Reductive versus non-reductive group actions in algebraic geometry

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Moduli spaces in algebraic geometry

Central problem of geometry/topology/physics: classification of objects.
Moduli problem is about grouping mathematical objects in equivalence classes.
Ingredients:

- objects;
- symmetry of the objects, that is, an equivalence relation $\sim$;
- families of objects parametrised by base spaces S (typically $\pi : \mathcal{X} \to S$ with $\pi^{-1}(s)$ the object parametrised by $s \in S$).

Solution of the moduli problem consists of:

1. a (coarse) moduli space $\mathcal{M} = \{\text{objects}\} / \sim$ with nice geometric structure.
2. universal family: family $\pi : \mathcal{U} \to \mathcal{M}$ of objects such that $\pi^{-1}(p)$ is an object representing the equivalence class $p \in \mathcal{M}$ and any family $\mathcal{X} \to S$ is a subfamily of $\mathcal{U}$ (that is, $\mathcal{X} = f^*\mathcal{U}$ for some $f : S \to \mathcal{M}$)
   Note: existence of a universal family makes sure that the classification remembers the structure of the curves. E.g continuity of π guarantees that objects which are similar will correspond to points in the moduli space which are close to each other.
   Problem: Universal family does not necessarily exist, so people study STACKS. If it does, we call it a fine moduli space. We only look at coarse moduli space today.

Triangles/similarities are parametrised by

$$\mathcal{M} = \{(a,b,c) : a \leq b \leq c, a+b+c = 1, c < a+b\}$$

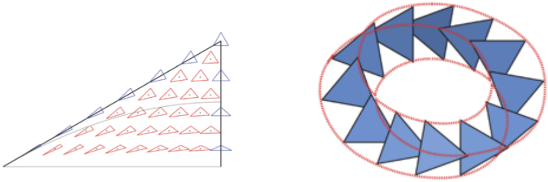


Figure : $\mathcal{M}$ and a nontrivial family of equilateral triangles over S^1

Let $\mathcal{T} \to S^1$ be a nontrivial family of equilateral triangles. The pull-back along the trivial map $S^1 \to$ (top vertex of $\mathcal{M}$) is the trivial family, this is $\neq \mathcal{T}$. This shows that there is no universal family of triangles over $\mathcal{M}$.
Reason for this: symmetries of $\pi^{-1}(p)$ are different as we move around with p on $\mathcal{M}$.

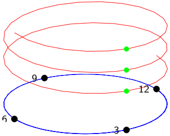
$\sim$ often comes from a group action on a parameter space of objects (group of symmetries):

$$x \sim y \Leftrightarrow \exists g \in G \text{ such that } g \cdot x = y$$

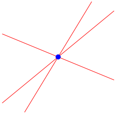
Moduli $= (\textit{parameter space})/(\textit{group action}) = \{\textit{orbits}\}$.

Examples:

1. **Clock:** $S^1 \cong \mathbb{R}/\mathbb{Z}$ where $\mathbb{Z}$ acts on $\mathbb{R}$ by translation:



   $k \cdot x = x + 2\pi k$ for $k \in \mathbb{Z}, x \in \mathbb{R}$.
2. **Complex projective space:** $\mathbb{P}^n \cong (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$
   Orbits of $\mathbb{C}^*$ on $\mathbb{C}^{n+1} \setminus \{0\}$=lines through the origin:



Moduli spaces in algebraic geometry are often constructed as quotients of algebraic varieties by actions of linear algebraic groups. We work over $\mathbb{C}$.
Observation: A linear algebraic group G is a semi-direct product of a **unipotent group** U (called **unipotent radical** of G) by a **reductive group** R:

$$G = U \rtimes R.$$

- Unipotent group: $(u-1)^n = 0$ for $u \in U, n >> 0$. Think of *group of strictly upper triangular matrices.*
- Reductive group: $R = K_{\mathbb{C}}$ is the **complexification** of a **maximal compact subgroup** K of R. That is,

  $$\mathrm{Lie}(K_{\mathbb{C}}) \cong \mathrm{Lie}(K) \otimes_{\mathbb{R}} \mathbb{C}.$$

  E.g. $K = U(n)$ and $K_{\mathbb{C}} = GL(n)$ or $K = SU(n)$ and $K_{\mathbb{C}} = SL(n)$.
  Alternative definition: G is reductive $\Leftrightarrow$ every representation of G is the direct sum of irreducibles.

(1) Automorphism group of complex projective space $\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$ is $PGL(n+1)$, which is reductive.

(2) Automorphism group of the weighted projective plane $\mathbb{P}(1,1,2) = (\mathbb{C}^3 \setminus \{0\})/\mathbb{C}^*$, where $\mathbb{C}^*$ acts on $\mathbb{C}^3$ with weights 1,1 and 2, is

$$\mathrm{Aut}(\mathbb{P}(1,1,2)) \cong R \ltimes U$$

with

- $R \cong GL(2) \times_{\mathbb{C}^*} \mathbb{C}^* \cong GL(2)$ reductive
- $U \cong (\mathbb{C}^+)^3$ unipotent acting via

$$(x,y,z) \mapsto (x,y,z+\lambda x^2+\mu xy+\nu y^2) \text{ for } (\lambda,\mu,\nu) \in \mathbb{C}^3.$$

$$\mathrm{Aut}(\mathbb{P}(1,1,2)) = \left(\begin{array}{ccc|c} & & & \nu \\ & \mathrm{Sym}^2 GL(2) & & \mu \\ & & & \lambda \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

Even very simple linear algebraic groups are **not compact**; e.g. $\mathbb{C}^*$. So when they act on algebraic varieties, the **orbits** are typically **not closed** and closures of orbits contain other orbits.

Example (1) $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1}$ by scalar multiplication.
Note: orbit $\{0\}$ is contained in the closure of *every* other orbit $\mathbb{C}^* v$ for $v \in \mathbb{C}^{n+1}$.
So the topological quotient $\mathbb{C}^{n+1}/\mathbb{C}^*$ does not have a nice geometric structure (it is not Hausdorff).
But $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^* \cong \mathbb{P}^n$ does.
This suggests we should drop 'small' orbits?

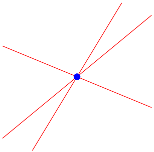


Figure : Orbits of $\mathbb{C}^*$ action on $\mathbb{C}^2$

Example (2): $\mathbb{C}^*$ acts on $\mathbb{C}^2$ with weights $1, -1$:

$$t \cdot (a, b) = (t^{-1}a, tb) \text{ for } t \in \mathbb{C}^*, (a, b) \in \mathbb{C}^2.$$

Note: the closures of O_2, O_3 contain O_1.
How to handle this orbit space?
Idea: G action on X induces G action on $A(X) = \{f : X \to \mathbb{C}\}$ via $(g \cdot f)(x) = f(g \cdot x)$. We expect: $A(X/G) = A(X)^G$ and $A(X)^G \hookrightarrow A(X)$ induces $X \to X/G$. Polynomial functions on $\mathbb{C}^2$ form the ring $\mathbb{C}[x, y]$. $\mathbb{C}^*$-action on $\mathbb{C}[x, y]$ is $t \cdot x = tx$ and $t \cdot y = t^{-1}y$ for all $t \in \mathbb{C}^*$. Then $\mathbb{C}[x, y]^{\mathbb{C}^*} = \mathbb{C}[xy] \cong \mathbb{C}[w]$ with the map $\mathbb{C}[w] \hookrightarrow \mathbb{C}[x, y]$ such that $w \mapsto xy$. This defines the quotient map $\mathbb{C}^2 \to \mathbb{C}^1$ such that $(a, b) \mapsto ab$. The orbits are $\{xy = s \neq 0\} \mapsto s$, $\{(a, 0), a \neq 0\} \mapsto 0$, $\{(0, b), b \neq 0) \mapsto 0$ and $\{(0, 0)\} \mapsto 0$

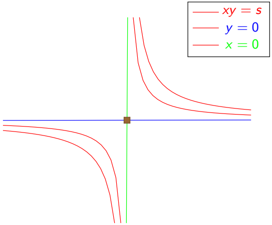


Figure : Orbits: $O_1 = (0, 0)$, $O_2 = \{(a, 0) : a \in \mathbb{C}^*\}$, $O_3 = \{(0, b) : b \in \mathbb{C}^*\}$, $O(s) = \{xy = s\}$

Conclusion: Quotient map is not injective on orbits, the categorical quotient $\mathbb{C}^1$ is not "the set of orbits". Two orbits map to the same point iff the closure of their orbits has a nontrivial intersection. $\mathbb{C}^1$ parametrises closed orbits.

- G complex reductive group
- X complex projective variety acted on by G We require a **linearisation** of the action (i.e. an ample line bundle L on X and a lift of the action to L; think of $X \subseteq \mathbb{P}^n$ and the action given by a representation $\rho : G \to GL(n+1)$).
- $$\begin{array}{ccccl} X & \rightsquigarrow & A(X) & = & \mathbb{C}[x_0, \ldots, x_n]/\mathcal{I}_X \\ \Big\downarrow & & & & = \bigoplus_{k=0}^{\infty} H^0(X, L^{\otimes k}) \\ & & \cup\!| & & \\ X/\!/G & \Leftarrow & A(X)^G & = & \mathbb{C}[y_0, \ldots, y_m]/\mathcal{I}^G \\ & & & & \text{algebra of invariants} \end{array}$$

G reductive implies that $A(X)^G$ is a *finitely generated* graded algebra and

$$X/\!/G = \mathrm{Proj}(A(X)^G) := \{[y_0 : \ldots : y_m] : f(y_0, \ldots, y_m) = 0 \text{ for } f \in \mathcal{I}^G\}$$

is a projective variety. The rational quotient map $X \dashrightarrow X/\!/G$ is defined as

$$x = [x_0 : \ldots : x_n] \mapsto [y_0(x) : y_1(x) : \ldots : y_m(x)],$$

this is well defined at points of X where not all invariants vanish.

The rational map $X \dashrightarrow X//G$ fits into a diagram

$$\begin{array}{rcccl} & X & -- \to & X//G & \text{cx proj variety} \\ & \bigcup & & \| & \\ \text{semistable} & X^{ss} & \xrightarrow{\text{onto}} & X//G & \\ & \bigcup & & \bigcup & \text{open} \\ \text{stable} & X^{s} & \longrightarrow & X^{s}/G & \end{array}$$

where the morphism $X^{ss} \to X//G$ is G-invariant and surjective.
Topologically: $\boxed{X//G = X^{ss}/\sim}$ where $x \sim y \Leftrightarrow \overline{Gx} \cap \overline{Gy} \cap X^{ss} \neq \emptyset$.
As a set: $X//G = \{\text{closed orbits of } G \text{ on } X\}$.

Question: Can we describe which points of $X \subseteq \mathbb{P}^n$ are stable and which are semistable for the G-action without having to know the invariant ring $A(X)^G$, that is, what all the G-invariant sections of powers of L are?
Answer: Hilbert-Mumford numerical criteria. This is source of effectiveness of reductive GIT.
Let $T = (\mathbb{C}^*)^r$ be a maximal torus of G, choose homogeneous coordinates on $\mathbb{P}^n$ so that T acts diagonally with weights $\alpha_0, \ldots, \alpha_n \in \mathrm{Lie}(T)^*$, that is

$$[x_0 : \ldots : x_n] \mapsto [t^{\alpha_0} x_0 : \ldots : t^{\alpha_n} x_n] \text{ for some } \alpha_i \in \mathbb{C}^r.$$

Then $x = [x_0 : \cdots : x_n] \in X$ is **semistable (respectively stable) for the action of** T iff 0 lies in (respectively lies in the interior of) the convex hull in $\mathrm{Lie}(T)^*$ of the set

$$\{\alpha_j : x_j \neq 0\}.$$

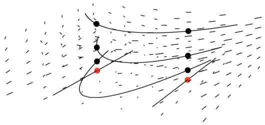
Moreover x is **semistable (respectively stable) for the action of** G iff gx is semistable (respectively stable) for the action of T for **every** $g \in G$.

Question: Can we define a sensible 'quotient' variety $X//G$ when G is not reductive? If so, can we understand it geometrically?

Problems:

1. $A(X)^G$ is not necessarily finitely generated (Nagata 1957) so $Proj(A(X)^G)$ is not a projective variety. In fact G is reductive if and only if $A(X)^G$ is finitely generated for all X.
2. Even if $A(X)^G$ fin. gen. the quotient map $q : X \to Proj(A(X)^G)$ is not necessarily surjective, the image is just constructible.
3. Topology of unipotent actions is problematic: Unipotent orbits cannot necessarily be separated with invariants.

   Example: $\mathbb{C}^+ = \{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{C}\}$ acts on $X = \mathrm{Sym}^2\mathbb{C}^2$. Let x_0, x_1, x_2 be basis of X. Then $A(X)^{\mathbb{C}^+} = \mathbb{C}[x_0, x_1^2 - x_0 x_2]$ and the orbit space looks like:



Definition

Call a unipotent linear algebraic group U **graded** if there is a homomorphism $\lambda : \mathbb{C}^* \to Aut(U)$ with the weights of the $\mathbb{C}^*$ action on $Lie(U)$ all **strictly positive**. Then let

$$\hat{U} = U \rtimes \mathbb{C}^* = \{(u,t) : u \in U, t \in \mathbb{C}^*\}$$

with multiplication $(u,t) \cdot (u',t') = (u(\lambda(t)(u')), tt')$.
We say that $H = U \rtimes R$ is graded if there is a central $\mathbb{C}^* \subset R$ such that

$$\hat{U} = U \rtimes \mathbb{C}^* \subset H \text{ is graded}$$

Example: Recall that

$$\mathrm{Aut}(\mathbb{P}(1,1,2)) = \left(\begin{array}{ccc|c} & \mathrm{Sym}^2 GL(2) & & \begin{matrix}\nu\\ \mu\\ \lambda\end{matrix} \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

The central one-parameter subgroup $\mathbb{C}^*$ of $R \cong GL(2)$ acts on $Lie(U)$ with all positive weights, and the associated extension is

$$\hat{U} = U \rtimes \mathbb{C}^* = \begin{pmatrix} t^2 & 0 & 0 & \nu \\ 0 & t^2 & 0 & \mu \\ 0 & 0 & t^2 & \lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Theorem (Berczi, Doran, Hawes, Kirwan 2015-16)

Let U be graded unipotent acting linearly on a projective variety X, and suppose that the action extends to $\hat{U} = U \rtimes \mathbb{C}^*$. Let $X^{\mathbb{C}^*}_{\min}$ be the union of those connected components of the fixed point set $X^{\mathbb{C}^*}$ where $\mathbb{C}^*$ acts on the fibres of L with minimum weight. Suppose that

$$(*) \quad x \in X^{\mathbb{C}^*}_{\min} \Rightarrow \dim \mathrm{Stab}_U(x) = \min_{y \in X} \dim \mathrm{Stab}_U(y)$$

Twist the action of $\hat{U}$ by a (rational) character so that 0 lies in the lowest bounded chamber for the $\mathbb{C}^*$ action on X. Then

1. *the ring $A(X)^{\hat{U}}$ of $\hat{U}$-invariants is* **finitely generated**, *so that $X//\hat{U} = Proj(A(X)^{\hat{U}})$;*
2. *$X^{ss,\hat{U}} = X^{s,\hat{U}}$ and $X//\hat{U} = X^{s,\hat{U}}/\hat{U}$ is a* **geometric quotient** *of $X^{s,\hat{U}}$ by $\hat{U}$.*

Moreover without condition (*) there is a projective completion of $X^{s,\hat{U}}/\hat{U}$ which is a geometric quotient by $\hat{U}$ of an open subset $\tilde{X}^{ss}$ of a $\hat{U}$-equivariant blow-up $\tilde{X}$ of X.

Application to quotients for the U-action:
Apply the $\hat{U}$-Theorem to the diagonal action of $\hat{U}$ on $X \times \mathbb{P}^1$ where

- $\hat{U}$ acts on $\mathbb{P}^1$ via $(u,t) \mapsto \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$
- linearisation is the product of L with $\mathcal{O}_{\mathbb{P}^1}(m)$ for $m >> 1$.

Then

$$(X \times \mathbb{P}^1)^{s,\hat{U}} \cap (X \times \mathbb{C}^*) = X^{\hat{s}} \times \mathbb{C}^*$$

where $X^{\hat{s}}$ is a U-invariant open subset of X with a geometric quotient $X^{\hat{s}}/U$ by U which is isomorphic to an open subset of $(X \times \mathbb{P}^1)^{s,\hat{U}}/\hat{U}$ and has a projective completion which is a geometric quotient by $\hat{U}$ of an open subset of a $\hat{U}$-equivariant blow-up of $X \times \mathbb{P}^1$. Also in this set-up there are **Hilbert–Mumford-like criteria for (semi)stability.**

- Moduli spaces of sheaves of fixed Harder–Narasimhan type over a nonsingular projective variety W
- Moduli of meromorphic connections on a curve–moduli of representations of quivers with multiplicities.
- Moduli of toric hypersurfaces.
- Moduli of map germs of curves into projective varieties–hyperbolicity questions.

Thanks for your attention.

Counts of nodal curves

Let S be a nonsingular projective surface and L a 5δ-ample line bundle on S. Let $N_\delta(L)$ denote the count of δ-nodal hypersurfaces in a generic linear system $\mathbb{P}^\delta \subset |L|$. According to Kleiman and Piene we can write $N_\delta = P_\delta/\delta!$ where P_δ satisfy the formal identity $\sum_{\delta \geq 0} \frac{P_\delta t^\delta}{r\delta!} = \exp(\sum_{q \geq 1} \frac{a_q t^q}{q!})$ for some integers $a_0, a_1, \ldots$. In particular,

$$P_0 = 1, P_1 = a_1, P_2 = a_1^2 + a_2, P_3 = a_1^3 + 3a_2a_1 + a_3, \ldots.$$

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Theorem (New formula for counts of δ-nodal curves, B.-Szenes 2016)

Introduce the variables indexed by boxes

$\mathcal{B} =$

z_{01}	z_{11}		$z_{\delta 1}$				
$\cdot$	z_{10}		$z_{\delta 0}$				$z_{2\delta,0}$

Then for $\delta > 1$ we have the following formula for a_δ

$$\operatorname*{Res}_{\mathbf{z}=\infty}\frac{(z_{10}\cdots z_{\delta 0})^{-1}\prod_{(a,b)<(a',b')}(z_{ab}-z_{a'b'})c_{2\delta}(L, L+z_{ab} : (a,b)\in\mathcal{B})\prod_{(a,b)\in\mathcal{B}} s_S\left(\frac{1}{z_{ab}}\right)dz}{\prod_{a+b\leq c\leq 2\delta}(z_{a0}-z_{b0})\prod_{\substack{(a,b)+(a',b')\leq(c,1)\leq(\delta,1)\\(a,b)+(a',b')\leq(c+1,0)\leq(\delta+1,0)}}(z_{ab}+z_{a'b'}-z_{a'b'})(\prod_{(a,b)\in\mathcal{B}} z_{ab})^2}$$

Problem:

- $f : M \to N$ holomorphic map between complex manifolds, $\dim(M) = m \leq \dim(N) = n$.
- A is a f.g. nilpotent algebra, e.g $A = z\mathbb{C}[z]/z^k$.
- $\alpha_f(A) = [\{p \in M : f_p \text{ has local algebra } A\}] \in H^*(M)$.
  Thom,1950's: $\alpha_f(A) = \mathrm{Tp}_A(c_1, c_2, \ldots)$ is a polynomial of the Chern classes c_i of $TN - f^*TM$.

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Theorem (Thom polynomial of A_k singularities, B.-Szenes 2012)

$$\mathrm{Tp}_k^{m-n} = \mathrm{Res}_{\mathbf{z}} \frac{(-1)^k \prod_{m<l\leq k}(z_m - z_l)\, Q_k(\mathbf{z})}{\prod_{m+r\leq l\leq k}(z_m + z_r - z_l)} \prod_{l=1}^{k} c\left(\frac{1}{z_l}\right) z_l^{m-n}\, dz_l,$$

where $c\left(\frac{1}{z_l}\right) = 1 + \frac{c_1}{z_l} + \frac{c_2}{z_l^2} + \ldots$ *is the total Chern class of* $TN - f^*TM$.

Conjecture[Green-Griffiths-Lang conjecture,1981] Every projective algebraic variety X of general type contains a proper algebraic subvariety $Y \subsetneq X$ such that every nonconstant entire holomorphic curve $f : \mathbb{C} \to X$ satisfies $f(\mathbb{C}) \subset Y$.

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3) Hyperbolic varieties

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Theorem (Degree of the jet differentials line bundle of Demailly, B. 2013)

For any homogeneous polynomial $P = P(u,h)$ of degree $\deg(P) = \dim \tilde{\mathcal{X}}_k = n + k(n-1)$ we have

$$\int_{\tilde{\mathcal{X}}_k} P(u,h) = \int_X \operatorname{Res}_{\mathbf{z}} \frac{Q_k(\mathbf{z}) \prod_{m<l}(z_m - z_l) P(z_1 + \ldots + z_k, h)}{\prod_{m+r \le l \le k}(z_m + z_r - z_l)(z_1 \ldots z_k)^n} \prod_{j=1}^{k} s\left(\frac{1}{z_j}\right)$$

Corollary (B. 2013)

The Green-Griffiths-Lang conjecture for hypersurfaces with polynomial degree follows from a positivity conjecture of Rimanyi on the coefficients of Thom

Thanks for your attention!