# Non-reductive GIT and invariant jet differentials
# Joint work with Frances Kirwan

Gergely Bérczi – Oxford

Rencontre Hyperbolicité, Paris-Orsay

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- Replace $L$ with $L^{\otimes k}$ $\Rightarrow$ we can assume that $X \subseteq \mathbb{P}^n$, the action of $G$ on $X$ extends to an action on $\mathbb{P}^n$ given $\rho : G \to GL(n+1)$, and $L$ is the hyperplane line bundle on $\mathbb{P}^n$.

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- $\mathcal{O}_L(X) = A(X) = \oplus_{k=0}^{\infty} H^0(X, L^{\otimes k})$ has an induced $G$-action. If $G$ is reductive, $A(X)^G$ is finitely generated graded algebra, and $A(X)^G \hookrightarrow A(X)$ implies

$$\begin{array}{ccccccccc}
X & \supset & X^{ss} & \supset & X^s \\
\downarrow & & \downarrow \text{onto} & & \downarrow \\
X//G \text{ proj. var.} & = & X//G & \supset (open) & X^s/G
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**Theorem (F. Kirwan, B. Doran)**

*Let $G = R \ltimes U$ act linearly on $X \subset \mathbb{P}^n$ Then one can define $X^s$ (stable points) and $X^{ss}$ (semistable points) with a geometric quotient $X^s \to X^s/G$ and an enveloping quotient $X^{ss} \to X//G$ s.t.*

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- The finitely generated semistable set of $X$ is $X^{fgss} = \bigcup_{f \in I^{fg}} X_f$ where

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**Remark**

*When $A(X)^G$ is finitely generated then $X^{fgss} = X^{nss}$ and $X//G = Proj(A(X)^G)$. But the quotient map is not surjective! And $X//G$ is not a set of equivalence classes in $X^{ss}$, no geometric description!*

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### Definition

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## Example

1. $Aut(\mathbb{P}(1,1,2)) = R \ltimes U$ *with* $R = GL(2) \times_{\mathbb{C}^*} \mathbb{C}^* = GL(2)$ *reductive and* $U = (\mathbb{C}^+)^3$ *unipotent, where*

$$(x,y,z) \mapsto (x,y,z+\lambda x^2+\mu xy+\nu y^2 \quad (\lambda,\mu,\nu)\in\mathbb{C}^3.$$

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2. *Jets of reparametrizations of curves at the origin* $\rightsquigarrow$ *jet differentials.*

**Theorem (G.B.,B. Doran,F. Kirwan –in progress)**

1. *Let U be a graded unipotent acting linearly on the affine variety X and assume the the action extends to* $\hat{U} = U \ltimes \mathbb{C}^*$*. Then* $A(X)^U = \oplus_{k \geq 0} H^0(X, L^{\otimes k})^U$ *is finitely generated, so* $X//U = Proj(A(X)^U)$.

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## Corollary (Popov-Pommerening conjecture)

*Let $G$ be a complex reductive group, and let $H \subset G$ be normalized by a maximal torus of $G$. Then every linear action of $H$ on an affine variety $X$ which extends to a linear action of $G$ has finitely generated invariants.*

# Jets of reparametrization germs

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$$= (f', \ldots, f^{(k)/k!}) \cdot \begin{pmatrix} \Phi_1 & \Phi_2 & \Phi_3 & \ldots & \Phi_k \\ 0 & \Phi_1^2 & \Phi_1\Phi_2 & \ldots & \\ 0 & 0 & \Phi_1^3 & \ldots & \\ . & . & . & . & . \\ & & & & \Phi_1^k \end{pmatrix},$$

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$\Phi_{i_1}\ldots\Phi_{i_l}$ is the matrix of the map $\mathrm{Sym}^{i_1+\cdots+i_l}(\mathbb{C}^p)\to\mathrm{Sym}^l\mathbb{C}^p$, which is induced by
$$\Phi_{i_1}\otimes\cdots\otimes\Phi_{i_l} : (\mathbb{C}^p)^{\otimes i_1}\otimes\cdots\otimes(\mathbb{C}^p)^{\otimes i_l}\to(\mathbb{C}^p)^{\otimes l}$$

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  where the polynomial in the $(i,j)$ entry is

$$p_{i,j}(\bar{\alpha}) = \sum_{a_1 + a_2 + \ldots + a_i = j} \alpha_{a_1}\alpha_{a_2} \ldots \alpha_{a_i}.$$

  The weights of $\mathbb{C}^*$ on $U$ are $1, 2, \ldots, n$.

- $V = J_k(n,m) = \{(p_1,\ldots,p_m) \in \mathrm{Poly}(\mathbb{C}^n,\mathbb{C}^m) : \deg p_i \leq k, p_i(0) = 0\}$

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- $\mathcal{D} = \mathrm{Diff}(\mathbb{C}^n,0) \times \mathrm{Diff}(\mathbb{C}^m,0) \curvearrowright J_d(n,m)$ with

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Note: $\Sigma_d(n,k)$ is a subset of the punctual Hilbert scheme of $d$ points on $\mathbb{C}^n$.

**Theorem (The test curve model of Porteous and Gaffney)**

$$\Sigma_k(n,m) \doteq \{\Psi \in J_k(n,m) | \exists\gamma \in J_k^{reg}(1,n) \text{ such that } \Psi \circ \gamma = 0\}.$$

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**Observation:** If $\varphi \in J_k^{reg}(1,1) = \mathbf{G}_k$, then

$$\Psi \circ \gamma = 0 \quad \Rightarrow \quad \Psi \circ (\gamma \circ \varphi) = 0$$

$$(\mathbb{C},0) \xrightarrow{\varphi} (\mathbb{C},0) \xrightarrow{\gamma} (\mathbb{C}^n,0) \xrightarrow{\Psi} (\mathbb{C}^m,0) \qquad (2)$$

**Theorem (The test curve model of Porteous and Gaffney)**

$$\Sigma_k(n,m) \doteq \{\Psi \in J_k(n,m) | \exists \gamma \in J_k^{reg}(1,n) \text{ such that } \Psi \circ \gamma = 0\}.$$

$$(\mathbb{C},0) \xrightarrow{\gamma} (\mathbb{C}^n,0) \xrightarrow{\Psi} (\mathbb{C}^m,0) \quad (1)$$

**Observation:** If $\varphi \in J_k^{reg}(1,1) = \mathbf{G}_k$, then

$$\Psi \circ \gamma = 0 \quad \Rightarrow \quad \Psi \circ (\gamma \circ \varphi) = 0$$

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**Proposition**

$\Sigma_k^0 = \{\Psi \in \Sigma_k : \dim\ker\Psi = 1\}$ *fibers with linear fibres over* $J_k^{reg}(1,n)/\mathbf{G}_k$.

If $\gamma = v_1 t + v_2 t^2 + \ldots + v_k t^k$ with $v_i \in \mathbb{C}^n$;
$\Psi(v) = Av + Bv^2 + \ldots$ with $A \in \mathrm{Hom}(\mathbb{C}^n, C^m)$, $B \in \mathrm{Hom}(\mathrm{Sym}^2(\mathbb{C}^n), \mathbb{C}^m), \ldots$
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$$A(v_1) = 0, \tag{3}$$
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The test curve model says:

$$\Sigma_k(n, m) \doteq \bigcup \{\mathrm{Sol}_\gamma \mid \gamma \in J_k^{reg}(1, n)\}.$$

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**Crucial observation:** The $G_k$ action on $\gamma = (v_1, \ldots, v_k)$ correspond to linear combinations of the equations.

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$$\rho(v_1, \ldots, v_k) = (v_1, v_2 + v_1^2, \ldots, \sum_{a_1+a_2+\ldots+a_i=j} v_{a_1} v_{a_2} \ldots v_{a_i}, \ldots),$$

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$$\rho^{flag} : \mathrm{Hom}^0(\mathbb{C}^k, \mathbb{C}^n)/\mathbf{G}_k \hookrightarrow \mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n)$$

$$\rho^{grass} : \mathrm{Hom}^0(\mathbb{C}^k, \mathbb{C}^n)/\mathbf{G}_k \hookrightarrow \mathrm{Grass}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n)$$

$$\rho^{proj} = Pluck \circ \rho^{grass} : \mathrm{Hom}^0(\mathbb{C}^k, \mathbb{C}^n)/\mathbf{G}_k \hookrightarrow \mathbb{P}(\wedge^k(\mathrm{Sym}^{\le k}\mathbb{C}^n))$$

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$$I = (\Delta_{\mathbf{i}_1,\ldots,\mathbf{i}_k}(f) : (\mathbf{i}_1,\ldots,\mathbf{i}_k) \in \binom{\mathrm{sym}^{\leq k}(n)}{k})$$

*be the ideal generated by the* $k \times k$ *minors of* $\rho(f' \ldots, f^{(k)}) \in \mathrm{Hom}(\mathbb{C}^k, \mathrm{Sym}^{\leq k}\mathbb{C}^n)$*. Then for* $k \geq 4$ *we get* $E_k^n = \sqrt{I}$*.*

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## Example

*$n = 3, k = 3$. There is a codimension 1 boundary component. Here a modification of the theorem holds. $E_3^3$ is generated by 16 independent invariants.*

$$J_3^{\mathrm{reg}}(1,2) = \{(f_i', f_i'', f_i''' : i = 1,2,3) \in (\mathbb{C}^3)^3; (f_1', f_2', f_3') \neq (0,0,0)\},$$

*Then invariants*$=3 \times 3$ *minors in* $\mathbb{C}[f_i^{(j)}]_{f_1'}$ *of:*

$$(f_1', f_2', f_3', f_1'', f_2'', f_3'', f_1''', f_2''', f_3''') \mapsto$$

$$\left( \begin{array}{cccccccccc} f_1' & f_2' & f_3' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ f_1'' & f_2'' & f_3'' & (f_1')^2 & 2f_1'f_2' & (f_2')^2 & 2f_1'f_3' & 2f_2'f_3' & (f_3')^2 & 0 \\ f_1''' & f_2''' & f_3''' & 2f_1'f_1'' & 2f_1'f_2'' + 2f_1''f_2' & 2f_2'f_2'' & 2f_1'f_3'' + 2f_3'f_1'' & 2f_2'f_3'' + 2f_2''f_3' & 2f_3'f_3'' & (f_1')^3 \end{array}\right.$$

Define

$$\phi : \mathrm{Hom}(\mathbb{C}^{\mathrm{sym}^{\leq k}(p)}, \mathbb{C}^n) \to \mathrm{Hom}(\mathbb{C}^{\mathrm{sym}^{\leq k}(p)}, \mathrm{Sym}^{\leq k}\mathbb{C}^n) \qquad (5)$$

$$(v_{10,\ldots 0}, v_{01\ldots 0}, \ldots, v_{0\ldots 0k}) \mapsto (\ldots, \sum_{\mathbf{s_1}+\mathbf{s_2}+\ldots+\mathbf{s_j}=\mathbf{s}} v_{\mathbf{s_1}} v_{\mathbf{s_2}} \ldots v_{\mathbf{s_j}}, \ldots),$$

where the generic component on the right hand side corresponds to $\mathbf{s} \in \mathbb{Z}^p_{\geq 0}$.

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*For $k = p = 2$ $\phi$ looks as follows*

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END OF THE FIRST TALK, THANKS FOR YOUR ATTENTION