Moduli of jets of holomorphic curves

Gergely Bérczi – Oxford

CIRM, Luminy

March 15, 2012.

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where the polynomial in the (i,j) entry is
$$p_{i,j}(\bar{\alpha}) = \sum_{a_1+a_2+\ldots+a_i=j} \alpha_{a_1}\alpha_{a_2}\ldots\alpha_{a_i}.$$

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2. Even if $\mathcal{O}(X)^G$ is finitely generated, the quotient map $X \to Proj(\mathcal{O}(X)^G)$ is not necessarily surjective. Moreover, $Proj(\mathcal{O}(X)^G)$ is not a set of equivalence classes in $X^{ss} \subset X$, no geometric description!

The contruction

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Remark For $k \leq n$ $\overline{\tilde{\rho}(J_k(1,n))} \subset \mathrm{Grass}$ is the closure of $\mathrm{GL}(n) \cdot \tilde{\rho}((e_1, \dots, e_n))$ where $\{e_1, \dots, e_n\}$ is a basis of $\mathbb{C}^n$. It is a finite union of $GL(n)$ orbits.

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- **Projective embedding:** $GL(k)/G_k \hookrightarrow \mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^k))$ as the orbit of

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Theorem (B-Kirwan,2011)

If $k \geq 4$ and $K = M(1 + \dots + k) + 1$ then the boundary orbits in $\overline{SL(k)p_k} \subset \wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^k) \otimes (\mathbb{C}^k)^{\otimes K}$ have codimension ≥ 2.

Corollary

U_k is a Grosshans subgroup of $\mathrm{SL}(k)$ (that is, $\mathcal{O}(\mathrm{SL}(k))^{U_k}$ is finitely generated). Therefore every linear action of U_k which extends to a linear action of $\mathrm{SL}(k)$ has finitely generated invariants.

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B-Szenes,2008: Equivariant localisation implies an iterated residue formula for (equivariant) intersection numbers of the non-reductive moduli space $J_k(1,n)/G_k$.

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Theorem (Thom)

Given $f : N^n \to M^m$ holomorphic map let $Z_A = \{p \in N : f_p \in \Sigma_A\}$. For generic f

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$$[Z_A] = Tp_A(TM, f^*TN) = PD^{(\mathbb{C}^*)^m \times (\mathbb{C}^*)^n}(\Sigma_A, J_k(n,m) \in H^*_{(\mathbb{C}^*)^n \times \mathbb{C}^*)^m(J_k(n,m)}$$

Theorem (Vergne)

$X \subset V$ a T-invariant subvariety, then

$$PD^T(X,V) = \int_X \mathrm{Thom}_T(V)$$

is an equivariant intersection number.

Link to singularities of maps

Theorem (Porteous, Gaffney)

$$\Sigma_k(n,m) \doteq \{\psi \in J_k(n,m) | \exists \gamma \in J_k^{reg}(1,n) \text{ such that } \psi \circ \gamma = 0\}.$$

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Proposition

$\Sigma_k^0 = \{\psi \in \Sigma_k : \dim\ker\psi = 1\}$ *fibers with linear fibres over* $J_k^{reg}(1,n)/G_k$.

$$(\mathbb{C},0) \xrightarrow{\gamma} (\mathbb{C}^n,0) \xrightarrow{\psi} (\mathbb{C}^m,0) \tag{3}$$

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- For $m = 1$ the map $\gamma \mapsto \mathrm{Sol}_\gamma$ gives us $J_k(1,n) \to \mathrm{Grass}(k, \mathrm{Sym}^{\leq k}\mathbb{C}^n)$ invariant under G_k. This is the map defined before.

Theorem (B-Szenes,2008)

Equivariant localisation on $J_k(1,n)/G_k$ leads us to

$$Tp_k^{m-n} = \mathrm{Res}_{\mathbf{z}=\infty} \frac{\prod_{m<l}(z_m - z_l)\, \mathcal{Q}_k(z_1 \dots z_k)}{\prod_{m+r\le l\le k}(z_m + z_r - z_l)} \cdot \prod_{l=1}^{k} c\left(\frac{1}{z_l}\right) z_l^{m-n}\, dz_l$$

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Conjecture (Green-Griffiths, 1979)

Let X be a projective variety of general type (i.e X has maximal Kodaira dimension, $\dim \oplus_{i=0}^{\infty} H^0(X, K^i) = \dim X$). Then there exists a proper subvariety $Y \subsetneq X$ such that for all non-constant holomorphic $f : \mathbb{C} \to X$ one has $f(\mathbb{C}) \subset Y$.

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Strategy–Green/Griffith/Demailly/Siu 1) Find differential equations which must be satisfied by (the jets of) any entire holomorphic curve. 2) Find enough independent equations s.t. their solution set is a proper subvariety.

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Theorem (Fundamental vanishing theorem)

(Green-Griffiths '78, Demailly '95, Siu '96) Let $P \in H^0(X, E_{k,m} \otimes \mathcal{O}(-A))$ be a global algebraic differential operator whose coefficients vanish on some ample divisor A. Then for any $f : \mathbb{C} \to X$, $P(f_{[k]}(\mathbb{C})) \equiv 0$.(note $f_{[k]}(\mathbb{C}) \subset J_kX$)

Link to hyperbolic varieties

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Theorem (B,2010)

Let $X \subset \mathbb{P}^{n+1}$ be a projective hypersurface. Positivity conjecture on Thom polynomials implies the existence $P(f', f'', \ldots, f^{(n)}) = 0$ when $\deg(X) > n^6$. This implies the polynomial Green-Griffiths conjecture for a generic projective hypersurface with $\deg(X) > n^6$.

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Moral: Intersection numbers of curvilinear Hilbert schemes turn up in several context.