

Non-reductive GIT, curve counting and hyperbolicity

Gergely Bérczi – Oxford

Based on joint work with F. Kirwan, B. Doran, T. Hawes, J. Jackson and A. Szenes

Institut Fourier, Grenoble

May 19, 2016

- X complex projective variety, on which linear algebraic group G acts.
- Linearisation: ample line bundle L on X and a lift of the action of G to L .
- Replacing L with $L^{\otimes k} \Rightarrow$ we can assume that $X \subseteq \mathbb{P}^n$, the action of G on X extends to an action on \mathbb{P}^n given $\rho : G \rightarrow GL(n+1)$, and L is the hyperplane line bundle on \mathbb{P}^n .
- $\mathcal{O}_L(X) = \bigoplus_{k=0}^{\infty} H^0(X, L^{\otimes k})$ has an induced G -action. If G is reductive, $\mathcal{O}_L(X)^G$ is finitely generated graded algebra, and $\mathcal{O}_L(X)^G \hookrightarrow \mathcal{O}_L(X)$ induces

$$\begin{array}{ccc}
 X & \dashrightarrow & X//G = \text{Proj} \mathcal{O}_L(X)^G \\
 \cup & & \parallel \\
 X^{ss} & \rightarrow & X//G \\
 \cup & & \cup \\
 X^s & \rightarrow & X^s/G \qquad \text{Geometric quotient}
 \end{array}$$

- Topologically: $X//G = X^{ss} / \sim$ where $x \sim y \Leftrightarrow \overline{Gx} \cap \overline{Gy} \cap X^{ss} \neq \emptyset$.
- Hilbert-Mumford numerical criterion: $T \subset G$ max torus acting diagonally via $t \cdot [x_0 : \dots : x_n] = [t^{\alpha_0} x_0 : \dots : t^{\alpha_n} x_n]$ for some $\alpha_i \in \mathbb{C}^r = \mathfrak{t}_{\mathbb{C}}^*$.

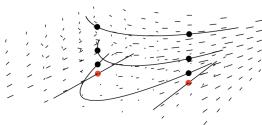
$$x = [x_0 : \dots : x_n] \in X_T^{s(ss)} \Leftrightarrow \{0\} \text{ sits inside (in the closure of) } \text{Conv}(\alpha_i : x_i \neq 0)$$

$x \in X$ is (semi)stable for G iff gx is T -(semi)stable for all $g \in G$.

Cornerstones of bad behaviour:

- 1 $\mathcal{O}_L(X)^G$ is not necessarily finitely generated (Nagata 1957) so $\text{Proj}(\mathcal{O}_L(X)^G)$ is not a projective variety.
- 2 Even if $\mathcal{O}_L(X)^G$ fin. gen. the quotient map $q : X \rightarrow \text{Proj}(\mathcal{O}_L(X)^G)$ is not nec. surjective, the image is just constructible.
- 3 Recall: every linear algebraic group can be written as $H = U \rtimes R$ where R is reductive and U is unipotent. Unipotent orbits cannot necessarily be separated with invariants.

Example: $\mathbb{C}^+ = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{C} \right\}$ acts on $X = \text{Sym}^2 \mathbb{C}^2$. Let x_0, x_1, x_2 be basis of X . Then $A(X)^{\mathbb{C}^+} = \mathbb{C}[x_0, x_1^2 - x_0x_2]$ and the orbit space looks like:



Q: Can we define a sensible 'quotient' variety $X//G$ when G is not reductive?

Definition

U (unipotent) group is graded if $\exists \lambda : \mathbb{C}^* \rightarrow \text{Aut}(U)$ with the weights of the \mathbb{C}^* action on $\text{Lie}(U)$ all positive. This defines the group extension

$$\hat{U} = U \rtimes \mathbb{C}^* \text{ with } (u, t) \cdot (u', t') = (u \cdot \lambda(t)(u'), tt')$$

We say that $H = U \rtimes R$ is graded if there is a central $\mathbb{C}^* \subset R$ such that

$$\hat{U} = U \rtimes \mathbb{C}^* \subset H \text{ is graded}$$

In most applications the acting non-reductive group is graded.

1) Moduli spaces of toric hypersurfaces

$$\begin{aligned} \text{Aut}(\mathbb{P}(1, 1, 2)) &= R \rtimes U \text{ where } R = GL(2) \text{ and } U = (\mathbb{C}^+)^3 \\ (x, y, z) &\mapsto (x, y, z + \lambda x^2 + \mu xy + \nu y^2) \text{ for } (\lambda, \mu, \nu) \in (\mathbb{C}^+)^3 \end{aligned}$$

$$\text{Aut}(\mathbb{P}(1, 1, 2)) = \left(\begin{array}{ccc|c} & & & \nu \\ & \text{Sym}^2 GL(2) & & \mu \\ & & & \lambda \\ \hline 0 & 0 & 0 & 1 \end{array} \right)$$

The central one-parameter subgroup $\mathbb{C}^* = \left(\begin{array}{ccc|c} t^2 & & & 0 \\ & t^2 & & 0 \\ & & t^2 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array} \right)$ of $R \cong GL(2)$ acts

on $\text{Lie}(U)$ with weight 2.

2) Jets of reparametrisation germs

- $J_k(1, n) = \{k\text{-jets of germs of holomorphic maps } (\mathbb{C}, 0) \rightarrow (\mathbb{C}^n, 0)\} = \{(f', f'', \dots, f^{(k)}) : f' \neq 0\}$
- $J_k(1, 1)$ is the group of k -jets of germs of biholomorphisms of $(\mathbb{C}, 0)$. $J_k(1, 1)$ acts on $J_k(1, n)$ via reparametrisation:
 - $f(z) = zf'(0) + \frac{z^2}{2!} f''(0) + \dots + \frac{z^k}{k!} f^{(k)}(0) \in J_k(1, n)$ and
 - $\varphi(z) = \alpha_1 z + \alpha_2 z^2 + \dots + \alpha_k z^k \in J_k(1, 1)$

Then

$$f \circ \varphi(z) = (f'(0)\alpha_1)z + (f'(0)\alpha_2 + \frac{f''(0)}{2!}\alpha_1^2)z^2 + \dots$$

$$= (f', \dots, f^{(k)}/k!) \cdot \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \dots & 2\alpha_1\alpha_{k-1} + \dots \\ 0 & 0 & \alpha_1^3 & \dots & 3\alpha_1^2\alpha_{k-2} + \dots \\ 0 & 0 & 0 & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \alpha_1^k \end{pmatrix} = \mathbb{C}^* \times \mathbf{U}_k$$

This group plays a central role in many applications: Enumerative geometry of Hilbert schemes of points, singularities of maps, hyperbolicity questions and in particular the Green-Griffiths-Lang conjecture...

- Let U be a graded unipotent lin. alg. group and $\hat{U} = U \rtimes \mathbb{C}^*$. Let X be an irreducible normal \hat{U} -variety and $L \rightarrow X$ an very ample linearisation of the \hat{U} action.
- $X_{\min}^{\mathbb{C}^*} := X \cap \mathbb{P}(H^0(X, L)_{\min}^*) = \left\{ x \in X \mid \begin{array}{l} x \text{ is a } \mathbb{C}^* \text{-fixed point and} \\ \mathbb{C}^* \text{ acts on } L^*|_x \text{ with min weight} \end{array} \right\}$
- $X_{\min} := \{x \in X \mid \lim_{t \rightarrow 0} t \cdot x \in X_{\min}^{\mathbb{C}^*}\}$ ($t \in \mathbb{G}_m \subseteq \hat{U}$) the Bialynicki-Birula stratum.

Theorem (B.,Doran, Hawes, Kirwan 2012-16)

Let $L \rightarrow X$ be an irreducible normal \hat{U} -variety with ample linearisation L . If

$$(*) x \in X_{\min}^{\mathbb{C}^*} \Rightarrow \text{Stab}_U(x) = 1$$

(or more generally $\dim \text{Stab}_U(x) = \min_{y \in X} \dim \text{Stab}_U(y)$) holds then

- 1 $X_{\min} \subseteq X^{s(U,L)}$, so we get a locally trivial U -quotient $X_{\min} \rightarrow X_{\min}/U$.
- 2 Twist the linearisation by a character $\chi : \hat{U} \rightarrow \mathbb{C}^*$ of \hat{U}/U such that 0 lies in the lowest bounded chamber for the \mathbb{C}^* action on X . Then for suitably divisible integers $r > 0$ $\mathcal{O}_X(X)^{\hat{U}} = \mathcal{O}_{(LX) \otimes r}(X)^{\hat{U}}$ is **finitely generated** and for the quotient

$$X//_{\chi} \hat{U} := \text{Proj}(\mathcal{O}_X(X)^{\hat{U}})$$

we have $X//\hat{U} = X^{ss,\hat{U}} / \sim$ where $x \sim y \Leftrightarrow \overline{\hat{U}x} \cap \overline{\hat{U}y} \cap X^{ss} \neq \emptyset$.

- 3 $X_{\min} \setminus (U \cdot X_{\min}^{\mathbb{C}^*}) = X^{s,\hat{U}} = X^{ss,\hat{U}}$ and $X//_{\chi} \hat{U} = X^{s,\hat{U}}/\hat{U}$ is a geometric quotient of $X^{s,\hat{U}}$ by \hat{U} . $X^{s,\hat{U}}, X^{ss,\hat{U}}$ are given by **Hilbert-Mumford type criterion**.

Remarks

- 1 Similar construction for any graded non-reductive group $H = U \rtimes R$.
- 2 If (*) does not hold then there is a sequence of blow-ups of X along \hat{U} -invariant projective subvarieties resulting in a projective variety \hat{X} with a linear action of \hat{U} (with respect to a power of an ample line bundle given by perturbing the pullback of L by small multiples of the exceptional divisors for the blow-ups) which satisfies (*). Namely, let d maximal s.t

$$X_{\min}^d = \{z \in X_{\min}^{\mathbb{C}^*} \mid \dim \text{Stab}_U(z) = d\} \neq \emptyset.$$

Then $U \cdot X_{\min}^d = \hat{U} \cdot X_{\min}^d$ is a closed subvariety of X_{\min}^0 and we blow-up this and perturb the pull-back of L with a small multiple of the ex. divisor to get ample \hat{U} -linearisation on the blown-up.

- 3 (joint with J. Jackson and F. Kirwan): Nice VGIT picture for $H = U \rtimes R$ action when R contains higher dimensional grading torus, that is, the grading \mathbb{C}^* can be moved inside R .
- 4 (joint with F. Kirwan) Cohomology of \hat{U} -quotients: Kirwan surjectivity for $H_0^*(X^{ss}) \rightarrow H^*(X//\hat{U})$ and residue formulae for cohomological pairings (intersection numbers) on $X//\hat{U}$.

- X/\mathbb{C} smooth variety of dimension n .
- $X^{[k]} = \{\xi \subset X : \dim(\xi) = 0, l(\xi) = \dim H^0(\xi, \mathcal{O}_\xi) = k\}$ Hilbert scheme of k points on X
- $X_p^{[k]} = \{\xi \in X^{[k]} : \text{supp}(\xi) = p\}$ Punctual Hilbert scheme at $p \in X$

Ultimate goal: Determine the geometric and topological invariants of $X^{[n]}$.

For surfaces these were extensively studied such as **Betti numbers** (Ellingsrud, Stromme, Göttsche), **Hodge numbers** (Sörgel, Göttsche), **cohomology ring** (Nakajima, Grojnowski), **Chern numbers of tautological bundles** (Kleiman-Piene, Lehn, Rennemo, Marian-Oprea-Pandharipande).

When $\dim(X) > 2$ not much is known: Rennemo, Tzeng, motivic DT invariants (Behrend-Bryan-Szendrői).

Tautological vector bundles:

F -rank r bundle (loc. free sheaf) on $X \rightsquigarrow F^{[k]}$ -rank rk bundle over $X^{[k]}$:

Fibre over $\xi \in X^{[k]}$ is $F \otimes \mathcal{O}_\xi = H^0(\xi, F|_\xi)$.

Equivalently: $F^{[k]} = q_* p^*(F)$ where $X^{[k]} \times X \supset \mathcal{Z} \xrightarrow{q} X^{[k]}$.

$$\begin{array}{c} \mathcal{Z} \\ \downarrow p \\ X \end{array}$$

Geometric subsets of Hilbert schemes

Following Rennemo (2014) we define the **geometric subset** of type $\mathbf{Q} = (Q_1, \dots, Q_s)$ as

$$P(\mathbf{Q}) = \{\xi \in X^{[k]} : \xi = \xi_1 \cup \dots \cup \xi_s \text{ where } \xi_i \in X_{p_i}^{[k_i]} \text{ is of type } Q_i\}.$$

A **type** $Q \subset X_p^{[k]}$ is a closed subset which is union of isomorphism classes:

$$\xi \in Q, \xi \simeq \xi' \Rightarrow \xi' \in Q.$$

Definition: For an algebra A of dimension k let $Q_A = \{\xi \in X^{[k]} : \xi \simeq A\}$.

Fix monomial ideals $I_{\lambda_1}, \dots, I_{\lambda_s}$ corresponding to the partitions (Young tableau) $\lambda_1, \dots, \lambda_s$ and let

$$Q_{\lambda_i} = \overline{\{\xi : \mathcal{O}_\xi \simeq \mathbb{C}[z_1, \dots, z_n]/I_{\lambda_i}\}}.$$

Let $P(\lambda_1, \dots, \lambda_s) = P(Q_{\lambda_1}, \dots, Q_{\lambda_s})$ denote the corresponding monomial geometric subset.

- Simplest case: $s = 1$ and $\lambda = \underbrace{\square \dots \square}_k$ then $A = \mathbb{C}[z]/z^k$ and

$$Q_\lambda = CX_p^{[k]} = \{\xi \in X_p^{[k]} : \xi \subset \mathcal{C}_p \text{ for some smooth curve } \mathcal{C} \subset X\}$$

$$P(\underbrace{\square \dots \square}_k) = CX^{[k]} = \cup_{p \in X} CX_p^{[k]} \text{ is the curvilinear component.}$$

- Most famous case (Göttsche): $X = S$ surface, $\lambda_1 = \dots = \lambda_s = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}$. Then

$$I_\lambda = (x^2, xy, y^2) \text{ and}$$

$$P(s \cdot \lambda) = P\left(\underbrace{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}, \dots, \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}}_s\right) \subset S^{[3s]}.$$

is $2s$ -dimensional closed variety and the count of s -nodal curves in the $5s$ -ample linear system $\mathbb{P}(|L|)$ is $\int_{P(s \cdot \lambda)} c_{2s}(L^{[3s]})$.

$$P(Q_1, \dots, Q_s) = \{\xi \in X^{[k]} : \xi = \xi_1 \cup \dots \cup \xi_s \text{ where } \xi_i \in X_{p_i}^{[k_i]} \text{ is of type } Q_i\}$$

Motivation: counting hypersurfaces with prescribed singularities on X .

Fact: Let T_1, \dots, T_s be analytic singularity types with expected codimension $d = \sum_{i=1}^s d_i$. Then there is a k and a geometric set $\mathcal{P} = P((T_i)) \subset X^{[k]}$ such that a generic hypersurface containing a $Z \in W$ has the specified singularities. Then we get that in a general $\mathbb{P}^d \subseteq |L|$ the number of hypersurfaces containing a subscheme $Z \in W$ equals $\int_{\mathcal{X}} c_{\dim(\mathcal{X})}(L^{[k]})$.

Theorem (Existence of universal polynomial, Rennemo, Tzeng 2013-14)

For any geometric subset $P(\mathbf{Q}) \subset X^{[k]}$ and Chern monomial $R(c_1, c_2, \dots)$ of degree $\dim(P(\mathbf{Q}))$ the tautological integral

$$\int_{P(\mathbf{Q})} R(c_1(F^{[n]}), c_2(F^{[n]}), \dots) = \tilde{R}(c_i(F), c_i(X))$$

is given by a universal polynomial of the Chern numbers of F and X .

Rest of the talk: Show that non-reductive moduli description combined with equivariant localisation results in closed iterated residue formulae for $\int_{P(\mathbf{Q})} R(c_i(F^{[k]}))$.

$$CX_p^{[k]} = \{\xi \in X_p^{[k]} : \xi \subset C_p \text{ for some smooth curve } C \subset X\} = \\ \overline{\{\xi \in X_p^{[k]} : \mathcal{O}_\xi \simeq z\mathbb{C}[z]/z^{k+1}\}} \text{ Curvilinear locus at } p$$

Def: $CX_p^{[k]}$ is the curvilinear component of the Hilbert scheme. This is a singular irreducible projective variety of dimension $(n-1)(k-1)$.

On surfaces ($n=2$) $\overline{CX_p^{[k]}} = X_p^{[k]}$, for $n > 2$ $\overline{CX_p^{[k]}}$ is a component of the punctual Hilbert scheme.

Non-reductive quotient model of curvilinear Hilbert schemes:

$\xi \in CX_p^{[k+1]} \rightsquigarrow \xi \subset C_p \subset X$ smooth curve germ $\rightsquigarrow f_\xi : (\mathbb{C}, 0) \rightarrow (X, p)$ k -jet of germ parametrising C_p . f_ξ is determined up to polynomial reparametrisation germs $\phi : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$. Let

$$J_k(1, n) = \{k\text{-jets } f : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^n, 0)\} = \{(f', f'', \dots, f^{[k]}) : f' \neq 0\}$$

$$CX_p^{[k+1]} = \{k\text{-jets } (\mathbb{C}, 0) \rightarrow (\mathbb{C}^n, 0)\} / \{k\text{-jets } (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)\} = J_k(1, n) / J_k(1, 1).$$

Observation: If $g \in J_k(n, 1)$ then $g \circ f_\xi = 0 \Rightarrow g \circ (f_\xi \circ \phi) = 0$

In other words: $\text{Ann}(f_\xi) \subset J_k(n, 1)$ is invariant under $J_k(1, 1)$. The map $f_\xi \mapsto \text{Ann}(f_\xi)$ defines an embedding

$$\rho : CX_p^{[k+1]} = J_k(1, n)/J_k(1, 1) \hookrightarrow \text{Grass}(\text{codim} = k, J_k(n, 1))$$

If $f_\xi = f_1 z + \dots + f_k z^k$ with $f_i \in \mathbb{C}^n$;

$g(v) = Av + Bv^2 + \dots$ with $A \in \text{Hom}(\mathbb{C}^n, \mathbb{C})$, $B \in \text{Hom}(\text{Sym}^2(\mathbb{C}^n), \mathbb{C}), \dots$

then $g \circ f_\xi = 0$ has the form:

$$A(f_1) = 0,$$

$$A(f_2) + B(f_1, f_1) = 0,$$

$$A(f_3) + 2B(f_1, f_2) + C(f_1, f_1, f_1) = 0,$$

Theorem (B-Szenes 2012, B 2015)

1 The dual of ρ can be written as

$$\rho : J_k(1, n)/J_k(1, 1) \hookrightarrow \text{Grass}(k, J_k(n, 1)^*)$$

$$(f', \dots, f^{(k)}) \mapsto \text{Span}(f', f'' + (f')^2, \dots, \sum_{a_1+a_2+\dots+a_j=j} f^{(a_1)} f^{(a_2)} \dots f^{(a_j)}, \dots),$$

2 For $k \leq n$ $\overline{\rho(J_k(1, n))} = \overline{\text{GL}(n) \cdot \rho(e_1, \dots, e_n)}$ where $\{e_1, \dots, e_n\}$ is a basis of \mathbb{C}^n .

Theorem (Integrals over the curvilinear Hilbert schemes, B. 2015)

$$\int_{\widetilde{CX}^{[k+1]}} R(c_i(F)) = \int_X \operatorname{Res}_{z=\infty} \frac{\prod_{i < j} (z_i - z_j) Q_k(\mathbf{z}) R(c_i(z_i + \theta_j, \theta_j)) d\mathbf{z}}{\prod_{i+j \leq l \leq k} (z_i + z_j - z_l) (z_1 \dots z_k)^n} \prod_{i=1}^k s_X\left(\frac{1}{z_i}\right)$$

Features of the residue formulae:

- 1 The residue gives a degree n symmetric polynomial in Chern roots of F and Segre classes of X reflecting the Gottsche conjecture on curvilinear components.
- 2 For fixed k the formula gives a universal generating series for the integrals as the dimension increases.
- 3 The geometry of Q_k . The GL_k -module of 3-tensors $\operatorname{Hom}(\mathbb{C}^k, \operatorname{Sym}^2 \mathbb{C}^k)$ has a diagonal decomposition

$$\operatorname{Hom}(\mathbb{C}^k, \operatorname{Sym}^2 \mathbb{C}^k) = \bigoplus \mathbb{C} q_l^{mr}, \quad 1 \leq m, r, l \leq k,$$

q_l^{mr} has weight $(z_m + z_r - z_l)$. Let

$$\epsilon = \sum_{m=1}^k \sum_{r=1}^{k-m} q_{m+r}^{mr} \subset W = \bigoplus_{1 \leq m+r \leq l \leq k} \mathbb{C} q_l^{mr} \subset \operatorname{Hom}(\mathbb{C}^k, \operatorname{Sym}^2 \mathbb{C}^k)$$

Then $Q_k(\mathbf{z}) = \operatorname{mdeg}(\overline{B_k} \epsilon, W)$. In particular $Q_{1,2,3} = 1, Q_4 = 2z_1 + z_2 - z_4$.

Next step Integration on $P(\lambda)$. We demonstrate this on $\lambda = (s, 2s) = \underbrace{\begin{array}{|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square \\ \hline \end{array}}_{2s}$

Two dimensional algebra \Rightarrow test surface to get non-reductive quotient model.

$\xi \in P(\lambda)_p \rightsquigarrow f_\xi : (\mathbb{C}^2, 0) \rightarrow (X, p)$ λ -jet of germ paramtrising the surface germ \mathcal{S}_ξ for which $\xi \in \mathcal{S}_\xi$.

$$J_\lambda(\mathbb{C}^2, \mathbb{C}^n) := \left\{ \frac{\partial f}{\partial x^i \partial y^j} \mid (i, j) \in \underbrace{\begin{array}{|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square \\ \hline \end{array}}_{2\delta} \right\}$$

f_ξ is determined up to λ -jets of reparametrisation germs $\phi : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$. So

$$P(\lambda)_p = J_\lambda(2, n) / J_\lambda(2, 2).$$

Observation: If $g \in J_{2s}(n, 2)$ then $g \circ f_\xi = 0 \Rightarrow g \circ (f_\xi \circ \phi) = 0$

If $f_\xi = \sum_{(i,j) \in \lambda} f_{ij} x^i y^j$ with $f_{ij} \in \mathbb{C}^n$ and $g(v) = Av + Bv^2 + \dots$ with

$A \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^2)$, $B \in \text{Hom}(\text{Sym}^2(\mathbb{C}^n), \mathbb{C}^2), \dots$

then $g \circ f_\xi = 0$ has the form:

$$\begin{aligned} A(f_{10}) &= 0 & A(f_{01}) &= 0 \\ A(f_{20}) + 2B(f_{10}, f_{10}) &= 0 & A(f_{11}) + B(f_{10}, f_{01}) &= 0 \\ &\dots & & \end{aligned}$$

We have $3s$ equations and induces an embedding

$$P(\lambda) \hookrightarrow \text{Grass}(3s, \text{Sym}^{\leq 2s} \mathbb{C}^n)$$

Then we can apply equivariant localisation, turn the formula into iterated residue and prove a vanishing theorem. (HARD WORK!!!)

Theorem (Tautological integrals on punctual monomial geometric subsets, B. 2016)

Introduce the variables indexed by boxes of λ :

$$\lambda = \begin{array}{|c|c|c|c|c|c|c|c|} \hline z_{01} & z_{11} & & z_{s1} & & & & \\ \hline \cdot & z_{10} & & z_{\delta 0} & & & & z_{2s,0} \\ \hline \end{array}$$

Let X be a smooth projective variety, F a bundle over X , $R(c_i(F^{[3s]}))$ a Chern polynomial of degree $\dim(P(\lambda)) = s + 3$. Then

$$\int_{P(\lambda)} R(c_i(F^{[3s]})) = \underset{\mathbf{z}=\infty}{\mathbf{Res}} \frac{\prod_{(a,b) < (a',b')} (z_{ab} - z_{a'b'}) R(c_i(\theta_j, \theta_j + z_{ab} : (a,b) \in \mathcal{B})) \prod_{(a,b) \in \mathcal{B}} s_S \left(\frac{1}{z_{ab}} \right) dz}{\prod_{a+b \leq c \leq 2s} (z_{a\mathbf{0}} - z_{b\mathbf{0}}) \prod_{\substack{(a,b)+(a',b') \leq (c,\mathbf{1}) \leq (s,\mathbf{1}) \\ (a,b)+(a',b') \leq (c+\mathbf{1},\mathbf{0}) \leq (s+\mathbf{1},\mathbf{0})}} (z_{ab} + z_{a'b'} - z_{a'b'}) \left(\prod_{(a,b) \in \mathcal{B}} z_{ab} \right)^2}$$

Similar formulae hold for other Young tableau λ 's. Non-reductive quotient model is derived using equations of test m -folds if λ is m -dimensional partition.

Let S be a nonsingular projective surface and L a $5s$ -ample line bundle on S . Let $N_s(L)$ denote the count of s -nodal hypersurfaces in a generic linear system $\mathbb{P}^s \subset |L|$. According to Kleiman and Piene we can write $N_s = P_s/s!$ where P_s satisfy the formal identity $\sum_{s \geq 0} \frac{P_s t^s}{s!} = \exp(\sum_{q \geq 1} \frac{a_q t^q}{q!})$ for some integers a_0, a_1, \dots . In particular,

$$P_0 = 1, P_1 = a_1, P_2 = a_1^2 + a_2, P_3 = a_1^3 + 3a_2 a_1 + a_3, \dots$$

Theorem (New formula for counts of s -nodal curves, B.-Szenes 2016)

Introduce the variables indexed by boxes

$$\mathcal{B} = \begin{array}{|c|c|c|c|c|c|c|c|} \hline z_{01} & z_{11} & & z_{s1} & & & & \\ \hline \cdot & z_{10} & & z_{s0} & & & & z_{2s,0} \\ \hline \end{array}$$

Then for $s > 1$ we have

$$a_s = \mathop{\text{Res}}_{z=\infty} \frac{(z_{10} \cdots z_{s0})^{-1} \prod_{(a,b) < (a',b')} (z_{ab} - z_{a'b'}) c_{2s}(L, L + z_{ab} : (a,b) \in \mathcal{B}) \prod_{(a,b) \in \mathcal{B}} s_S \left(\frac{\mathbf{1}}{z_{ab}} \right) dz}{\prod_{a+b \leq c \leq 2s} (z_{a0} - z_{b0}) \prod_{\substack{(a,b)+(a',b') \leq (c,\mathbf{1}) \leq (s,\mathbf{1}) \\ (a,b)+(a',b') \leq (c+\mathbf{1},\mathbf{0}) \leq (s+\mathbf{1},\mathbf{0})}} (z_{ab} + z_{a'b'} - z_{a'b'}) \left(\prod_{(a,b) \in \mathcal{B}} z_{ab} \right)^2}$$

Problem:

- $f : M \rightarrow N$ holomorphic map between complex manifolds, $\dim(M) = m \leq \dim(N) = n$.
- A is a f.g. nilpotent algebra, e.g $A = \mathbb{C}[z]/z^k$.
- $\alpha_f(A) = [\{p \in M : f_p \text{ has local algebra } A\}] \in H^*(M)$.
Thom, 1950's: $\alpha_f(A) = \text{Tp}_A(c_1, c_2, \dots)$ is a polynomial of the Chern classes c_i of $TN - f^*TM$.

Theorem (Thom polynomial of A_k singularities, B.-Szenes 2012)

$$\text{Tp}_k^{m-n} = \text{Res}_{\mathbf{z}} \frac{(-1)^k \prod_{m < l \leq k} (z_m - z_l) Q_k(\mathbf{z})}{\prod_{m+r \leq l \leq k} (z_m + z_r - z_l)} \prod_{l=1}^k c\left(\frac{1}{z_l}\right) z_l^{m-n} dz_l,$$

where $c\left(\frac{1}{z_l}\right) = 1 + \frac{c_1}{z_l} + \frac{c_2}{z_l^2} + \dots$ is the total Chern class of $TN - f^*TM$.

Conjecture (Green-Griffiths-Lang conjecture, 1981)

Every projective algebraic variety X of general type contains a proper algebraic subvariety $Y \subsetneq X$ such that every nonconstant entire holomorphic curve $f : \mathbb{C} \rightarrow X$ satisfies $f(\mathbb{C}) \subset Y$.

Related to the stronger concept of a hyperbolic variety of Kobayashi: a projective variety X is hyperbolic if there is no non-constant entire holomorphic curve in X , i.e. any holomorphic map $f : \mathbb{C} \rightarrow X$ must be constant.

Theorem (Degree of the jet differentials line bundle of Demailly, B. 2013)

For any homogeneous polynomial $P = P(u, h)$ of degree

$$\deg(P) = \dim \tilde{\mathcal{X}}_k = n + k(n - 1)$$

we have

$$\int_{\tilde{\mathcal{X}}_k} P(u, h) = \int_X \operatorname{Res}_z \frac{Q_k(\mathbf{z}) \prod_{m < l} (z_m - z_l) P(z_1 + \dots + z_k, h)}{\prod_{m+r \leq l \leq k} (z_m + z_r - z_l) (z_1 \dots z_k)^n} \prod_{j=1}^k s_X \left(\frac{1}{z_j} \right)$$

Corollary (B. 2013)

The Green-Griffiths-Lang conjecture for hypersurfaces with polynomial degree follows from a positivity conjecture of Rimanyi on the coefficients of Thom polynomials.