Nonreductive GIT and tautologial integrals on curvilinear Hilbert schemes

Gergely Bérczi – Oxford

NS@50
Chennai, 12 October 2015

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- $\mathcal{O}_L(X) = \oplus_{k=0}^{\infty} H^0(X, L^{\otimes k})$ has an induced G-action. If G is reductive, $\mathcal{O}_L(X)^G$ is finitely generated graded algebra, and $\mathcal{O}_L(X)^G \hookrightarrow \mathcal{O}_L(X)$ induces

$$\begin{array}{ccc} X & \dashrightarrow & X//G = \mathrm{Proj}\mathcal{O}_L(X)^G \\ \cup & & \| \\ X^{ss} & \twoheadrightarrow & X//G \\ \cup & & \cup \\ X^s & \to & X^s/G \end{array} \qquad \text{Geometric quot.}$$

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Hilbert-Mumford numerical criterion: $T \subset G$ max torus.
$x = [x_0 : \ldots : x_n] \in X_T^{ss(s)} \leftrightarrow$ The origin sits inside (in the closure of) $\mathrm{Conv}(w(x_i) : x_i \neq 0) \subset \mathfrak{t}_{\mathbb{C}}^*$. $x \in X$ is (semi)stable for G iff gx is T-(semi)stable for all $g \in G$.

Cornerstones of bad behaviour:

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A: One can define open subsets X^s (stable points) and X^{ss} (semistable points) with a geometric quotient $X^s \to X^s/G$ and an enveloping quotient $X^{ss} \to X//G$ s.t. if G is finitely generated then $X//G = Proj(\mathcal{O}_L(X)^G)$ and

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No topological description of $X//G$ and no Hilbert-Mumford criterion.

There are several possible definitions capturing different features of the reductive case (see Doran-Kirwan 2008)

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Definition

$G \subset R$ reductive. $\iota : X \hookrightarrow R \times_G X$ (quasi-projective, Vinberg). An ample reductive envelope is a projective completion $\overline{R \times_G X}$ with ample R-linearisation extending L. Then $X^{\overline{ss}} = \iota^{-1}(\overline{R \times_G X})^{ss,R}$.

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Examples:

- **Moduli spaces of toric hypersurfaces** E.g $\mathrm{Aut}(\mathbb{P}(1,1,2)) = R \ltimes U$ with $R = GL(2) \times \mathbb{C}^*$ reductive and $U = (\mathbb{C}^+)^3$ unipotent, where U acts as

$$(x,y,z) \mapsto (x,y,z+\lambda x^2 + \mu xy + \nu y^2 \quad (\lambda,\mu,\nu) \in \mathbb{C}^3.$$

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- **Jets of reparametrisations of curves** $\rightsquigarrow$ jet differentials, see later
- **GIT stratification** $X = \cup_{\beta\in\mathcal{B}} S_\beta$ such that (i) $S_0 = X^{ss}$ (ii) $\overline{S}_\beta = \cup_{|\gamma|>|\beta|} S_\gamma$ (iii) $S_\beta \simeq G \times_{P_\beta} Y_\beta$ and would be nice to have
$$S_\beta/G = Y_\beta/P_\beta = \overline{Y}_\beta//P_\beta$$

Moduli of degree 4 **hypersurfaces in** $\mathbb{P}(1,1,2)$: $\mathrm{Aut}(\mathbb{P}(1,1,2))$ acts on

$$H_4 = \mathrm{Span}(x^4, x^3y, x^2y^2, xy^3, y^4, x^2z, xyz, y^2z, z^2)$$

via

$$(x,y,z) \mapsto (x, y, z + \lambda x^2 + \mu xy + \nu y^2 \quad (\lambda, \mu, \nu) \in \mathbb{C}^3.$$

With respect to this basis, the $(\mathbb{C}^+)^3$ action is linearly represented as:

$$Aut(\mathbb{P}[1,1,2]) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & \lambda^2 \\ 0 & 1 & 0 & 0 & 0 & \mu & \lambda & 0 & 2\lambda\mu \\ 0 & 0 & 1 & 0 & 0 & \nu & \mu & \lambda & 2\nu+\mu^2 \\ 0 & 0 & 0 & 1 & 0 & 0 & \nu & \mu & 2\mu\nu \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \nu & \nu^2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2\lambda \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2\mu \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2\nu \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Examples: Reparametrisation groups

Jets of reparametrisation germs

- $J_k(1,n) = \{k\text{-jets of germs of holomorphic maps } (\mathbb{C},0) \to (\mathbb{C}^n,0)\} =$
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- $J_k(1,1)$ is the group of k-jets of germs of biholomorphisms of $(\mathbb{C},0)$. If $f(z) = zf'(0) + \frac{z^2}{2!}f''(0) + \ldots + \frac{z^k}{k!}f^{(k)}(0)$ and $\varphi \in J_k(1,1)$ has the form

$$\varphi(z) = \alpha_1 z + \alpha_2 z^2 + \ldots + \alpha_k z^k \text{ with some } \alpha_1, \ldots, \alpha_k \in \mathbb{C}, \alpha_1 \neq 0$$

Then

$$f \circ \varphi(z) = (f'(0)\alpha_1)z + (f'(0)\alpha_2 + \frac{f''(0)}{2!}\alpha_1^2)z^2 + \ldots$$

$$= (f', \ldots, f^{(k)}/k!) \cdot \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \ldots & 2\alpha_1\alpha_{k-1} + \ldots \\ 0 & 0 & \alpha_1^3 & \ldots & 3\alpha_1^2\alpha_{k-2} + \ldots \\ 0 & 0 & 0 & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \alpha_1^k \end{pmatrix}$$

where the polynomial in the (i,j) entry is

$$p_{i,j}(\underline{\alpha}) = \sum_{a_1+a_2+\ldots+a_i=j} \alpha_{a_1}\alpha_{a_2}\ldots\alpha_{a_i}.$$

Theorem (B.,Doran, Hawes, Kirwan 2015)

Let U be a graded unipotent lin. alg. group over $k = \bar{k}$ and $\hat{U} = U \rtimes \mathbb{G}_m$. Let X be an irreducible normal $\hat{U}$-variety and $L \to X$ an ample linearisation of the $\hat{U}$ action. Twist the linearisation by a character $\chi : \hat{U} \to \mathbb{G}_m$ of $\hat{U}/U$ such that 0 lies just above the lowest weight for the $\mathbb{G}_m$ action on X.

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If (*) fails then we can perform an analogue of the partial resolution construction of Kirwan for reductive groups to get blow-up $\tilde{X}\to X$ with $\tilde{X}$ satisfying (*) for suitable ample linearisation.

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If (*) fails then we can perform an analogue of the partial resolution construction of Kirwan for reductive groups to get blow-up $\tilde{X} \to X$ with $\tilde{X}$ satisfying (*) for suitable ample linearisation.

Remarks: 1) Can apply the Thm to diagonal action of $\hat{U}$ on $X \times \mathbb{P}^1$ and linearisation $L \otimes \mathcal{O}_{\mathbb{P}^1}(m)$ with m big to define

$$X^s \times \{[1:1]\} = (X \times \mathbb{P}^1)^{s,\hat{U}} \cap (X \times \{[1:1]\})$$

with similar nice properties.

Theorem (B.,Doran, Hawes, Kirwan 2015)

Let U be a graded unipotent lin. alg. group over $k=\bar{k}$ and $\hat{U}=U\rtimes\mathbb{G}_m$. Let X be an irreducible normal $\hat{U}$-variety and $L\to X$ an ample linearisation of the $\hat{U}$ action. Twist the linearisation by a character $\chi:\hat{U}\to\mathbb{G}_m$ of $\hat{U}/U$ such that 0 lies just above the lowest weight for the $\mathbb{G}_m$ action on X. If

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2) Similar construction for any graded non-reductive group $U\rtimes R$.

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$f_\xi(z) = zf'(0) + \frac{z^2}{2!}f''(0) + \ldots + \frac{z^k}{k!}f^{(k)}(0) \in J_k(1, n)$ and
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Then $f \circ \varphi(z) =$

$$= (f'(0)\alpha_1)z + (f'(0)\alpha_2 + \frac{f''(0)}{2!}\alpha_1^2)z^2 + \ldots + \left(\sum_{i_1+\ldots+i_l=k} \frac{f^{(l)}(0)}{l!}\alpha_{i_1}\ldots\alpha_{i_l} \right) z^k =$$

$$= (f'(0), \ldots, f^{(k)}(0)/k!) \cdot \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \ldots & 2\alpha_1\alpha_{k-1} + \ldots \\ 0 & 0 & \alpha_1^3 & \ldots & 3\alpha_1^2\alpha_{k-2} + \ldots \\ 0 & 0 & 0 & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \alpha_1^k \end{pmatrix}$$

the (i,j) entry is $p_{i,j}(\bar{\alpha}) = \sum_{a_1+a_2+\ldots+a_i=j} \alpha_{a_1}\alpha_{a_2}\ldots\alpha_{a_i}$.

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Ultimate goal: Determine the geometric and topological invariants of $X^{[n]}$.
For surfaces these were extensively studied such as **Betti numbers** (Ellingsrud, Stromme, Göttsche), **Hodge numbers** (Sörgel, Göttsche), **cohomology ring** (Nakajima, Grojnowski), **Chern numbers of tautological bundles** (Lehn, Rennemo, Marian-Oprea-Pandharipande).

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A type $Q \subset X_p^{[k]}$ is a constructible subset which is union of isomorphism classes: $\xi \in Q, \xi \simeq \xi' \Rightarrow \xi' \in Q$.

Ultimate goal: Determine the geometric and topological invariants of $X^{[n]}$.
For surfaces these were extensively studied such as **Betti numbers** (Ellingsrud, Stromme, Göttsche), **Hodge numbers** (Sörgel, Göttsche), **cohomology ring** (Nakajima, Grojnowski), **Chern numbers of tautological bundles** (Lehn, Rennemo, Marian-Oprea-Pandharipande).
When $\dim(X) > 2$ not much is known: Rennemo, Tzeng, motivic DT invariants (Behrend-Bryan-Szendroi).
Tautological integrals:
F–rank r bundle (loc. free sheaf) on $X \rightsquigarrow F^{[k]}$–rank rk bundle over $X^{[k]}$:
Fibre over $\xi \in X^{[k]}$ is $F \otimes \mathcal{O}_\xi = H^0(\xi, F|_\xi)$.

Equivalently: $F^{[k]} = q_* p^*(F)$ where $X^{[k]} \times X \supset \mathcal{Z} \xrightarrow{q} X^{[k]}$.

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{q} & X^{[k]} \\ \downarrow{\scriptstyle p} & & \\ X & & \end{array}$$

Following Rennemo (2014) we call $\mathcal{X} \subset X^{[k]}$ a **geometric subset** when

$$\mathcal{X} = \{\xi \in X^{[k]} : \xi = \xi_1 \cup \ldots \cup \xi_s \text{ where } \xi_i \in X_{p_i}^{[k_i]} \text{ is of type } Q_i\}.$$

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is the curvilinear component.
Want: Closed formula for $\int_{\widetilde{CX}^{[k]}} P(c_i(F^{[n]}))$ for some geometric partial resolution $\widetilde{CX}^{[k]} \to \overline{CX}^{[k]}$.

Geometric resolutions of $CX^{[k]}$ in two steps

Step 1: $X^{[k_1,\ldots,k_t]}=\{(\xi_1\subset\xi_2\subset\ldots\subset\xi_t):\xi_i\in X^{[k_i]}\}\subset X^{[k_1]}\times\ldots\times X^{[k_t]}$

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Define

$$\tilde{\rho} : CX_p^{[k+1]} \hookrightarrow X^{[2,\ldots,k]}$$

$$\xi \mapsto \text{ Unique flag } (\xi_1 \subset \ldots \subset \xi_k) \text{ such that } \xi_i \in CX_p^{[i]}$$

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$$f = f_\xi \mapsto (\mathrm{Ann}(f_{[1]})^* \subset \mathrm{Ann}(f_{[2]})^* \subset \ldots \subset \mathrm{Ann}(f_{[k]})^*) =$$

$$= (f') \subset (f', f'' + (f')^2) \subset \ldots \subset (f', f'' + (f')^2, \ldots, f^{[k]} + \sum_{\Sigma a_i = k} (f^{[i]})^{a_i})$$

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Step 2 Assume $k\le n$. Let $\pi:J_k(n,1)^*\simeq\oplus_{i=1}^k\mathrm{Sym}^i\mathbb{C}^n\to\mathbb{C}^n$ be the projection.

$$\widetilde{CX}_p^{[k+1]}=\{(\tilde{\xi}_1\subset\ldots\subset\tilde{\xi}_k),(V_1\subset\ldots\subset V_k):\pi(\tilde{\xi}_i)\subset V_i\}\subset\widehat{CX}_p^{[k+1]}\times\mathrm{Flag}(k,J_k(n,1)$$

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Equivalently: Let $B_n \subset \mathrm{GL}(n)$ be the upper Borel and $z_k = \rho(e_1, \ldots, e_k)$ the base point in $\mathrm{Flag}(k, J_k(n,1)^*)$. Then

$$\widetilde{CX}_p^{[k+1]} = \mathrm{GL}(n) \times_{B_n} \overline{B_n \cdot z_k} \to \overline{\mathrm{GL}(n) \cdot z_k} = \widehat{CX}_p^{[k+1]}$$

Set-up: $k \leq n$. $\widetilde{CX}^{[k+1]} \to CX^{[k+1]} \to X$ partial resolution. Fibre over $p \in X$

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Let $T = (\mathbb{C}^*)^n$ act on $T_p^* X_p$ with weights $\lambda_1, \ldots, \lambda_n$.

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Equivariant localisation on $\widetilde{CX}_p^{[k]}$

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Still, we can prove the following vanishing property

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Theorem (B-Szenes 2012, B. 2015)

Let $n \leq k$. Only one fixed point contributes to the iterated residue. This distinguished fixed point corresponds to the sequence $\pi = (1, 2, \ldots, k)$ which is the ideal $\xi = m_k^2 \subset m_n^2$

Let $Q_k(z_1,\ldots,z_k) = \mathrm{mdeg}(T_{[1],\ldots,[k]}\mathcal{B}_k, T_{[1],\ldots,[k]}\mathrm{Flag})$. Then

Theorem

For any n, k (we can drop $k \le n$ condition!)

$$\int_{\widetilde{CX}^{[k+1]}} P(c_i(F)) = \int_X \mathrm{Res}_{\mathbf{z}=\infty} \frac{\prod_{i<j}(z_i - z_j) Q_k(z) P(c_i(z_i + \theta_j, \theta_j)) d\mathbf{z}}{\prod_{i+j\le l\le k}(z_i + z_j - z_l)(z_1 \ldots z_k)^n} \prod_{i=1}^{k} s_X\left(\frac{1}{z_i}\right)$$

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Features:

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q_l^{mr} has weight $(z_m + z_r - z_l)$. Let

$$\epsilon = \sum_{m=1}^{k} \sum_{r=1}^{k-m} q_{m+r}^{mr} \subset W = \bigoplus_{1 \le m+r \le l \le k} \mathbb{C}q_l^{mr} \subset \mathrm{Hom}(\mathbb{C}^k, \mathrm{Sym}^2\mathbb{C}^k)$$

Then $Q_k(z) = \mathrm{mdeg}(\overline{B_k\epsilon}, W)$. In particular $Q_{1,2,3} = 1, Q_4 = 2z_1 + z_2 - z_4$.