Non-reductive GIT and applications

Gergely Bérczi – Oxford
Based on joint work with F. Kirwan, B. Doran, T. Hawes, J. Jackson and A. Szenes

QGM Aarhus

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- $\mathcal{O}_L(X) = \oplus_{k=0}^{\infty} H^0(X, L^{\otimes k})$ has an induced G-action. If G is reductive, $\mathcal{O}_L(X)^G$ is finitely generated graded algebra, and $\mathcal{O}_L(X)^G \hookrightarrow \mathcal{O}_L(X)$ induces

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- Hilbert-Mumford numerical criterion: $T \subset G$ max torus acting diagonally via $t \cdot [x_0 : \ldots : x_n] = [t^{\alpha_0} x_0 : \ldots : t^{\alpha_n} x_n]$ for some $\alpha_i \in \mathbb{C}^r = \mathfrak{t}^*_{\mathbb{C}}$.

$x = [x_0 : \ldots : x_n] \in X_T^{s(ss)} \Leftrightarrow \{0\}$ sits inside(in the closure of) $\mathrm{Conv}(\alpha_i : x_i \neq 0)$

$x \in X$ is (semi)stable for G iff gx is T-(semi)stable for all $g \in G$.

What is G is not reductive?

Cornerstones of bad behaviour:

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3. Recall: every linear algebraic group can be written as $H = U \rtimes R$ where R is reductive and U is unipotent. Every orbit for a unipotent group action on a quasi-affine variety is closed+no points with non-trivial but finite isotropy groups. BUT! Unipotent orbits cannot necessarily be separated with invariants (see example in my talk tomorrow!)

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A: (Kirwan-Doran 2007, B.-Doran-Hawes-Kirwan 2015) One can define open subsets X^s (stable points) and X^{ss} (semistable points) with a geometric quotient $X^s \to X^s/G$ and an enveloping quotient $X^{ss} \to X//G$ s.t. if G is finitely generated then $X//G = Proj(\mathcal{O}_L(X)^G)$ and

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No topological description of $X//G$ and no Hilbert-Mumford criterion.

There are several possible definitions capturing different features of the reductive case (see Doran-Kirwan 2008, Fauntleroy 1985) but there is a canonical notion of stability.

Definition (Semi-stability and stability for unipotent actions, Doran-Kirwan 2007)

Let U be a unipotent group acting on the smooth projective variety X w.r.t an ample line bundle L. Let $I = \mathcal{O}_L(X)^U = \cup_{m \geq 0} H^0(X, L^{\otimes m})^U$ be the ring of invariants.

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Graded algebraic groups

Definition

U (unipotent) group is graded if $\exists\ \lambda : \mathbb{C}^* \to Aut(U)$ *with the weights of the* $\mathbb{C}^*$ *action on* $Lie(U)$ *all positive. This defines the group extension*

$$\hat{U} = U \rtimes \mathbb{C}^* \text{ with } (u,t)\cdot(u',t') = (u \cdot \lambda(t)(u'), tt')$$

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In most applications the acting non-reductive group is graded.
1) **Moduli spaces of toric hypersurfaces**

$$\mathrm{Aut}(\mathbb{P}(1,1,2)) = R \ltimes U \text{ where } R = GL(2) \text{ and } U = (\mathbb{C}^+)^3$$

$$(x,y,z) \mapsto (x,y,z+\lambda x^2 + \mu xy + \nu y^2) \text{ for } (\lambda,\mu,\nu) \in (\mathbb{C}^+)^3$$

$$\mathrm{Aut}(\mathbb{P}(1,1,2)) = \left(\begin{array}{ccc|c} & & & \nu \\ & \mathrm{Sym}^2 GL(2) & & \mu \\ & & & \lambda \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

The central one-parameter subgroup $\mathbb{C}^* = \begin{pmatrix} t^2 & & & \\ & t^2 & & \\ & & t^2 & \\ 0 & 0 & 0 & 1 \end{pmatrix}$ of $R \cong GL(2)$ acts on $Lie(U)$ with weight 2.

2) **Jets of reparametrisation germs**

- $J_k(1,n) = \{k\text{-jets of germs of holomorphic maps } (\mathbb{C},0) \to (\mathbb{C}^n,0)\} =$
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- $J_k(1,1)$ is the group of k-jets of germs of biholomorphisms of $(\mathbb{C},0)$. $J_k(1,1)$ acts on $J_k(1,n)$ via reparametrisation:
  - $f(z) = zf'(0) + \frac{z^2}{2!}f''(0) + \ldots + \frac{z^k}{k!}f^{(k)}(0) \in J_k(1,n)$ and
  - $\varphi(z) = \alpha_1 z + \alpha_2 z^2 + \ldots + \alpha_k z^k \in J_k(1,1)$

Then

$$f \circ \varphi(z) = (f'(0)\alpha_1)z + (f'(0)\alpha_2 + \frac{f''(0)}{2!}\alpha_1^2)z^2 + \ldots$$

$$= (f', \ldots, f^{(k)}/k!) \cdot \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \ldots & 2\alpha_1\alpha_{k-1} + \ldots \\ 0 & 0 & \alpha_1^3 & \ldots & 3\alpha_1^2\alpha_{k-2} + \ldots \\ 0 & 0 & 0 & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \alpha_1^k \end{pmatrix} = \mathbb{C}^* \rtimes \mathbf{U}_k$$

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This group plays a central role in many applications: Enumerative geometry of Hilbert schemes of points, singularities of maps, hyperbolicity questions and in particular the Green-Griffiths-Lang conjecture, see later.

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$$X = \bigsqcup_{\beta \in \mathcal{B}} S_\beta$$

into locally closed subsets S_β indexed by the finite subset $\mathcal{B}$ of a +ve Weyl chamber in $\mathrm{Lie}(K)$

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- $S_\beta \cong G \times_{P_\beta} Y_\beta^{ss}$ where $P_\beta = \{g \in G |\lim_{t \to 0} \beta(t) g \beta(t^{-1})$ exists $\}$ is parabolic subgroup and

  $Y_\beta = \{[x_0 : \ldots : x_n] \in X | x_i = 0$ if $\alpha_i \cdot \beta < ||\beta||$ and $x_i \neq 0$ for some $\alpha_i \cdot \beta = ||\beta||^2\}$

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To construct a quotient of (an open subset of) S_β by G we can study the linear action on $\overline{Y}_\beta$ of the parabolic subgroup P_β. The parabolic is a graded group.

3) **GIT stratification** Given a linear G-action, X has a stratification

$$X = \bigsqcup_{\beta \in \mathcal{B}} S_\beta$$

into locally closed subsets S_β indexed by the finite subset $\mathcal{B}$ of a +ve Weyl chamber in $\mathrm{Lie}(K)$

$\mathcal{B} = \{\beta \in \mathfrak{t}_+ \textit{is the closest point to } 0 \textit{ of the convex hull of some subset of the weights.}\}$

- $S_0 = X^{ss}$
- For each $\beta \in \mathcal{B}$ the closure of S_β is contained in $\bigcup_{\gamma \geqslant \beta} S_\gamma$
- $S_\beta \cong G \times_{P_\beta} Y_\beta^{ss}$ where $P_\beta = \{g \in G | \lim_{t \to 0} \beta(t) g \beta(t^{-1})$ exists $\}$ is parabolic subgroup and

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4) **Representations of quivers with multiplicities:** describing moduli of meromorphic connections on the trivial bundle over $\mathbb{P}^1$ as moduli of representations of quivers with multiplicities (Crawley-Boevey, Boalch, Yamakawa, Hausel). Acting group is block-parabolic.

The $\hat{U}$ Theorem

- Let U be a graded unipotent lin. alg. group and $\hat{U} = U \rtimes \mathbb{C}^*$. Let X be an irreducible normal $\hat{U}$-variety and $L \to X$ an very ample linearisation of the $\hat{U}$ action.

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- $X^{\mathbb{C}^*}_{\mathbf{min}} := X \cap \mathbb{P}(H^0(X,L)^*_{\mathbf{min}}) = \left\{ \; x \in X \;\middle|\; \begin{array}{c} x \text{ is a } \mathbb{C}^*\text{-fixed point and} \\ \mathbb{C}^* \text{ acts on } L^*|_x \text{ with min weight} \end{array} \right\}$
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1. Apply the $\hat{U}$-Thm to diagonal action of $\hat{U}$ on $X \times \mathbb{P}^1$ and linearisation $L \otimes \mathcal{O}_{\mathbb{P}^1}(m)$ with m big to define

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5. (joint with F. Kirwan) Cohomology of $\hat{U}$-quotients: Kirwan surjectivity for $H^*_{\hat{U}}(X^{ss}) \to H^*(X//\hat{U})$ and residue formulae for cohomological pairings (intersection numbers) on $X//\hat{U}$.

In progress:

- (with F. Kirwan and J. Jackson and V. Hoskins) Constructing moduli spaces of sheaves of fixed Harder–Narasimhan type over a nonsingular projective variety W.
- (with F. Kirwan and J. Jackson) Moduli of meromorphic connections on a curve–constructing moduli of representations of quivers with multiplicities after P. Boalch, Yamakawa, Hausel-Wong-Wyn.
- (with F. Kirwan) Moduli space of hypersurfaces in toric varieties.

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We demonstrate the power of this approach on 3 examples:

1. Curve counting and hypersurface counting–Tautological integrals on geometric subsets of Hilbert schemes.
2. Topology of singularities of maps–Thom polynomials
3. Hyperbolicity of varieties–Green-Griffiths-Lang conjecture and intersection numbers of the Demailly jet differentials bundle.

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Non-reductive quotient model of curvilinear Hilbert schemes:

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1) Curvilinear Hilbert schemes on smooth varieties

- $X/\mathbb{C}$ smooth variety of dimension n.
- $X^{[k]} = \{\xi \subset X : \dim(\xi) = 0, l(\xi) = \dim H^0(\xi, \mathcal{O}_\xi) = k\}$ Hilbert scheme of k points on X
- $X_p^{[k]} = \{\xi \in X^{[k]} : \mathrm{supp}(\xi) = p\}$ Punctual Hilbert scheme at $p \in X$
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The natural embedding $\rho : J_k(1,n)/J_k(1,1) \hookrightarrow \mathrm{Grass}(k, J_k(n,1)^*)$ then gives a parametrised subvariety

$$\overline{CX_p^{[k+1]}} = \overline{im(\rho)} \subset \mathrm{Grass}(k, J_k(n,1)^*) \text{ where } J_k(n,1)^* \simeq \oplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n.$$

Tautological integrals

Ultimate goal: Determine the geometric and topological invariants of $X^{[n]}$.
For surfaces these were extensively studied such as **Betti numbers** (Ellingsrud, Stromme, Göttsche), **Hodge numbers** (Sörgel, Göttsche), **cohomology ring** (Nakajima, Grojnowski), **Chern numbers of tautological bundles** (Kleiman-Piene, Lehn, Rennemo, Marian-Oprea-Pandharipande).

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E.g for a nilpotent algebra A of dim k $Q_A = \{\xi \in X^{[k]} : \xi \simeq A\}$.

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For any geometric subset $\mathcal{X}(\mathbf{Q}) \subset X^{[k]}$ and Chern monomial $P(c_1, c_2, \ldots)$ of degree $\dim(\mathcal{X}(\mathbf{Q}))$ *the tautological integral*

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$$\int_{\widetilde{CX}^{[k+1]}} P(c_i(F)) = \int_X \operatorname{Res}_{\mathbf{z}=\infty} \frac{\prod_{i<j}(z_i - z_j) Q_k(\mathbf{z}) P(c_i(z_i + \theta_j, \theta_j)) d\mathbf{z}}{\prod_{i+j \le l \le k}(z_i + z_j - z_l)(z_1 \dots z_k)^n} \prod_{i=1}^{k} s_X(\frac{1}{z_i})$$

Features of the residue formulae:

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$$\epsilon = \sum_{m=1}^{k} \sum_{r=1}^{k-m} q_{m+r}^{mr} \subset W = \bigoplus_{1 \le m+r \le l \le k} \mathbb{C} q_l^{mr} \subset \mathrm{Hom}(\mathbb{C}^k, \mathrm{Sym}^2\mathbb{C}^k)$$

Then $Q_k(z) = \mathrm{mdeg}(\overline{B_k \epsilon}, W)$. In particular $Q_{1,2,3} = 1, Q_4 = 2z_1 + z_2 - z_4$.

Let S be a nonsingular projective surface and L a 5δ-ample line bundle on S. Let $N_\delta(L)$ denote the count of δ-nodal hypersurfaces in a generic linear system $\mathbb{P}^\delta \subset |L|$. According to Kleiman and Piene we can write $N_\delta = P_\delta/\delta!$ where P_δ satisfy the formal identity $\sum_{\delta \geq 0} \frac{P_\delta t^\delta}{r\delta!} = \exp(\sum_{q \geq 1} \frac{a_q t^q}{q!})$ for some integers $a_0, a_1, \ldots$. In particular,

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Theorem (New formula for counts of δ-nodal curves, B.-Szenes 2016)

Introduce the variables indexed by boxes

$\mathcal{B} =$

z_{01}	z_{11}		$z_{\delta 1}$				
$\cdot$	z_{10}		$z_{\delta 0}$				$z_{2\delta,0}$

Then for $\delta > 1$ we have the following formula for a_δ

$$\operatorname*{Res}_{\mathbf{z}=\infty} \frac{(z_{\mathbf{1}\mathbf{0}}\cdots z_{\delta\mathbf{0}})^{-\mathbf{1}} \prod_{(a,b)<(a',b')} (z_{ab}-z_{a'b'})\, c_{\mathbf{2}\delta}(L, L+z_{ab} : (a,b)\in\mathcal{B}) \prod_{(a,b)\in\mathcal{B}} s_S\left(\frac{\mathbf{1}}{z_{ab}}\right) d\mathbf{z}}{\prod_{a+b\le c\le \mathbf{2}\delta} (z_{a\mathbf{0}}-z_{b\mathbf{0}}) \prod_{\substack{(a,b)+(a',b')\le(c,\mathbf{1})\le(\delta,\mathbf{1}) \\ (a,b)+(a',b')\le(c+\mathbf{1},\mathbf{0})\le(\delta+\mathbf{1},\mathbf{0})}} (z_{ab}+z_{a'b'}-z_{a'b'})(\prod_{(a,b)\in\mathcal{B}} z_{ab})^{\mathbf{2}}}$$

Problem:

- $f : M \to N$ holomorphic map between complex manifolds, $\dim(M) = m \leq \dim(N) = n$.
- A is a f.g. nilpotent algebra, e.g $A = z\mathbb{C}[z]/z^k$.
- $\alpha_f(A) = [\{p \in M : f_p \text{ has local algebra } A\}] \in H^*(M)$.
  Thom,1950's: $\alpha_f(A) = \mathrm{Tp}_A(c_1, c_2, \ldots)$ is a polynomial of the Chern classes c_i of $TN - f^*TM$.

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Theorem (Thom polynomial of A_k singularities, B.-Szenes 2012)

$$\mathrm{Tp}_k^{m-n} = \mathrm{Res}_{\mathbf{z}} \frac{(-1)^k \prod_{m<l\le k}(z_m - z_l)\, Q_k(\mathbf{z})}{\prod_{m+r\le l\le k}(z_m + z_r - z_l)} \prod_{l=1}^{k} c\left(\frac{1}{z_l}\right) z_l^{m-n}\, dz_l,$$

where $c\left(\frac{1}{z_l}\right) = 1 + \frac{c_1}{z_l} + \frac{c_2}{z_l^2} + \ldots$ *is the total Chern class of* $TN - f^*TM$.

Conjecture (Green-Griffiths-Lang conjecture,1981)

Every projective algebraic variety X of general type contains a proper algebraic subvariety $Y \subsetneq X$ such that every nonconstant entire holomorphic curve $f : \mathbb{C} \to X$ satisfies $f(\mathbb{C}) \subset Y$.

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Related to the stronger concept of a hyperbolic variety of Kobayashi: a projective variety X is hyperbolic if there is no non-constant entire holomorphic curve in X, i.e. any holomorphic map $f : \mathbb{C} \to X$ must be constant. Hyperbolic algebraic varieties have attracted considerable attention, in part because of their conjectured diophantine properties: Lang conjectured that any hyperbolic complex projective variety over a number field K can contain only finitely many rational points over K.

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Theorem (Degree of the jet differentials line bundle of Demailly, B. 2013)

For any homogeneous polynomial $P = P(u, h)$ of degree $\deg(P) = \dim \tilde{\mathcal{X}}_k = n + k(n-1)$ we have

$$\int_{\tilde{\mathcal{X}}_k} P(u,h) = \int_X \mathrm{Res}_{\mathbf{z}} \frac{Q_k(\mathbf{z}) \prod_{m<l}(z_m - z_l) P(z_1 + \ldots + z_k, h)}{\prod_{m+r \le l \le k}(z_m + z_r - z_l)(z_1 \ldots z_k)^n} \prod_{j=1}^{k} s\left(\frac{1}{z_j}\right)$$

Corollary (B. 2013)

The Green-Griffiths-Lang conjecture for hypersurfaces with polynomial degree follows from a positivity conjecture of Rimanyi on the coefficients of Thom polynomials.

Thanks for your attention!