Non-reductive GIT and curve counting

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Mumford's reductive GIT in a nutshell

- X complex projective variety, on which linear algebraic group G acts.
- Linearisation: ample line bundle L on X and a lift of the action of G to L.
- Replacing L with $L^{\otimes k}$ $\Rightarrow$ we can assume that $X \subseteq \mathbb{P}^n$, the action of G on X extends to an action on $\mathbb{P}^n$ given $\rho : G \to GL(n+1)$, and L is the hyperplane line bundle on $\mathbb{P}^n$.
- $\mathcal{O}_L(X) = \oplus_{k=0}^{\infty} H^0(X, L^{\otimes k})$ has an induced G-action. If G is reductive, $\mathcal{O}_L(X)^G$ is finitely generated graded algebra, and $\mathcal{O}_L(X)^G \hookrightarrow \mathcal{O}_L(X)$ induces

$$
\begin{array}{ccccc}
X & \dashrightarrow & X//G = \mathrm{Proj}\mathcal{O}_L(X)^G & & \\
\cup & & \| & & \\
X^{ss} & \twoheadrightarrow & X//G & & \\
\cup & & \cup & & \\
X^s & \to & X^s/G & & \text{Geometric quotient}
\end{array}
$$

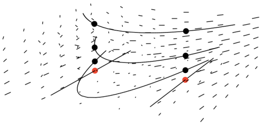
- Topologically: $X//G = X^{ss}/\sim$ where $x \sim y \Leftrightarrow \overline{Gx} \cap \overline{Gy} \cap X^{ss} \neq \emptyset$.
- Hilbert-Mumford numerical criterion: $T \subset G$ max torus acting diagonally via $t \cdot [x_0 : \ldots : x_n] = [t^{\alpha_0} x_0 : \ldots : t^{\alpha_n} x_n]$ for some $\alpha_i \in \mathbb{C}^r = \mathfrak{t}_{\mathbb{C}}^*$.

$$x = [x_0 : \ldots : x_n] \in X_T^{s(ss)} \Leftrightarrow \{0\} \text{ sits inside(in the closure of) } \mathrm{Conv}(\alpha_i : x_i \neq 0)$$

$x \in X$ is (semi)stable for G iff gx is T-(semi)stable for all $g \in G$.

Cornerstones of bad behaviour:

1. $\mathcal{O}_L(X)^G$ is not necessarily finitely generated (Nagata 1957) so $Proj(\mathcal{O}_L(X)^G)$ is not a projective variety.
2. Even if $\mathcal{O}_L(X)^G$ fin. gen. the quotient map $q : X \to Proj(\mathcal{O}_L(X)^G)$ is not nec. surjective, the image is just constructible.
3. Recall: every linear algebraic group can be written as $H = U \rtimes R$ where R is reductive and U is unipotent. Unipotent orbits cannot necessarily be separated with invariants.
   Example: $\mathbb{C}^+ = \{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{C}\}$ acts on $X = \mathrm{Sym}^2\mathbb{C}^2$. Let x_0, x_1, x_2 be basis of X. Then $A(X)^{\mathbb{C}^+} = \mathbb{C}[x_0, x_1^2 - x_0x_2]$ and the orbit space looks like:



Q: Can we define a sensible 'quotient' variety $X//G$ when G is not reductive?

Definition

U (unipotent) group is graded if $\exists\, \lambda : \mathbb{C}^* \to Aut(U)$ with the weights of the $\mathbb{C}^*$ action on $Lie(U)$ all positive. This defines the group extension

$$\hat{U} = U \rtimes \mathbb{C}^* \text{ with } (u,t)\cdot(u',t') = (u\cdot\lambda(t)(u'), tt')$$

We say that $H = U \rtimes R$ is graded if there is a central $\mathbb{C}^* \subset R$ such that

$$\hat{U} = U \rtimes \mathbb{C}^* \subset H \text{ is graded}$$

In most applications the acting non-reductive group is graded.
1) **Moduli spaces of toric hypersurfaces**

$$\mathrm{Aut}(\mathbb{P}(1,1,2)) = R \ltimes U \text{ where } R = GL(2) \text{ and } U = (\mathbb{C}^+)^3$$

$$(x,y,z) \mapsto (x,y,z+\lambda x^2+\mu xy+\nu y^2) \text{ for } (\lambda,\mu,\nu) \in (\mathbb{C}^+)^3$$

$$\mathrm{Aut}(\mathbb{P}(1,1,2)) = \left(\begin{array}{ccc|c} & \mathrm{Sym}^2 GL(2) & & \begin{matrix}\nu\\ \mu\\ \lambda\end{matrix} \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

The central one-parameter subgroup $\mathbb{C}^* = \begin{pmatrix} t^2 & & & 0 \\ & t^2 & & 0 \\ & & t^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ of $R \cong GL(2)$ acts on $Lie(U)$ with weight 2.

2) **Jets of reparametrisation germs**

- $J_k(1,n) = \{k\text{-jets of germs of holomorphic maps } (\mathbb{C},0) \to (\mathbb{C}^n,0)\} = $
$= \{(f', f'', \ldots, f^{(k)}) : f' \neq 0\}$
- $J_k(1,1)$ is the group of k-jets of germs of biholomorphisms of $(\mathbb{C},0)$. $J_k(1,1)$ acts on $J_k(1,n)$ via reparametrisation:
  - $f(z) = zf'(0) + \frac{z^2}{2!}f''(0) + \ldots + \frac{z^k}{k!}f^{(k)}(0) \in J_k(1,n)$ and
  - $\varphi(z) = \alpha_1 z + \alpha_2 z^2 + \ldots + \alpha_k z^k \in J_k(1,1)$

  Then

$$f \circ \varphi(z) = (f'(0)\alpha_1)z + (f'(0)\alpha_2 + \frac{f''(0)}{2!}\alpha_1^2)z^2 + \ldots$$

$$= (f', \ldots, f^{(k)}/k!) \cdot \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \ldots & 2\alpha_1\alpha_{k-1} + \ldots \\ 0 & 0 & \alpha_1^3 & \ldots & 3\alpha_1^2\alpha_{k-2} + \ldots \\ 0 & 0 & 0 & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \alpha_1^k \end{pmatrix} = \mathbb{C}^* \rtimes \mathbf{U}_k$$

This group plays a central role in many applications: Enumerative geometry of Hilbert schemes of points, singularities of maps, hyperbolicity questions and in particular the Green-Griffiths-Lang conjecture...

The $\hat{U}$ Theorem

- Let U be a graded unipotent lin. alg. group and $\hat{U} = U \rtimes \mathbb{C}^*$. Let X be an irreducible normal $\hat{U}$-variety and $L \to X$ an very ample linearisation of the $\hat{U}$ action.
- $X_{\min}^{\mathbb{C}^*} := X \cap \mathbb{P}(H^0(X,L)^*_{\min}) = \left\{ x \in X \;\middle|\; \begin{array}{c} x \text{ is a } \mathbb{C}^*\text{-fixed point and} \\ \mathbb{C}^* \text{ acts on } L^*|_x \text{ with min weight} \end{array} \right\}$
- $X_{\min} := \{x \in X \mid \lim_{t\to 0} t \cdot x \in X_{\min}^{\mathbb{C}^*}\}$ $(t \in \mathbb{G}_m \subseteq \hat{U})$ the Bialynicki-Birula stratum.

Theorem (B.,Doran, Hawes, Kirwan 2012-16)

Let $L \to X$ be an irreducible normal $\hat{U}$-variety with ample linearisation L. If

$$(*)\ x \in X_{\min}^{\mathbb{C}^*} \Rightarrow \mathrm{Stab}_U(x) = 1$$

(or more generally $\dim \mathrm{Stab}_U(x) = \min_{y\in X} \dim \mathrm{Stab}_U(y)$*) holds then*

1. $X_{\min} \subseteq X^{s(U,L)}$*, so we get a locally trivial U-quotient* $X_{\min} \to X_{\min}/U$.
2. *Twist the linearisation by a character $\chi : \hat{U} \to \mathbb{C}^*$ of $\hat{U}/U$ such that 0 lies in the lowest bounded chamber for the $\mathbb{C}^*$ action on X. Then for suitably divisible integers $r > 0$* $\mathcal{O}_\chi(X)^{\hat{U}} = \mathcal{O}_{(L^\chi)^{\otimes r}}(X)^{\hat{U}}$ *is* ***finitely generated*** *and for the quotient*

$$X/\!/_\chi \hat{U} := \mathrm{Proj}(\mathcal{O}_\chi(X)^{\hat{U}})$$

we have $X/\!/\hat{U} = X^{ss,\hat{U}}/\sim$ *where* $x \sim y \Leftrightarrow \overline{\hat{U}x} \cap \overline{\hat{U}y} \cap X^{ss} \neq \emptyset$.

3. $X_{\min} \setminus (U \cdot X_{\min}^{\mathbb{C}^*}) = X^{s,\hat{U}} = X^{ss,\hat{U}}$ *and* $X/\!/_\chi \hat{U} = X^{s,\hat{U}}/\hat{U}$ *is a geometric quotient of* $X^{s,\hat{U}}$ *by* $\hat{U}$. $X^{s,\hat{U}}, X^{ss,\hat{U}}$ *are given by* ***Hilbert-Mumford type criterion.***

Remarks

1. Similar construction for any graded non-reductive group $H = U \rtimes R$.
2. If (*) does not hold then there is a sequence of blow-ups of X along $\hat{U}$-invariant projective subvarieties resulting in a projective variety $\hat{X}$ with a linear action of $\hat{U}$ (with respect to a power of an ample line bundle given by perturbing the pullback of L by small multiples of the exceptional divisors for the blow-ups) which satisfies (*). Namely, let d maximal s.t

$$X^d_{\mathrm{min}} = \{z \in X^{\mathbb{C}^*}_{\mathrm{min}} \mid \dim \mathrm{Stab}_U(z) = d\} \neq \emptyset.$$

   Then $U \cdot X^d_{\mathrm{min}} = \hat{U} \cdot X^d_{\mathrm{min}}$ is a closed subvariety of X^0_{min} and we blow-up this and perturb the pull-back of L with a small multiple of the ex. divisor to get ample $\hat{U}$-linearisation on the blown-up.
3. (joint with J. Jackson and F. Kirwan): Nice VGIT picture for $H = U \rtimes R$ action when R contains higher dimensional grading torus, that is, the grading $\mathbb{C}^*$ can be moved inside R.
4. (joint with F. Kirwan) Cohomology of $\hat{U}$-quotients: symplectic implosion for $\hat{U}$-quotients and Kirwan surjectivity for $H^*_{\hat{U}}(X^{ss}) \to H^*(X//\hat{U})$ and residue formulae for cohomological pairings (intersection numbers) on $X//\hat{U}$.

Hilbert schemes and tautological bundles

- $X/\mathbb{C}$ smooth variety of dimension n.
- $X^{[k]} = \{\xi \subset X : \dim(\xi) = 0, l(\xi) = \dim H^0(\xi, \mathcal{O}_\xi) = k\}$ Hilbert scheme of k points on X
- $X_p^{[k]} = \{\xi \in X^{[k]} : \mathrm{supp}(\xi) = p\}$ Punctual Hilbert scheme at $p \in X$

Ultimate goal: Determine the geometric and topological invariants of $X^{[n]}$.
For surfaces these were extensively studied such as **Betti numbers** (Ellingsrud, Stromme, Göttsche), **Hodge numbers** (Sörgel, Göttsche), **cohomology ring** (Nakajima, Grojnowski,Lehn), **Chern numbers of tautological bundles** (Kleiman-Piene, Lehn, Rennemo, Marian-Oprea-Pandharipande).
When $\dim(X) > 2$ not much is known:

- Rennemo, Tzeng: see in a minute
- motivic DT invariants (Behrend-Bryan-Szendröi).

Tautological vector bundles:
F–rank r bundle (loc. free sheaf) on $X \rightsquigarrow F^{[k]}$–rank rk bundle over $X^{[k]}$:
Fibre over $\xi \in X^{[k]}$ is $F \otimes \mathcal{O}_\xi = H^0(\xi, F|_\xi)$.

Equivalently: $F^{[k]} = q_* p^*(F)$ where

$$X^{[k]} \times X \supset \begin{array}{ccc} \mathcal{Z} & \xrightarrow{q} & X^{[k]} \\ \downarrow{\scriptstyle p} & & \\ X & & \end{array}.$$

Following Rennemo (2014) we define the **geometric subset** of type $\mathbf{Q} = (Q_1, \ldots, Q_s)$ as

$$P(\mathbf{Q}) = \{\xi \in X^{[k]} : \xi = \xi_1 \cup \ldots \cup \xi_s \text{ where } \xi_i \in X_{p_i}^{[k_i]} \text{ is of type } Q_i\}.$$

A **type** $Q \subset X_p^{[k]}$ is a closed subset which is union of isomorphism classes:

$$\xi \in Q, \xi \simeq \xi' \Rightarrow \xi' \in Q.$$

Definition: For an algebra A of dimension k let $Q_A = \{\xi \in X^{[k]} : \xi \simeq A\}$.
Fix monomial ideals $I_{\lambda_1}, \ldots, I_{\lambda_s}$ corresponding to the partitions (Young tableau) $\lambda_1, \ldots, \lambda_s$ and let

$$Q_{\lambda_i} = \overline{\{\xi : \mathcal{O}_\xi \simeq \mathbb{C}[z_1, \ldots, z_n]/I_{\lambda_i}\}}.$$

Let $P(\lambda_1, \ldots, \lambda_s) = P(Q_{\lambda_1}, \ldots, Q_{\lambda_s})$ denote the corresponding monomial geometric subset.

- Simplest case: $s = 1$ and $\lambda = \underbrace{\square\square\cdots\square}_{k}$ then $A = \mathbb{C}[z]/z^k$ and

$$Q_\lambda = CX_p^{[k]} = \{\xi \in X_p^{[k]} : \xi \subset \mathcal{C}_p \text{ for some smooth curve } \mathcal{C} \subset X\}$$

$$P(\underbrace{\square\square\cdots\square}_{k}) = CX^{[k]} = \cup_{p \in X} CX_p^{[k]} \text{ is the curvilinear component.}$$

- Most famous case (Göttsche): $X = S$ surface, $\lambda_1 = \ldots = \lambda_s = \begin{smallmatrix}\square & \\ \square & \square\end{smallmatrix}$. Then

  $I_\lambda = (x^2, xy, y^2)$ and

$$P(s \cdot \lambda) = P\left(\underbrace{\begin{smallmatrix}\square & \\ \square & \square\end{smallmatrix}, \ldots, \begin{smallmatrix}\square & \\ \square & \square\end{smallmatrix}}_{s}\right) \subset S^{[3s]}.$$

  is $2s$-dimensional closed variety and the count of s-nodal curves in the $5s$-ample linear system $\mathbb{P}(|L|)$ is $\int_{P(s\cdot\lambda)} c_{2s}(L^{[3s]})$.

$$P(Q_1,\ldots,Q_s)=\{\xi\in X^{[k]}:\xi=\xi_1\sqcup\ldots\sqcup\xi_s \text{ where } \xi_i\in X_{p_i}^{[k_i]} \text{ is of type } Q_i\}$$

Motivation: counting hypersurfaces with prescribed singularities on X.
Fact: Let $T_1,\ldots,T_s$ be analytic singularity types with expected codimension $d=\sum_{i=1}^s d_i$. Then there is a k and a geometric set $W=P(T_1,\ldots,T_s)\subset X^{[k]}$ such that a generic hypersurface containing a $Z\in W$ has the specified singularities. Therefore a general $\mathbb{P}^d\subseteq|L|$ the number of hypersurfaces containing a subscheme $Z\in W$ equals $\int_W c_{\dim(W)}(L^{[k]})$.

Theorem (Existence of universal polynomial, Rennemo, Tzeng 2013-14)

For any geometric subset $P(\mathbf{Q})\subset X^{[k]}$ and Chern monomial $R(c_1,c_2,\ldots)$ of degree $\dim(P(\mathbf{Q}))$ *the tautological integral*

$$\int_{P(\mathbf{Q})} R(c_i(F^{[k]}))=\tilde{R}(c_i(F),c_i(X))$$

is given by a universal polynomial of the Chern numbers of F and X.

Rest of the talk: Show that non-reductive moduli description combined with equivariant localisation results in closed iterated residue formulae for $\int_{P(\mathbf{Q})} R(c_i(F^{[k]}))$.

$CX_p^{[k]} = \{\xi \in X_p^{[k]} : \xi \subset \mathcal{C}_p$ for some smooth curve $\mathcal{C} \subset X\} =$
$\{\xi \in X_p^{[k]} : \mathcal{O}_\xi \simeq z\mathbb{C}[z]/z^{k+1}\}$ Curvilinear locus at p

Def: $\overline{CX_p^{[k]}}$ is the curvilinear component of the Hilbert scheme. This is a singular irreducible projective variety of dimension $(n-1)(k-1)$.

On surfaces ($n = 2$) $\overline{CX_p^{[k]}} = X_p^{[k]}$, for $n > 2$ $\overline{CX_p^{[k]}}$ is a component of the punctual Hilbert scheme.

Non-reductive quotient model of curvilinear Hilbert schemes:

$\xi \in CX_p^{[k+1]} \rightsquigarrow \xi \subset \mathcal{C}_p \subset X$ smooth curve germ $\rightsquigarrow f_\xi : (\mathbb{C}, 0) \to (X, p)$ k-jet of germ parametrising $\mathcal{C}_p$. f_ξ is determined up to polynomial reparametrisation germs $\phi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$. Let

$$J_k(1, n) = \{k\text{-jets } f : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)\} = \{(f', f'', \ldots, f^{[k]}) : f' \neq 0\}$$

$$CX_p^{[k+1]} = \{k\text{-jets } (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)\}/\{k\text{-jets } (\mathbb{C}, 0) \to (\mathbb{C}, 0)\} = J_k(1, n)/J_k(1, 1).$$

Observation: If $g \in J_k(n,1)$ then $g \circ f_\xi = 0 \Rightarrow g \circ (f_\xi \circ \phi) = 0$
In other words: $Ann(f_\xi) \subset J_k(n,1)$ is invariant under $J_k(1,1)$. The map $f_\xi \mapsto Ann(f_\xi)$ defines an embedding

$$\rho : CX_p^{[k+1]} = J_k(1,n)/J_k(1,1) \hookrightarrow \mathrm{Grass}(\mathrm{codim} = k, J_k(n,1))$$

If $f_\xi = f_1 z + \ldots + f_k z^k$ with $f_i \in \mathbb{C}^n$;
$g(v) = Av + Bv^2 + \ldots$ with $A \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C})$, $B \in \mathrm{Hom}(\mathrm{Sym}^2(\mathbb{C}^n), \mathbb{C}), \ldots$
then $g \circ f_\xi = 0$ has the form:

$$A(f_1) = 0,$$
$$A(f_2) + B(f_1, f_1) = 0,$$
$$A(f_3) + 2B(f_1, f_2) + C(f_1, f_1, f_1) = 0,$$

Theorem (B-Szenes 2012, B 2015)

1. *The dual of ρ can be written as*

$$\rho : J_k(1,n)/J_k(1,1) \hookrightarrow \mathrm{Grass}(k, J_k(n,1)^*)$$
$$(f', \ldots, f^{(k)}) \mapsto \mathrm{Span}(f', f'' + (f')^2, \ldots, \sum_{a_1+a_2+\ldots+a_i=j} f^{(a_1)} f^{(a_2)} \ldots f^{(a_i)}, \ldots),$$

2. *For $k \leq n$ $\overline{\rho(J_k(1,n))} = \overline{\mathrm{GL}(n) \cdot \rho(e_1, \ldots, e_n)}$ where $\{e_1, \ldots, e_n\}$ is a basis of $\mathbb{C}^n$.*

Theorem (Integrals over the curvilinear Hilbert schemes, B. 2015)

$$\int_{\widetilde{CX}^{[k+1]}} R(c_i(F)) = \int_X \mathrm{Res}_{\mathbf{z}=\infty} \frac{\prod_{i<j}(z_i - z_j) Q_k(\mathbf{z}) R(c_i(z_i+\theta_j,\theta_j)) d\mathbf{z}}{\prod_{i+j\le l\le k}(z_i+z_j-z_l)(z_1\ldots z_k)^n} \prod_{i=1}^k s_X(\frac{1}{z_i})$$

Features of the residue formulae:

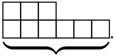
1. The residue gives a degree n symmetric polynomial in Chern roots of F and Segre classes of X reflecting the Gottsche conjecture on curvilinear components.
2. For fixed k the formula gives a universal generating series for the integrals as the dimension increases.
3. The geometry of Q_k. The GL_k-module of 3-tensors $\mathrm{Hom}(\mathbb{C}^k, \mathrm{Sym}^2\mathbb{C}^k)$ has a diagonal decomposition

$$\mathrm{Hom}(\mathbb{C}^k, \mathrm{Sym}^2\mathbb{C}^k) = \bigoplus \mathbb{C}q_l^{mr},\ 1 \le m, r, l \le k,$$

q_l^{mr} has weight $(z_m + z_r - z_l)$. Let

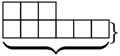
$$\epsilon = \sum_{m=1}^{k}\sum_{r=1}^{k-m} q_{m+r}^{mr} \subset W = \bigoplus_{1\le m+r\le l\le k} \mathbb{C}q_l^{mr} \subset \mathrm{Hom}(\mathbb{C}^k, \mathrm{Sym}^2\mathbb{C}^k)$$

Then $Q_k(z) = \mathrm{mdeg}(\overline{B_k\epsilon}, W)$. In particular $Q_{1,2,3} = 1, Q_4 = 2z_1 + z_2 - z_4$.

Next step Integration on $P(\lambda)$. We demonstrate this on $\lambda = (s, 2s) =$ .

$2s$

Two dimensional algebra $\Rightarrow$ test surface to get non-reductive quotient model.
$\xi \in P(\lambda)_p \rightsquigarrow f_\xi : (\mathbb{C}^2, 0) \to (X, p)$ λ-jet of germ parametrising the surface germ $\mathcal{S}_\xi$ for which $\xi \in \mathcal{S}_\xi$.

$$J_\lambda(\mathbb{C}^2, \mathbb{C}^n) := \{\frac{\partial f}{\partial^i x \partial^j y} | (i,j) \in$$  $$\}$$

2δ

f_ξ is determined up to λ-jets of reparametrisation germs $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$. So

$$P(\lambda)_p = J_\lambda(2, n)/J_\lambda(2, 2).$$

Observation: If $g \in J_{2s}(n, 2)$ then $g \circ f_\xi = 0 \Rightarrow g \circ (f_\xi \circ \phi) = 0$
If $f_\xi = \sum_{(i,j)\in\lambda} f_{ij} x^i y^j$ with $f_{ij} \in \mathbb{C}^n$ and $g(v) = Av + Bv^2 + \ldots$ with $A \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^2)$, $B \in \mathrm{Hom}(\mathrm{Sym}^2(\mathbb{C}^n), \mathbb{C}), \ldots$
then $g \circ f_\xi = 0$ has the form:

$$A(f_{10}) = 0 \qquad A(f_{01}) = 0$$
$$A(f_{20}) + 2B(f_{10}, f_{10}) = 0 \qquad A(f_{11}) + B(f_{10}, f_{01}) = 0$$
$$\ldots$$

We have $3s$ equations and induces an embedding

$$P(\lambda) \hookrightarrow \mathrm{Grass}(3s, \mathrm{Sym}^{\le 2s}\mathbb{C}^n)$$

Then we can apply equivariant localisation, turn the formula into iterated residue and prove a vanishing theorem. (HARD WORK!!!)

Theorem (Tautological integrals on punctual monomial geometric subsets, B. 2016)

Introduce the variables indexed by boxes of λ:

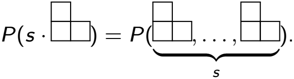
$\lambda =$

z_{01}	z_{11}		z_{s1}				
$\cdot$	z_{10}		$z_{\delta 0}$				$z_{2s,0}$

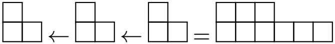
Let X be a smooth projective variety, F a bundle over X, $R(c_i(F^{[3s]})$ a Chern polynomial of degree $\dim(P(\lambda)) = s+3$. *Then*

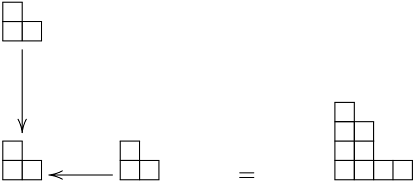
$$\int_{P(\lambda)} R(c_i(F^{[\mathbf{3}s]}) = \operatorname*{\mathbf{Res}}_{\mathbf{z}=\infty} \frac{\prod_{(a,b)<(a',b')} (z_{ab} - z_{a'b'}) R(c_i(\theta_j, \theta_j + z_{ab} : (a,b) \in \mathcal{B})) \prod_{(a,b)\in\mathcal{B}} s_S\left(\frac{\mathbf{1}}{z_{ab}}\right) d\mathbf{z}}{\prod_{a+b\leq c\leq \mathbf{2}s} (z_{a\mathbf{0}} - z_{b\mathbf{0}}) \prod_{\substack{(a,b)+(a',b')\leq(c,\mathbf{1})\leq(s,\mathbf{1}) \\ (a,b)+(a',b')\leq(c+\mathbf{1},\mathbf{0})\leq(s+\mathbf{1},\mathbf{0})}} (z_{ab} + z_{a'b'} - z_{a'b'}) (\prod_{(a,b)\in\mathcal{B}} z_{ab})^{\mathbf{2}}}$$

Similar formulae hold for other Young tableau λ's. Non-reductive quotient model is derived using equations of test m-folds if λ is m-dimensional partition.

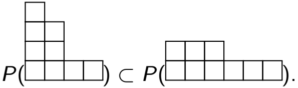
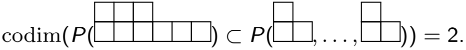
Goal: Study 

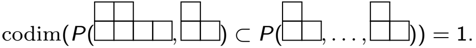
Monomial ideals can come together different ways:

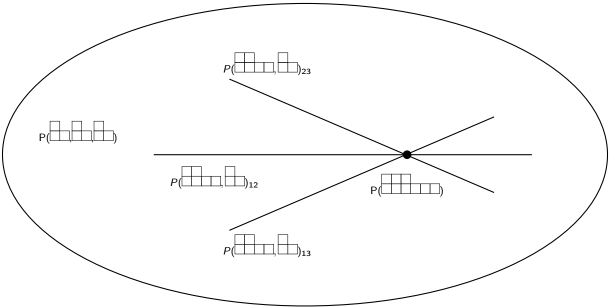
1) along a line: 

2) along different axes: 

Observations:

- 
- 

- 



Observations: If S is toric with number of T-fixed points $\geq s$ (we can assume this by taking appropriate blow-ups..) then

- $P(\ydiagram{4,2}, \ydiagram{2,1})_{ij}$, $P(\ydiagram{6,3})$ are T-invariant subvarieties of $P(\ydiagram{2,1}, \ydiagram{2,1}, \ydiagram{2,1})$.
- The pair $(S_0^{[s]} \subset S^{[s]})$ is T-equivariant to $(P(\underbrace{\ydiagram{6,3}}_{2s}) \subset P(s \cdot \ydiagram{2,1}))$. Due to Haiman $S_0^{[s]}$ is complete intersection in $S^{[s]}$

Lemma (Localisation at invariant subvarieties)

Suppose that M is a compact manifold and T is a complex torus acting smoothly on M, and the fixed point set M^T of the T-action on M is finite. Let $M_1, \dots, M_r$ torus-invariant pairwise disjoint submanifolds such that $M^T \subset \cup_{i=1}^r M_i$. Then for for an equivariantly closed form $\alpha \in H_T^\bullet(M)$

$$\int_M \alpha = \sum_{i=1}^r \int_{M_i} \frac{\alpha^{[\dim(M_i)]}|_{M_i}}{\mathrm{Euler}^T(N(M_i))}.$$

Here $\mathrm{Euler}^T(N(M_i)$ is the T-equivariant Euler class of the T-equivariant normal bundle $N(M_i)$ of M_i in M and $\alpha^{[\dim(M_i)]}$ is the differential-form-degree-$\dim(M_i)$ part of α.

According to Kleiman and Piene we can write $\int_{P(s\cdot \ydiagram{1,2})} c_{2s}(L^{[3s]}) = N_s = P_s/s!$ where

P_s satisfy the formal identity $\sum_{s\geq 0} \frac{P_s t^s}{rs!} = \exp(\sum_{q\geq 1} \frac{a_q t^q}{q!})$ for some integers $a_0, a_1, \dots$. In particular,

$$P_0 = 1, P_1 = a_1, P_2 = a_1^2 + a_2, P_3 = a_1^3 + 3a_2a_1 + a_3, \dots.$$

Observation: Terms of P_s $\leftrightarrow$ partitions of s $\leftrightarrow$ Torus-invariant submanifolds in the localisation formula. In particular: $a_s = \int_{P(s,2s)} \frac{c_{2s}(L^{[3s]})}{Euler^T(P(\underbrace{\ydiagram{3,6}}_{2s}) \subset (P(s\cdot \ydiagram{1,2}))}$.

The residue formula

Let S be a nonsingular projective surface and L a $5s$-ample line bundle on S. Let $N_s(L)$ denote the count of s-nodal hypersurfaces in a generic linear system $\mathbb{P}^s \subset |L|$. According to Kleiman and Piene we can write $N_s = P_s/s!$ where P_s satisfy the formal identity $\sum_{s\geq 0} \frac{P_s t^s}{rs!} = \exp(\sum_{q\geq 1} \frac{a_q t^q}{q!})$ for some integers $a_0, a_1, \ldots$. In particular,

$$P_0 = 1, P_1 = a_1, P_2 = a_1^2 + a_2, P_3 = a_1^3 + 3a_2a_1 + a_3, \ldots.$$

Theorem (New formula for counts of s-nodal curves, B.-Szenes 2016)

Introduce the variables indexed by boxes

$\mathcal{B} =$

z_{01}	z_{11}		z_{s1}				
$\cdot$	z_{10}		z_{s0}				$z_{2s,0}$

Then for $s > 1$ we have

$$a_s = \operatorname*{Res}_{\mathbf{z}=\infty} \frac{(z_{\mathbf{10}}\cdots z_{s\mathbf{0}})^{-\mathbf{1}} \prod\limits_{(a,b)<(a',b')} (z_{ab} - z_{a'b'}) c_{\mathbf{2}s}(L, L + z_{ab} : (a,b) \in \mathcal{B}) \prod\limits_{(a,b)\in\mathcal{B}} s_S\left(\frac{\mathbf{1}}{z_{ab}}\right) d\mathbf{z}}{\prod\limits_{a+b\leq c\leq \mathbf{2}s} (z_{a\mathbf{0}} - z_{b\mathbf{0}}) \prod\limits_{\substack{(a,b)+(a',b')\leq(c,\mathbf{1})\leq(s,\mathbf{1}) \\ (a,b)+(a',b')\leq(c+\mathbf{1},\mathbf{0})\leq(s+\mathbf{1},\mathbf{0})}} (z_{ab} + z_{a'b'} - z_{a'b'})(\prod\limits_{(a,b)\in\mathcal{B}} z_{ab})^{\mathbf{2}}}$$

- $s = 1$. Variables:

z_{01}	
$\cdot$	z_{10}

$$a_1 = 3L^2 + 2Lc_1(S) + c_2(S) = \operatorname*{Res}_{\mathbf{z}=\infty} \frac{(z_{10}-z_{01})^2 c_2(L, L+z_{10}, L+z_{01})d\mathbf{z}}{2(z_{10}z_{01})^2} s(1/z_{10})s(1/z_{01})$$

where:

  - $c_2(L, L+z_{10}, L+z_{01})$ denotes the second elementary symmetric polynomial formes from the formal Chern roots $L, L+z_{10}, L+z_{01}$ and
  - $s_S = 1/c_S$ is the total Segre class of S and in particular $s_0 = 1, s_1 = c_1(S), s_2 = c_1^2 - c_2$.
- $s = 2$. Variables:

z_{01}	z_{11}		
$\cdot$	z_{10}	z_{20}	z_{30}

The following identity is now computer-checked

$$-42L^2 - 39Lc_1(S) - 6c_1^2(S) - 7c_2(S) = a_2 =$$

$$\operatorname*{Res}_{\mathbf{z}=\infty} \frac{\prod_{(a,b)<(a',b')\in\mathcal{B}}(z_{ab}-z_{a'b'})c_4(L, L+z_{10}, L+z_{20}, L+z_{30}, L+z_{01}, L+z_{11})\prod_{(a,b)\in\mathcal{B}} s_S\left(\frac{1}{z_{ab}}\right)dz}{z_{10}(2z_{10}-z_{20})(z_{10}+z_{20}-z_{30})(2z_{10}-z_{30})(z_{10}+z_{01}-z_{30})(z_{10}+z_{01}-z_{11})(2z_{10}-z_{11})(z_{10}z_{20}z_{30}z_{01}z_{1}\cdots}$$

where again

  - $c_4(L, L+z_{10}, L+z_{20}, L+z_{30}, L+z_{01}, L+z_{11})$ denotes the fourth elementary symmetric polynomial formes from the formal Chern roots $L, L+z_{ab}, (a,b) \in \mathcal{B}$.
  - $s_S = 1/c_S$ is the total Segre class of S and in particular $s_0 = 1, s_1 = c_1(S), s_2 = c_1^2 - c_2$.