Thom polynomials of singularities and the Green-Griffiths conjecture
Joint work with Frances Kirwan

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- $\mathcal{D} = \mathrm{Diff}(\mathbb{C}^n, 0) \times \mathrm{Diff}(\mathbb{C}^m, 0) \curvearrowright J_k(n,m)$ with

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Thom's principle: there is a well-defined polynomial

$$MD_A^{n \to m} \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_m]^{S_n \times S_m}$$

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Haefliger and Kosinski:

$$c(q) = c_0 + c_1 q + c_2 q^2 + \ldots = \frac{c(f^*(TM))}{c(TN)} = \frac{\prod_{s=1}^m (1 + \theta_s q)}{\prod_{i=1}^n (1 + \lambda_i q)}$$

Then

$$MD_A(TN, f^*(TM)) = Tp_A^{m-n}(c_1, c_2, \ldots)$$

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3. $H^*_G(V) = H^*_G(pt)$ is the G-equivariant cohomology ring of V. $H^*_{GL(d)}(pt) = \mathbb{C}[x_1, \ldots, x_d]^{S_d}$.

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Vergne: there is an equivariant Thom class:

$$Thom_G(V) \in H_G^{\dim V}(V)$$

s.t

$$\mathrm{mdeg}[\Sigma, V] = \int_\Sigma Thom_G(V).$$

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normalized For T-invariant linear subspaces of V the invariant is defined to equal to the product of weights in the normal direction.

Example: $(\mathbb{C}^*)^3$ acts on $\mathbb{C}^4$ with weights $\eta_1, \ldots \eta_4$. Let $\eta_1 + \eta_2 = \eta_3 + \eta_4$, and

$$\Sigma = \mathrm{Spec}(\mathbb{C}[y_1, y_2, y_3, y_4]/(y_1 y_2 - y_3 y_4)).$$

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Take the flat deformation

$$\Sigma_t = \mathrm{Spec}(\mathbb{C}[y_1, y_2, y_3, y_4]/(y_1 y_2 - t y_3 y_4)),$$

For $t = 0$ $\Sigma_0 = \{y_1 y_2 = 0\}$, so normalisation says

$$\mathrm{mdeg}[\Sigma, \mathbb{C}^4] = \eta_1 + \eta_2 = \eta_3 + \eta_4.$$

Recall:

- $V = J_k(n,m) = \{(p_1,\ldots,p_m) \in \mathrm{Poly}(\mathbb{C}^n,\mathbb{C}^m) : \deg p_i \le k, p_i(0)=0\}$
- $\Sigma_k = \left\{p \in J_k(n,m) : \mathbb{C}[x_1,\ldots,x_n]/\langle p_1,\ldots,p_m\rangle \cong z\mathbb{C}[z]/z^{k+1}\right\}$
- $\mathcal{D} = \mathrm{Diff}(\mathbb{C}^n,0) \times \mathrm{Diff}(\mathbb{C}^m,0) \curvearrowright J_k(n,m)$ with

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Note that $\mathrm{GL}_n \times \mathrm{GL}_m \subset \mathcal{D}$

Thom's principle again:

$$MD_k^{n\to m} = \mathrm{mdeg}^{\mathrm{GL}_n\times\mathrm{GL}_m}[\Sigma_k, J_k(n,m)].$$

Theorem (The test curve model of Porteous and Gaffney)

$$\Sigma_k(n,m) \doteq \left\{\Psi \in J_k(n,m) | \exists \gamma \in J_k^{reg}(1,n) \text{ such that } \Psi \circ \gamma = 0\right\}.$$

$$(\mathbb{C},0) \xrightarrow{\gamma} (\mathbb{C}^n,0) \xrightarrow{\Psi} (\mathbb{C}^k,0) \tag{1}$$

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Proposition

$\Sigma_k^0 = \{\Psi : \dim \ker \Psi = 1\} \doteq \Sigma_k$ *fibers with linear fibres over* $J_k^{reg}(1,n)/\mathbf{G}_k$.

- identify $J_k(1,n)$ with $\mathrm{Hom}(\mathbb{C}^k,\mathbb{C}^n)$;

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- Then $\mathbf{G}_k$ acts on $J_k(1,n)$ on the right by

$$\left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \ldots & 2\alpha_1\alpha_{k-1} + \ldots \\ 0 & 0 & \alpha_1^3 & \ldots & 3\alpha_1^2\alpha_{k-2} + \ldots \\ 0 & 0 & 0 & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \alpha_1^k \end{pmatrix} : \alpha_1 \in \mathbb{C}^* \alpha_i \in \mathbb{C} \right\};$$

where the polynomial in the (i,j) entry is

$$p_{i,j}(\bar{\alpha}) = \sum_{a_1+a_2+\ldots+a_i=j} \alpha_{a_1}\alpha_{a_2}\ldots\alpha_{a_i}.$$

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$$\rho : \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n) \to \mathrm{Hom}(\mathbb{C}^k, \mathrm{Sym}_{\leq k}\mathbb{C}^n)$$

$$\rho(v_1, \ldots, v_k) = (v_1, v_2 + v_1^2, \ldots, \sum_{a_1+a_2+\ldots+a_i=j} v_{a_1} v_{a_2} \ldots v_{a_i}, \ldots),$$

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It descends to an injective map

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3. $\mathrm{im}(\varrho) \subset \mathrm{Grass} \subset \mathbb{P}[\wedge^k(\mathrm{Sym}_{\leq k}\mathbb{C}^k)]$ *is a subset of the affine chart* A_k *defined by* $\det(v_1, \ldots, v_k) \neq 0$.

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4. *The boundary orbits of* $\overline{\mathrm{im}(\varrho)} \subset A_k$ *have codimension at least* 2, *so* $\overline{\mathrm{im}(\rho)} = \mathrm{Spec}(\mathbb{C}[A_k]^{\mathbf{G}_k})$.

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$$Tp_k^{m-n} = \int_{\Sigma_k} \mathrm{Thom}_k(n,m) = \int_{\mathrm{Flag}_k(\mathbb{C}^n)} \int_{F \in \mathrm{Flag}_k(\mathrm{Sym}_{\leq k}\mathbb{C}^n)} \mathrm{Thom}|_F$$

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$$Tp_k^{m-n} = \mathrm{Res}_{\mathbf{z}=\infty} \frac{\prod_{m<l}(z_m - z_l)\, \mathcal{Q}_k(z_1 \dots z_k)}{\prod_{m+r\le l\le k}(z_m + z_r - z_l)} \cdot \prod_{l=1}^{k} c\left(\frac{1}{z_l}\right)\, z_l^{m-n}\, dz_l$$

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Conjecture (Rimanyi, 1998)

$$Tp_k^{m-n} \in \mathbb{N}[c_1, \dots, c_{k(m-n+1)}] \; i.e \; \frac{\prod_{m<l}(z_m - z_l)\, \mathcal{Q}_k(z_1 \dots z_k)}{\prod_{m+r \le l \le k}(z_m + z_r - z_l)} > 0.$$

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  (degeneracy of the Kobayashi pseudo-metric)
- Hyperbolic varieties are especially interesting for their expected diophantine properties:

Conjecture (S. Lang)

If a projective variety X defined over $\mathbb{Q}$ is hyperbolic, then $X(\mathbb{Q})$ is finite.

Conjecture (Green-Griffith-Lang, 1979)

Let X be a projective variety of general type. Then there exists an algebraic variety $Y \subsetneq X$ such that for all non-constant holomorphic $f : \mathbb{C} \to X$ one has $f(\mathbb{C}) \subset Y$.

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- Diverio, Merkel, Rousseau (2009) $deg(X) > 2^{n^5} \Rightarrow GGL$.

Strategy(Demailly '95, Siu '96, DMR '09)

- $$f : \mathbb{C} \to X, \quad t \to f(t) = (f_1(t), f_2(t), \dots, f_n(t))$$

  a curve written in some local holomorphic coordinates $(z_1, \dots, z_n)$ on X. Define the bundle

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- The group of reparametrizations $\mathbf{G}_k=J_k^{reg}(1,1)$ acts fiberwise on J_kX. The fibres of J_kX can be identified with $J_k(1,n)$, the action is linearised as before.
- $\mathbf{G}_k=\mathbb{C}^*\ltimes U_k$, and for $\lambda\in\mathbb{C}^*$

  $$(\lambda\cdot f)(t)=f(\lambda\cdot t),\ \text{so}\ \lambda\cdot(f',f'',\ldots f^{(k)})=(\lambda f',\lambda^2f'',\ldots\lambda^kf^{(k)}).$$

- Consider algebraic differential operators = polynomial functions on J_kX. Locally in multi-index notation

$$Q(f', f'', \ldots, f^{(k)}) = \sum_{\alpha_i \in \mathbb{N}^n} a_{\alpha_1,\alpha_2,\ldots\alpha_k}(f(t))(f'(t)^{\alpha_1} f''(t)^{\alpha_2} \cdots f^{(k)}(t)$$

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  where $a_{\alpha_1, \alpha_2, \ldots \alpha_k}(z)$ are holomorphic coefficients on X and $t \to z = f(t)$ is a curve.

- Q is homogeneous of weighted degree m under the $\mathbb{C}^*$ action iff

$$Q(\lambda f', \lambda^2 f'', \ldots \lambda^d f^{(d)}) = \lambda^m Q(f', f'', \ldots f^{(k)}).$$

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- *(Demailly, '95)*
  The bundle of invariant jet differentials of order k *and weighted degree* m *is the subbundle* $E_{k,m} \subset E^{GG}_{k,m}$*, whose elements are invariant under arbitrary changes of parametrization, i.e. for* $\phi \in \mathbf{G}_d$

$$Q((f\circ\phi)', (f\circ\phi)'', \ldots, (f\circ\phi)^{(k)}) = \phi'(0)^m Q(f', f'', \ldots, f^{(k)}).$$

Therefore:

$$\oplus_m (E_{k,m})_x = \oplus_m (E^{GG}_{k,m})^{\mathbb{U}}_x = \mathcal{O}((J_k X)_x)^{\mathbf{G}_k} = \mathcal{O}(J_k(1,n)/\mathbf{G}_k)$$

Applying our previous contstruction fibrewise we get

Proposition

1. *The quotient $J_kX/\mathbf{G}_k$ has the structure of a locally trivial bundle over X, and there is a holomorphic embedding*

$$\phi^{\mathbb{P}} : J_kX/\mathbf{G}_k \hookrightarrow \mathbb{P}(\wedge^k(T_X^* \oplus \mathrm{Sym}^2(T_X^*) \oplus \ldots \oplus \mathrm{Sym}^k(T_X^*))$$

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2. $$(\pi_k)_*\mathcal{O}_{\mathcal{X}_k}(m) = \mathcal{O}(E_{k,m\binom{k+1}{2}})$$

where $\pi_k : \mathbb{P}(\wedge^k(T_X^* \oplus \mathrm{Sym}^2(T_X^*) \oplus \ldots \oplus \mathrm{Sym}^k(T_X^*))) \to X$ *is the projection.*

Theorem (Fundamental vanishing theorem)

(Green-Griffiths '78, Demailly '95, Siu '96) Let $P \in H^0(X, E_{k,m} \otimes \mathcal{O}(-A))$ be a global algebraic differential operator whose coefficients vanish on some ample divisor A. Then for any $f : \mathbb{C} \to X$, $P(f_{[k]}(\mathbb{C})) \equiv 0$.(note $f_{[k]}(\mathbb{C}) \subset J_k X$)

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then $f_{[k]}(\mathbb{C}) \subset Z_\sigma$, where $Z_\sigma \subset \mathcal{X}_d$ is the zero divisor.
2. *If σ_j is a basis of of sections then the image $f(\mathbb{C})$ lies in $Y = \pi_k(\bigcap Z_{P_j})$, hence GGL holds if there are enough independent differential equations so that $Y = \pi_k(\bigcap(Z_{P_j})) \subsetneq X$.*

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Theorem (DMR, 2009)

Assume that $n = k$, and there exist a $\delta = \delta(n) > 0$ and $D = D(n, \delta)$ such that

$$H^0(\mathcal{X}_k, \mathcal{O}_{\mathcal{X}_k}(m) \otimes \pi^* K_X^{-\delta m}) \simeq H^0(X, E_{k,m\binom{k+1}{2}} T_X^* \otimes K_X^{-\delta m}) \neq 0$$

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whenever $\deg(X) > D(n, \delta)$ *provided that* $m > m_{D,\delta,n}$ *is large enough. Then GGL holds for*

$$\deg(X) \geq \max(D(n, \delta), \frac{n^2 + 2n}{\delta} + n + 2).$$

We use the algebraic Morse inequalities of Demailly. Let $L \to X$ be a holomorphic line bundle given as

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$$\sum_{j=0}^{q}(-1)^{q-j}h^j(X, L^{\otimes m} \otimes E) \leq r\frac{m^n}{n!}\sum_{j=0}^{q}(-1)^{q-j}\binom{n}{j}F^{n-j} \cdot G^j + o(m^n).$$

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$q = 2$ asserts

$$F^n - nF^{n-1}G > 0 \Rightarrow H^0(L^{\otimes m}) \neq 0 \text{ for } m \gg 0.$$

Proposition (B-Kirwan)

F and G are nef bundles in the following equality:

$$\underbrace{\mathcal{O}_{\mathcal{X}_n}(1) \otimes \pi^* K_X^{-\delta\binom{n+1}{2}}}_{L} = \underbrace{(\mathcal{O}_{\mathcal{X}_n}(1) \otimes \pi^* \mathcal{O}_X(2n^2))}_{F} \otimes$$

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- $h = c_1(\mathcal{O}_X(1)), c_1(K_X) = -c_1(X) = (d-n-2)h$, and $\mathcal{O}_{\mathcal{X}_n}(1) = \det \tau$, where $\tau \to \mathcal{X}$ is the tautological n-bundle.

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- $\dim(\mathcal{X}_n) = n^2$, and we want to compute the integral

$$\int_{\mathcal{X}_n} (c_1(\det\tau) + 2n^2\pi^* h)^{n^2} -$$

$$-n^2(c_1(\det\tau)+2n^2\pi^* h)^{n^2-1}(2n^2\pi^* h+\delta\binom{n+1}{2}(d-n-2)h)$$

Applying the snowman-model, and after proving the stronger vanishing property we get

Residue formula for the Demailly intersection number

$$I = \int_X \mathrm{Res}_{\mathbf{z}=\infty} \frac{\prod_{m<l}(z_m - z_l)\, \mathcal{Q}_n(z_1 \ldots z_n) R(\mathbf{z}, h, d, \delta)}{\prod_{l=1}^n \prod_{m=1}^{l-1} \prod_{r=1}^{\min(m,l-m)} (z_m + z_r - z_l)(z_1 \ldots z_n)^n} \cdot \prod_{l=1}^n \left(1 + \frac{dh}{z_l}\right) \prod_{l=1}^n \left(1 - \frac{h}{z_l} + \frac{h^2}{z_l^2} - \ldots\right)^{n+2}$$

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where

$$R(\mathbf{z}, h, d, \delta) = (-z_1 - \dots - z_n + 2n^2 h)^{n^2} - - n^2(-z_1 - \dots - z_n + 2n^2 h)^{n^2-1}(2n^2 h + \delta \binom{n+1}{2}(d-n-2)h)$$

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- $p(n, d, \delta) = a_n(n, \delta)d^n + \ldots + a_0(n, \delta)$ is a degree-n polynomial in $d = \deg(X)$.
- The leading coefficient is

$$a_n(n, \delta) = \left(1 - n^2 \binom{n+1}{2} \delta\right) \Theta(n),$$

where

$$\Theta(n) = \mathrm{coeff}_1 \frac{Q(\mathbf{z}) \prod_{m<l} (z_m - z_l)(z_1 + \ldots + z_n)^{n^2}}{\prod_{m+r \leq l \leq n} (z_m + z_r - z_l)(z_1 \ldots z_n)^n}$$

Corollary

For $\delta < \frac{2}{n^3(n+1)}$ the leading coefficient of the Demailly intersection number is positive.

More information about $Q(\mathbf{z})$ is needed!

Theorem (B, 2009)

Rimanyi conjecture for Thom polynomials of A_n singularities $\Rightarrow$ GGL is true for $d = \deg(X) > n^6$.

Remark: This method works when X is a complete intersection.

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Thank you for your attention