \sum_{m=1}^{k} \sum_{r=1}^{k-m} q_{m+r}^{mr} \subset W = \bigoplus_{1 \le m+r \le l \le k} \mathbb{C} q_l^{mr} \subset \mathrm{Hom}(\mathbb{C}^k, \mathrm{Sym}^2\mathbb{C}^k)$$

Then $Q_k(z) = \mathrm{mdeg}(\overline{B_k\epsilon}, W)$. In particular $Q_1 = Q_1 = Q_2 = 1$, $Q_4 = 2z_1 + z_2 - z_4$.

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$$V^U = \wedge^n(\mathbb{C}^n \oplus S^{k_2}\mathbb{C}^n \oplus \ldots \oplus S^{k_n}\mathbb{C}^n)$$

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Stabiliser of z^U in $\mathrm{GL}(n)$ is U, so $\mathrm{GL}(n)/U = \mathrm{GL}(n)z^U$ and this gives

$$\mathrm{GL}(n)/\hat{U} \hookrightarrow \mathbb{P}(V^U), \mathrm{Flag}_n(V^U)$$

Tautological integrals over $\overline{\mathrm{GL}(n)/\hat{U}}$ have similar iterated residue forms with contribution from only one point, namely the distinguished flag

$$(e_1) \subset (e_1, e_2) \subset \ldots \subset \mathbb{C}^n \in \mathrm{Flag}_n(V^U)$$