Nonreductive GIT and tautologial integrals on curvilinear Hilbert schemes

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Joint work with B. Doran, T. Hawes, F. Kirwan

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Topologically: $X//G = X^{ss}/\sim$ where $x \sim y \Leftrightarrow \overline{Gx} \cap \overline{Gy} \cap X^{ss} \neq \emptyset$.

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Hilbert-Mumford numerical criterion: $T \subset G$ max torus.
$x = [x_0 : \ldots : x_n] \in X_T^{ss(s)} \leftrightarrow$ The origin sits inside (in the closure of) $\mathrm{Conv}(w(x_i) : x_i \neq 0) \subset \mathfrak{t}_{\mathbb{C}}^*$.
$x \in X$ is (semi)stable for G iff gx is T-(semi)stable for all $g \in G$.

What is G is not reductive?

Cornerstones of bad behaviour:

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A: One can define open subsets X^s (stable points) and X^{ss} (semistable points) with a geometric quotient $X^s \to X^s/G$ and an enveloping quotient $X^{ss} \to X//G$ s.t. if G is finitely generated then $X//G = Proj(\mathcal{O}_L(X)^G)$ and

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No topological description of $X//G$ and no Hilbert-Mumford criterion.

There are several possible definitions capturing different features of the reductive case (see Doran-Kirwan 2008, Fauntleroy 1985). Recall that every linear algebraic group $H = U \rtimes R$ is the semi-direct product of its maximal unipotent radical U and a reductive R.

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Definition (Semi-stability and stability for unipotent actions)

Let U be a unipotent group acting on the smooth projective variety X w.r.t an ample line bundle L. Let $I = \mathcal{O}_L(X)^U = \cup_{m \geq 0} H^0(X, L^{\otimes m})^U$ be the ring of invariants.

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U (unipotent) group is graded if $\exists\ \lambda : \mathbb{C}^* \to Aut(U)$ with the weights of the $\mathbb{C}^*$ action on $Lie(U)$ all positive. This defines the group extension

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Examples:
1) **Moduli spaces of toric hypersurfaces** E.g $\mathrm{Aut}(\mathbb{P}(1,1,2)) = R \ltimes U$ with $R = GL(2)\times\mathbb{C}^*$ reductive and $U = (\mathbb{C}^+)^3$ unipotent, where U acts as

$$(x,y,z) \mapsto (x,y,z+\lambda x^2+\mu xy+\nu y^2 \quad (\lambda,\mu,\nu)\in\mathbb{C}^3.$$

Induced action on

$$H_4 = \mathrm{Span}(x^4, x^3y, x^2y^2, xy^3, y^4, x^2z, xyz, y^2z, z^2)$$

$$Aut(\mathbb{P}[1,1,2]) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & \lambda^2 \\ 0 & 1 & 0 & 0 & 0 & \mu & \lambda & 0 & 2\lambda\mu \\ 0 & 0 & 1 & 0 & 0 & \nu & \mu & \lambda & 2\nu+\mu^2 \\ 0 & 0 & 0 & 1 & 0 & 0 & \nu & \mu & 2\mu\nu \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \nu & \nu^2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2\lambda \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2\mu \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2\nu \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Examples

2) **GIT stratification** $X = \cup_{\beta\in\mathcal{B}} S_\beta$ such that (i) $S_0 = X^{ss}$ (ii) $\overline{S}_\beta = \cup_{|\gamma|>|\beta|} S_\gamma$ (iii) $S_\beta \simeq G \times_{P_\beta} Y_\beta$ and would be nice to have

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3) **Jets of reparametrisation germs**

- $J_k(1,n) = \{k\text{-jets of germs of holomorphic maps } (\mathbb{C},0) \to (\mathbb{C}^n,0)\} =$
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- $J_k(1,1)$ is the group of k-jets of germs of biholomorphisms of $(\mathbb{C},0)$. If $f(z) = zf'(0) + \frac{z^2}{2!}f''(0) + \ldots + \frac{z^k}{k!}f^{(k)}(0)$ and $\varphi \in J_k(1,1)$ has the form

  $$\varphi(z) = \alpha_1 z + \alpha_2 z^2 + \ldots + \alpha_k z^k \text{ with some } \alpha_1, \ldots, \alpha_k \in \mathbb{C}, \alpha_1 \neq 0$$

  Then

  $$f \circ \varphi(z) = (f'(0)\alpha_1)z + (f'(0)\alpha_2 + \frac{f''(0)}{2!}\alpha_1^2)z^2 + \ldots$$

  $$= (f', \ldots, f^{(k)}/k!) \cdot \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \ldots & 2\alpha_1\alpha_{k-1} + \ldots \\ 0 & 0 & \alpha_1^3 & \ldots & 3\alpha_1^2\alpha_{k-2} + \ldots \\ 0 & 0 & 0 & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \alpha_1^k \end{pmatrix}$$

- More generally: **Representations of quivers with multiplicities:** describing moduli of meromorphic connections on the trivial bundle over $\mathbb{P}^1$. (Crawley-Boevey, Boalch, Yamakawa, Hausel, etc).

The $\hat{U}$ Theorem

- Let U be a graded unipotent lin. alg. group over $k = \bar{k}$ and $\hat{U} = U \rtimes \mathbb{G}_m$. Let X be an irreducible normal $\hat{U}$-variety and $L \to X$ an very ample linearisation of the $\hat{U}$ action.

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- $X^{\mathbb{G}_m}_{\min} := X\cap\mathbb{P}(H^0(X,L)^*_{\min}) = \left\{ x\in X \;\middle|\; \begin{array}{c} x \text{ is a } \mathbb{G}_m\text{-fixed point and} \\ \mathbb{G}_m \text{ acts on } L^*|_x \text{ with min weight} \end{array} \right\}$
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(*) All points in $X_{\min}^{\mathbb{G}_m}$ have trivial stabiliser in U then

1. $X^0_{\min}\subseteq X^{s(U,L)}$, *so we get a locally trivial U-quotient* $X^0_{\min}\to X^0_{\min}/U$.

The $\hat{U}$ Theorem

- Let U be a graded unipotent lin. alg. group over $k = \bar{k}$ and $\hat{U} = U \rtimes \mathbb{G}_m$. Let X be an irreducible normal $\hat{U}$-variety and $L \to X$ an very ample linearisation of the $\hat{U}$ action.
- $X^{\mathbb{G}_m}_{\min} := X \cap \mathbb{P}(H^0(X,L)^*_{\min}) = \left\{ x \in X \;\middle|\; \begin{array}{c} x \text{ is a } \mathbb{G}_m\text{-fixed point and} \\ \mathbb{G}_m \text{ acts on } L^*|_x \text{ with min weight} \end{array} \right\}$
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Theorem (B.,Doran, Hawes, Kirwan 2015)

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2. *Twist the linearisation by a character $\chi : \hat{U} \to \mathbb{G}_m$ of $\hat{U}/U$ such that 0 lies just above the lowest weight for the $\mathbb{G}_m$ action on X. Then*

$$X^0_{\min} \setminus (U \cdot X^{\mathbb{G}_m}_{\min}) = X^{s(L^{(\chi)})} = X^{ss,fg(L^{(\chi)})}.$$

3. *the enveloping quotient $X//_{L^{(\chi)}}\hat{U}$ is a projective variety and for suitably divisible integers $r > 0$ $k[X,(L^{(\chi)})^{\otimes r}]^{\hat{U}}$ is finitely generated, with* $X//_{L^{(\chi)}}\hat{U} = \mathrm{Proj}(k[X,(L^{(\chi)})^{\otimes r}]^{\hat{U}})$;

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Remarks

1. Apply the $\hat{U}$-Thm to diagonal action of $\hat{U}$ on $X \times \mathbb{P}^1$ and linearisation $L \otimes \mathcal{O}_{\mathbb{P}^1}(m)$ with m big to define

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4. (joint with J. Jackson and F. Kirwan): Nice VGIT picture for $H = U \rtimes R$ action when R contains higher dimensional torus.

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On surfaces ($n = 2$) $\overline{CX_p^{[k]}} = X_p^{[k]}$, for $n > 2$ $\overline{CX_p^{[k]}}$ is a component of the punctual Hilbert scheme.

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$f_\xi(z) = zf'(0) + \frac{z^2}{2!}f''(0) + \ldots + \frac{z^k}{k!}f^{(k)}(0) \in J_k(1,n)$ and
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Then $f \circ \varphi(z) =$

$$= (f'(0)\alpha_1)z + (f'(0)\alpha_2 + \frac{f''(0)}{2!}\alpha_1^2)z^2 + \ldots + \left(\sum_{i_1+\ldots+i_l=k} \frac{f^{(l)}(0)}{l!}\alpha_{i_1}\ldots\alpha_{i_l} \right) z^k =$$

$$= (f'(0), \ldots, f^{(k)}(0)/k!) \cdot \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \ldots & 2\alpha_1\alpha_{k-1} + \ldots \\ 0 & 0 & \alpha_1^3 & \ldots & 3\alpha_1^2\alpha_{k-2} + \ldots \\ 0 & 0 & 0 & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \alpha_1^k \end{pmatrix}$$

the (i,j) entry is $p_{i,j}(\bar{\alpha}) = \sum_{a_1+a_2+\ldots+a_i=j} \alpha_{a_1}\alpha_{a_2}\ldots\alpha_{a_i}$.

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Recall: $f_\xi \in J_k(1,n), \phi \in J_k(1,1)$, $CX_p^{[k+1]} = J_k(1,n)/J_k(1,1)$.

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In other words: $Ann(f_\xi) \subset J_k(n,1)$ is invariant under $J_k(1,1)$. The map $f_\xi \mapsto Ann(f_\xi)$ defines an embedding

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Tautological integrals

Ultimate goal: Determine the geometric and topological invariants of $X^{[n]}$.
For surfaces these were extensively studied such as **Betti numbers** (Ellingsrud, Stromme, Göttsche), **Hodge numbers** (Sörgel, Göttsche), **cohomology ring** (Nakajima, Grojnowski), **Chern numbers of tautological bundles** (Lehn, Rennemo, Marian-Oprea-Pandharipande).

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is the curvilinear component.
Want: Closed formula for $\int_{\widetilde{CX}^{[k]}} P(c_i(F^{[n]}))$ for some geometric partial resolution $\widetilde{CX}^{[k]} \to \overline{CX}^{[k]}$.

Step 1: $X^{[k_1,\ldots,k_t]} = \{(\xi_1 \subset \xi_2 \subset \ldots \subset \xi_t) : \xi_i \in X^{[k_i]}\} \subset X^{[k_1]} \times \ldots \times X^{[k_t]}$

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Define

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$$= (f') \subset (f', f'' + (f')^2) \subset \ldots \subset (f', f'' + (f')^2, \ldots, f^{[k]} + \sum_{\Sigma a_i = k} (f^{[i]})^{a_i})$$

Geometric resolutions of $CX^{[k]}$ in two steps

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Step 2 Assume $k \leq n$. Let $\pi : J_k(n,1)^* \simeq \oplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n \to \mathbb{C}^n$ be the projection.

$$\widetilde{CX}_p^{[k+1]} = \{(\tilde{\xi}_1 \subset \ldots \subset \tilde{\xi}_k), (V_1 \subset \ldots \subset V_k) : \pi(\tilde{\xi}_i) \subset V_i\} \subset \widehat{CX}_p^{[k+1]} \times \mathrm{Flag}(k, J_k(n,1)$$

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Equivalently: Let $B_n \subset \mathrm{GL}(n)$ be the upper Borel and $z_k = \rho(e_1, \ldots, e_k)$ the base point in $\mathrm{Flag}(k, J_k(n,1)^*)$. Then

$$\widetilde{CX}_p^{[k+1]} = \mathrm{GL}(n) \times_{B_n} \overline{B_n \cdot z_k} \to \overline{\mathrm{GL}(n) \cdot z_k} = \widehat{CX}_p^{[k+1]}$$

Set-up: $k \le n$. $\widetilde{CX}^{[k+1]} \to CX^{[k+1]} \to X$ partial resolution. Fibre over $p \in X$

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Let $T = (\mathbb{C}^*)^n$ act on $T_p^* X_p$ with weights $\lambda_1, \ldots, \lambda_n$.

$$\int_{\widetilde{CX}_p^{[k+1]}} P(c_i(F^{[k+1]})) = \int_{\mathcal{F}} \int_{\overline{B_n \cdot z_k}} P(c_i(F^{[k+1]})) = \sum_{\sigma \in \mathcal{F}} \frac{\int_{\overline{B_n \cdot z_k}} P(c_i(F^{[k+1]}))}{\prod_{\substack{1 \le i \le k \\ i < j}} (\lambda_{\sigma(j)} - \lambda_{\sigma(i)})} =$$

Equivariant localisation on $\widetilde{CX}_p^{[k]}$

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$$= \mathrm{Res}_{\mathbf{z}=\infty} \frac{\prod_{i<j}(z_i - z_j) \int_{\overline{B_n \cdot z_k}} P(c_i(F^{[k+1]}))}{\prod_{\substack{1 \le i \le n \\ 1 \le j \le k}} (\lambda_i - z_j)} d\mathbf{z} =$$

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Remains: How to compute $\int_{\overline{B_n \cdot z_k}} P(c_i(F))$?

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$\mathcal{B}_k^T$ is parametrised by sequences of partitions $\pi = (\pi_1, \ldots, \pi_k)$ where $\Sigma \pi_i = i$.
At $p = \xi_p$

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$\mathcal{B}_k^T$ is parametrised by sequences of partitions $\pi = (\pi_1, \ldots, \pi_k)$ where $\Sigma\pi_i = i$.

At $p = \xi_p$

$$c(\xi_p \otimes F) = \prod_{j=1}^{r}(1+\theta_j)\prod_{i=1}^{k}\prod_{j=1}^{r}(1+z_{\pi_i}+\theta_j)$$

Integration on $\overline{B_n z_k}$

Remains: How to compute $\int_{\overline{B_n \cdot z_k}} P(c_i(F))$?

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Still, we can prove the following vanishing property

Theorem (B-Szenes 2012, B. 2015)

Let $n \le k$. Only one fixed point contributes to the iterated residue. This distinguished fixed point corresponds to the sequence $\pi = (1, 2, \ldots, k)$ which is the ideal $\xi = m_k^2 \subset m_n^2$

Let $Q_k(z_1,\ldots,z_k) = \mathrm{mdeg}(T_{[1],\ldots,[k]}\mathcal{B}_k, T_{[1],\ldots,[k]}\mathrm{Flag})$. Then

Theorem (B. 2015)

For any n, k (we can drop $k \le n$ condition!)

$$\int_{\widetilde{CX}^{[k+1]}} P(c_i(F)) = \int_X \mathrm{Res}_{\mathbf{z}=\infty} \frac{\prod_{i<j}(z_i - z_j) Q_k(\mathbf{z}) P(c_i(z_i + \theta_j, \theta_j)) d\mathbf{z}}{\prod_{i+j \le l \le k}(z_i + z_j - z_l)(z_1 \ldots z_k)^n} \prod_{i=1}^{k} s_X\left(\frac{1}{z_i}\right)$$

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Features:

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$$\mathrm{Hom}(\mathbb{C}^k, \mathrm{Sym}^2\mathbb{C}^k) = \bigoplus \mathbb{C}q_l^{mr}, \ 1 \le m, r, l \le k,$$

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q_l^{mr} has weight $(z_m+z_r-z_l)$. Let

$$\epsilon=\sum_{m=1}^{k}\sum_{r=1}^{k-m} q_{m+r}^{mr}\subset W=\bigoplus_{1\le m+r\le l\le k}\mathbb{C}q_l^{mr}\subset \mathrm{Hom}(\mathbb{C}^k,\mathrm{Sym}^2\mathbb{C}^k)$$

Then $Q_k(z)=\mathrm{mdeg}(\overline{B_k\epsilon},W)$. In particular $Q_{1,2,3}=1, Q_4=2z_1+z_2-z_4$.

Hint of other results

Theorem (Thom polynomial of A_k singularities of holomorphic maps $f: N^n \to M^m$, B.-Szenes 2012)

$$\mathrm{Tp}_k^{m-n} = \mathrm{Res}_{\mathbf{z}} \frac{(-1)^k \prod_{m<l\leq k}(z_m - z_l)\, Q_k(\mathbf{z})}{\prod_{m+r\leq l\leq k}(z_m + z_r - z_l)} \prod_{l=1}^{k} c\left(\frac{1}{z_l}\right) z_l^{m-n}\, dz_l,$$

where $c\left(\frac{1}{z_l}\right) = 1 + \frac{c_1}{z_l} + \frac{c_2}{z_l^2} + \ldots$ *is the total Chern class of* $TN - f^*TM$.

Theorem (Degree of tautological line bundles on jet differentials bundle of Demailly, B. 2013)

For any homogeneous polynomial $P = P(u,h)$ *of degree* $\deg(P) = \dim \tilde{\mathcal{X}}_k = n + k(n-1)$ *we have*

$$\int_{\tilde{\mathcal{X}}_k} P(u,h) = \int_X \mathrm{Res}_{\mathbf{z}} \frac{Q_k(\mathbf{z}) \prod_{m<l}(z_m - z_l) P(z_1 + \ldots + z_k, h)}{\prod_{m+r\leq l\leq k}(z_m + z_r - z_l)(z_1 \ldots z_k)^n} \prod_{j=1}^{k} s\left(\frac{1}{z_j}\right)$$