Equivariant localization and iterated residues

Gergely Bérczi – Oxford

Warsaw

January 18, 2011.

Motivation: Any group action on a topological space carries information about the space.
Let X be a G-space, i.e a top. space with a G-action. G-equivariant cohomology for free action is $H^*(X/G)$. For non-free action the quotient X/G is not well-behaved and $H^*(X/G)$ does not carries enough information. We need to "resolve" the action by replacing X with $X \times EG$. This has a free (diagonal) G-action, and

$$H^*_G(X) = H^*(EG \times_G X)$$

Example

$H^*_G(pt) = H^*(BG) = \mathbb{C}[\mathfrak{h}]^W$, and $H^*_{\mathfrak{g}}(X)$ is a $H^*_G(pt)$-module. For example $H^*_{GL_n}(pt) = S^W = \mathbb{C}[x_1, \dots, x_n]^{S_n}$.

For a smooth manifold M we can define equivariant differential forms.

$$\Omega_G(M) = \{\alpha : \mathfrak{g} \to \Omega(M) : \alpha(gX) = g\alpha(X), g \in G\} = (\mathbb{C}[\mathfrak{g}] \otimes \Omega(M))^G$$

where $(g \cdot \alpha)(X) = g \cdot (\alpha(g^{-1} \cdot X))$.
Define

$$(d_G\alpha)(X) = (d - \iota(X_M))\alpha(X)$$

which increases the degree by one if the $\mathbb{Z}$-grading is given by

$$\deg(P \otimes \alpha) = 2\deg(P) + \deg(\alpha).$$

for $P \in \mathbb{C}[\mathfrak{g}], \alpha \in \Omega(M)$. Then

$$H_G^*(M) = H_{d_G}^*$$

Note that $\alpha \in \Omega_G(M)$ is equivariantly closed if

$$\alpha(X) = \alpha(X)_0 + \ldots + \alpha(X)_n \text{ s.t } \iota(X_M)\alpha(X)_i = d\alpha(X)_{i-2}$$

Equivariant localization

We can integrate equivariant forms:

$$\int_M : \Omega_G(M) \to \mathbb{C}[\mathfrak{g}]^G$$

by the formula

$$\left(\int_M \alpha\right)(X) = \int_M \alpha(X) = \int_M \alpha_{[n]}(X)$$

General principle: information is stored at the fixed points of the action.

Theorem (Atiyah/Bott/Berline/Vergne)

$G = T$.

$$\int_M \alpha = (2\pi)^l \sum_{p \in M^T} \frac{\alpha_0(p)}{\mathrm{Euler}^T(T_p M)}$$

In other words:

$$\int_M \alpha(X) = (2\pi)^l \sum_{p \in M^T} \frac{\alpha(X)_0(p)}{\prod_i \lambda_i}$$

where λ_i *are the weights of the Lie action*

$$X : \xi \in T_p M \to [X_M(p), \xi] \in T_p M.$$

How many lines intersect 2 given lines and go through a point in $\mathbb{C}^3$?

$$C_2(R) = \{V \in Gr(2,4) : V \subset R\}, \ C_1(L) = \{V \in Gr(2,4) : L \subset V\}$$

Answer:

$$C_1(L_1) \cap C_1(L_2) \cap C_2(R) = \int_{Gr(2,4)} c_1(\tau)^2 c_2(\tau)$$

where τ is the tautological rank 2 bundle over $Gr(2,4)$.
Apply equivariant localization!

- $T^4 \subset GL(4)$ acts on $\mathbb{C}^4$ with weights $\mu_1, \mu_2, \mu_3, \mu_4 \in \mathfrak{t}^* \subset H_T^*(pt)$.
- The induced action on $Gr(2,4)$ has $\binom{4}{2}$ fixed points, the coordinate subspaces indexed by (i,j)
- The weights on $T_{(i,j)}Gr$ are $\mu_s - \mu_i, \mu_s - \mu_j$ with $s \neq i,j$. ABBV localization gives

$$\int_{Gr(2,4)} c_1^2 c_2 = \sum_{\sigma \in S_4/S_2} \sigma \cdot \frac{(\mu_1+\mu_2)^2 \mu_1 \mu_2}{(\mu_3-\mu_1)(\mu_4-\mu_1)(\mu_3-\mu_2)(\mu_4-\mu_2)} = 2$$

- $z_1, \ldots, z_d$ coordinates on $\mathbb{C}^d$.
- $\omega_1, \ldots, \omega_N$ affine linear forms on $\mathbb{C}^d$; $\omega_i = a_i^0 + a_i^1 z_1 + \ldots + a_i^d z_d$.
- $h(\mathbf{z})$ a function $h(z_1 \ldots z_d)$, and $d\mathbf{z} = dz_1 \wedge \cdots \wedge dz_d$ holomorphic d-form.
- iterated residue at infinity:

$$\operatorname*{Res}_{\mathbf{z_1}=\infty} \ldots \operatorname*{Res}_{\mathbf{z_d}=\infty} \frac{h(\mathbf{z})\ d\mathbf{z}}{\prod_{i=1}^N \omega_i} \stackrel{\text{def}}{=} \left(\frac{1}{2\pi i}\right)^d \int_{|z_1|=R_1} \cdots \int_{|z_d|=R_d} \frac{h(\mathbf{z})\ d\mathbf{z}}{\prod_{i=1}^N \omega_i}, \qquad (1)$$

where $1 \ll R_1 \ll \ldots \ll R_d$. The torus $\{|z_m| = R_m;\ m = 1 \ldots d\}$ is oriented in such a way that $\mathrm{Res}_{z_1=\infty} \ldots \mathrm{Res}_{z_d=\infty}\ d\mathbf{z}/(z_1 \cdots z_d) = (-1)^d$.

In practice, the iterated residue 1 may be computed using the following **algorithm**: for each i, use the expansion

$$\frac{1}{\omega_i} = \sum_{j=0}^{\infty} (-1)^j \frac{(a_i^0 + a_i^1 z_1 + \ldots + a_i^{q(i)-1} z_{q(i)-1})^j}{(a_i^{q(i)} z_{q(i)})^{j+1}}, \tag{2}$$

where $q(i)$ is the largest value of m for which $a_i^m \neq 0$, then multiply the product of these expressions with $(-1)^d h(z_1 \cdots z_d)$, and then take the coefficient of $z_1^{-1} \ldots z_d^{-1}$ in the resulting Laurent series.

Example

- $\frac{1}{z_1 - z_2}$ *has two different Laurent expansion..*
- $\mathrm{Res}_{z=\infty} \frac{1}{(z_1 - z_2)(2z_1 - z_2)} =$ $\mathrm{coeff}_{(z_1 z_2)^{-1}} \frac{1}{z_2}^2 (1 + \frac{z_1}{z_2} + \frac{z_1^2}{z_2^2} + \ldots)(1 + \frac{2z_1}{z_2} + \frac{4z_1^2}{z_2^2} + \ldots) = 3$
- *Grothendieck residue $\neq$ iterated residue in general.*

$$\int_{Gr(2,4)} c_1^2 c_2 = \sum_{\sigma \in S_4/S_2} \sigma \cdot \frac{(\mu_1+\mu_2)^2 \mu_1 \mu_2}{(\mu_3-\mu_1)(\mu_4-\mu_1)(\mu_3-\mu_2)(\mu_4-\mu_2)}$$

Take the differential form

$$\omega = \frac{(z_2-z_1)^2(z_1+z_2)^2 z_1 z_2 \, d\mathbf{z}}{\prod_{i=1}^4(\mu_i - z_1)\prod_{i=1}^4(\mu_i - z_2)}$$

$$\operatorname*{Res}_{z_2=\infty} \omega = \sum_{i=1}^4 \underbrace{-\frac{(\mu_i - z_1)^2(\mu_i+z_1)^2 \mu_i z_1 dz_1}{\prod_{j=1}^4(\mu_j - z_1)\prod_{j\neq i}(\mu_j-\mu_i)}}_{z_2=\mu_i} = \sum_{i=1}^4 -\frac{(\mu_i - z_1)(\mu_i+z_1)^2 \mu_i z_1 dz_1}{\prod_{j\neq i}(\mu_j - z_1)\prod_{j\neq i}(\mu_j-\mu_i)}$$

$$\operatorname*{Res}_{\mathbf{z}=\infty} \omega = \sum_{i=1}^4 \sum_{j\neq i} -\frac{(\mu_i-\mu_j)(\mu_i+\mu_j)^2\mu_i\mu_j}{\prod_{k\neq i,j}(\mu_k-\mu_j)\prod_{j\neq i}(\mu_j-\mu_i)} =$$

$$= \sum_{i=1}^4 \sum_{j\neq i} \frac{(\mu_i+\mu_j)^2\mu_i\mu_j}{\prod_{k\neq i,j}(\mu_k-\mu_j)\prod_{k\neq i,j}(\mu_k-\mu_i)} = 2$$

$$\Sigma_1 \doteq \{A \in \mathrm{Hom}(n,k);\ \dim\ker A = 1\} = \{A \in \mathrm{Hom}(n,k)\exists![v] \in \mathbb{P}^{n-1} :\ Av = 0\}.$$

We have the fibration $\pi : \Sigma_1 \to \mathbb{P}^{n-1}$.
Apply AB/BV localization with $M = \mathbb{P}^{n-1}$.
Fixed-point data: n fixed points on $\mathbb{P}^{n-1}$: $p_1 \dots p_n$, corresponding to the coordinate axes. The weights of $T_{p_i}\mathbb{P}^{n-1}$ are $\{\lambda_s - \lambda_i;\ s \neq i\}$. The fiber at p_i is the set of matrices A with all entries in the ith column vanishing.
Normalization axiom $\Rightarrow$ mdeg of the fiber at p_i is $\prod_{j=1}^k (\eta_j - \lambda_i)$, so:

$$\mathrm{mdeg}[\Sigma_1, \mathrm{Hom}(n,k)] = \int_{\Sigma_1} \mathrm{Thom}_{(\mathbb{C}^*)^{n+k}}(\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^k)) =$$

$$= \int_{\mathbb{P}^{n-1}} \int_{fiber} \mathrm{Thom}_{(\mathbb{C}^*)^{n+k}} = \sum_{i=1}^n \frac{\prod_{j=1}^k (\theta_j - \lambda_i)}{\prod_{s \neq i} (\lambda_s - \lambda_i)}$$

Consider the rational differential form

$$-\frac{\prod_{j=1}^{k}(\theta_j - z)}{\prod_{i=1}^{n}(\lambda_i - z)}\, dz.$$

The residues of this form at finite poles: $\{z = \lambda_i;\ i = 1 \ldots n\}$ exactly recover the terms of the sum. Apply the residue theorem, and change variables: $z = -1/q$, and we get

$$\mathrm{mdeg}[\Sigma_1, \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^k)] = \mathrm{res}_{q=0} \frac{\prod_{j=1}^{k}(1 + q\theta_j)}{\prod_{i=1}^{n}(1 + q\lambda_i)} \frac{dq}{q^{k-n+2}} = c_{k-n+1}.$$

We have the following *iterated residue theorem*.

Proposition (Berczi-Szenes, 2008)

For a polynomial $Q(\mathbf{z})$ *on* $\mathbb{C}^d$, *we have*

$$\sum_{\sigma \in S_n/S_{n-d}} \frac{Q(\lambda_{\sigma\cdot 1}\cdots\lambda_{\sigma\cdot d})}{\prod_{1\le m\le d}\prod_{i=m+1}^{n}(\lambda_{\sigma\cdot i}-\lambda_{\sigma\cdot m})} = \operatorname*{Res}_{\mathbf{z}=\infty} \frac{\prod_{1\le m<l\le d}(z_m - z_l)\, Q(\mathbf{z})\, d\mathbf{z}}{\prod_{l=1}^{d}\prod_{i=1}^{n}(\lambda_i - z_l)} \tag{3}$$