Thom polynomials of singularities via equivariant localization

Gergely Bérczi – Oxford

Warsaw

January 19, 2011.

Singularities of maps

Set up: $k \leq n \leq m$ fixed

- Let A be a nilpotent algebra, $\dim A/\mathbb{C} = k$. We will take $A_k = z\mathbb{C}[z]/z^{k+1}$
- $V = J_k(n,m) =$
  $\{p = (p_1, \ldots, p_m) \in \mathrm{Poly}(\mathbb{C}^n, \mathbb{C}^m) : \deg p_i \leq k, p_i(0) = 0\}$
- $\Sigma = \{p \in J_k(n,m) : \mathbb{C}[x_1, \ldots, x_n]/\langle p_1, \ldots, p_m\rangle = A\}$
- $\mathcal{D} = \mathrm{Diff}(\mathbb{C}^n, 0) \times \mathrm{Diff}(\mathbb{C}^m, 0) \curvearrowright J_d(n,m)$ with

$$(A,B)p = BpA^{-1}$$

Motivation: $f : N^n \to M^m$ (N^n, M^m complex manifolds)

$$Z(f) = \left\{p \in N | \widehat{f_p} \in \Sigma\right\}$$

Thom's principle: there is a well-defined polynomial

$$MD_A^{n \to m} \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_m]^{S_n \times S_m}$$

such that

$$[Z(f)] = MD_A(TN, f^*(TM)) \in H^*(N, \mathbb{C}).$$

Haefliger and Kosinski:

$$c(q) = c_0 + c_1 q + c_2 q^2 + \ldots = \frac{c(f^*(TM))}{c(TN)} = \frac{\prod_{m=1}^k (1 + \theta_m q)}{\prod_{i=1}^n (1 + \lambda_i q)}$$

Then

$$MD_A(TN, f^*(TM)) = Tp_A^{k-n}(c_1, c_2, \ldots)$$

MD stands for: Multidegree, Equivariant Hilbert polynomial, Equivariant Poincaré dual

What is this? Algebraic invariant of G-invariant subvarieties of a complex vector space.

Set up:

1. $V = \mathbb{C}^N$ complex vector space, with a G-action.
2. $\Sigma \subset V$ is a G-invariant closed subvariety.
3. $H_G^*(V) = H_G^*(pt)$ is the G-equivariant cohomology ring of V. $H_{GL(d)}^*(pt) = \mathbb{C}[x_1, \ldots, x_d]^{S_d}$.

Definition/Description

$$\mathrm{mdeg}[\Sigma, V] \in H_G^{\mathrm{codim}(\Sigma \in V)}(pt)$$

Vergne's integral definition-topology
$EG \times_G \Sigma \subset EG \times_G V$ represents a homology cycle. Then

$$\mathrm{mdeg}[\Sigma, V] = PD(EG \times_G \Sigma \subset EG \times_G V)$$

So multidegree=equivariant Poincaré dual.
Vergne: there is an equivariant Thom class:

$$Thom_G(V) \in H_G^{\dim V}(V)$$

s.t

$$\mathrm{mdeg}[\Sigma, V] = \int_\Sigma Thom_G(V).$$

Sturmfeld's axiomatic definition

$\mathrm{mdeg}[\Sigma, V]$ is characterized by the following axioms:

additive: If $\Sigma \doteq \cup \Sigma_i$

$$\mathrm{mdeg}[\Sigma, V] = \sum_{i=1}^{c} \mathrm{mult}(\Sigma_i) \cdot \mathrm{mdeg}[\Sigma_i, W],$$

degenerative: $\mathrm{mdeg}[\Sigma, V] = \mathrm{mdeg}[\mathrm{Groeb}(\Sigma), V]$

normalized: For T-invariant linear subspaces of V the invariant is defined to be equal to the product of weights in the normal direction.

Example: $(\mathbb{C}^*)^3$ acts on $\mathbb{C}^4$ with weights $\eta_1, \ldots \eta_4$. Let $\eta_1 + \eta_2 = \eta_3 + \eta_4$, and

$$\Sigma = \mathrm{Spec}(\mathbb{C}[y_1, y_2, y_3, y_4]/(y_1 y_2 - y_3 y_4)).$$

$$\Sigma_t = \mathrm{Spec}(\mathbb{C}[y_1, y_2, y_3, y_4]/(y_1 y_2 - t y_3 y_4)),$$

For $t = 0$ $\Sigma_0 = \{y_1 y_2 = 0\}$, so normalisation says

$$\mathrm{mdeg}[\Sigma, \mathbb{C}^4] = \eta_1 + \eta_2 = \eta_3 + \eta_4.$$

Thom's principle again:

$$MD_d^{n\to m} = \mathrm{mdeg}^{\mathrm{GL}_n \times \mathrm{GL}_m}[\Sigma_k, J_k(n,m)].$$

Theorem (The test curve model of Porteous and Gaffney)

$$\Sigma_k(n,m) \doteq \{\Psi \in J_k(n,m) | \exists \gamma \in J_k^{reg}(1,n) \text{ such that } \Psi \circ \gamma = 0\}.$$

$$(\mathbb{C},0) \xrightarrow{\gamma} (\mathbb{C}^n,0) \xrightarrow{\Psi} (\mathbb{C}^m,0) \tag{1}$$

Observation: If $\varphi \in J_k^{reg}(1,1) = \mathbf{G}_k$, then

$$\Psi \circ \gamma = 0 \quad \Rightarrow \quad \Psi \circ (\gamma \circ \varphi) = 0$$

$$(\mathbb{C},0) \xrightarrow{\varphi} (\mathbb{C},0) \xrightarrow{\gamma} (\mathbb{C}^n,0) \xrightarrow{\Psi} (\mathbb{C}^m,0) \tag{2}$$

Proposition

$\Sigma_k^0 = \{\Psi : \dim\ker\Psi = 1\} \doteq \Sigma_k$ *fibers with linear fibres over* $J_k^{reg}(1,n)/\mathbf{G}_k$.

If $\gamma = v_1 t + v_2 t^2 + \ldots + v_d t^d$ with $v_i \in \mathbb{C}^n$;
$\Psi(v) = Av + Bv^2 + \ldots$ with $A \in \mathrm{Hom}(\mathbb{C}^n, C^k)$, $B \in \mathrm{Hom}(\mathrm{Sym}^2(\mathbb{C}^n), \mathbb{C}^k), \ldots$
then $\Psi \circ \gamma = 0$ has the form:

$$\begin{aligned} A(v_1) &= 0, \\ A(v_2) + B(v_1, v_1) &= 0, \\ A(v_3) + 2B(v_1, v_2) + C(v_1, v_1, v_1) &= 0, \\ &\ldots \end{aligned} \tag{3}$$

The Theorem says:

$$\Sigma_d(n,k) \doteq \bigcup \{\mathrm{Sol}_\gamma \mid \gamma \in J_d^{reg}(1,n)\}.$$

- identify $J_k(1,n)$ with $\mathrm{Hom}(\mathbb{C}^k,\mathbb{C}^n)$;
- Let $\mathrm{Sym}_{\leq k}\mathbb{C}^n = \oplus_{i=1}^k \mathrm{Sym}^i\mathbb{C}^n$.
  Define a map

$$\rho : \mathrm{Hom}(\mathbb{C}^k,\mathbb{C}^n) \to \mathrm{Hom}(\mathbb{C}^k,\mathrm{Sym}_{\leq k}\mathbb{C}^n)$$

$$\rho(v_1,\ldots,v_k) = (v_1, v_2 + v_1^2, \ldots, \sum_{a_1+a_2+\ldots+a_i=j} v_{a_1}v_{a_2}\ldots v_{a_i}, \ldots),$$

Let $\mathrm{Hom}^0(\mathbb{C}^k,\mathbb{C}^n) = \{v_0 \neq 0\} = J_k^{reg}(1,n)$. Then ρ descends to

$$\rho^{flag} : \mathrm{Hom}^0(\mathbb{C}^k,\mathbb{C}^n)/\mathbf{G}_k \hookrightarrow \mathrm{Flag}_k(\mathrm{Sym}_{\leq k}\mathbb{C}^n)$$

$$\rho^{grass} : \mathrm{Hom}^0(\mathbb{C}^k,\mathbb{C}^n)/\mathbf{G}_k \hookrightarrow \mathrm{Grass}_k(\mathrm{Sym}_{\leq k}\mathbb{C}^n)$$

$$\rho^{proj} = Pluck \circ \rho^{grass} : \mathrm{Hom}^0(\mathbb{C}^k,\mathbb{C}^n)/\mathbf{G}_k \hookrightarrow \mathbb{P}(\wedge^k(\mathrm{Sym}_{\leq k}\mathbb{C}^n))$$

The computation: double localization+vanishing theorem:

$$\begin{array}{ccc} \Sigma_k(n,m) & & \subset J_k(n,m) \\ \downarrow & & \\ J_k^{reg}(1,n)/\mathbf{G}_k & & \subset \mathrm{Flag}_k(\mathrm{Sym}_{\le k}\mathbb{C}^n) \\ \downarrow & & \\ J_k^{reg}(1,n)/B_k = \mathrm{Flag}_k(\mathbb{C}^n) & & \end{array} \tag{4}$$

$$Tp_k^{m-n} = \int_{\Sigma_k} \mathrm{Thom}_k(n,m) = \int_{\mathrm{Flag}_k(\mathbb{C}^n)} \int_{F \in \mathrm{Flag}_k(\mathrm{Sym}_{\le k}\mathbb{C}^n)} \mathrm{Thom}|_F$$

Toy example: $d = 1$

$\Sigma_1 \doteq \{A \in \mathrm{Hom}(n,k);\ \dim\ker A = 1\} = \{A \in \mathrm{Hom}(n,k) \exists ! [v] \in \mathbb{P}^{n-1} : Av = 0\}$.

We have the fibration $\pi : \Sigma_1 \to \mathbb{P}^{n-1}$.
Apply AB/BV localization with $M = \mathbb{P}^{n-1}$.

$$\mathrm{mdeg}[\Sigma_1, \mathrm{Hom}(n,k)] = \int_{\Sigma_1} \mathrm{Thom}_{(\mathbb{C}^*)^{n+k}}(\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^k)) =$$

$$= \int_{\mathbb{P}^{n-1}} \int_{fiber} \mathrm{Thom}_{(\mathbb{C}^*)^{n+k}} = \sum_{i=1}^{n} \frac{\prod_{j=1}^{k}(\theta_j - \lambda_i)}{\prod_{s \neq i}(\lambda_s - \lambda_i)}$$

Consider the rational differential form

$$-\frac{\prod_{j=1}^{k}(\theta_j - z)}{\prod_{i=1}^{n}(\lambda_i - z)}\, dz.$$

The residues of this form at finite poles: $\{z = \lambda_i;\ i = 1 \ldots n\}$ exactly recover the terms of the sum. Apply the residue theorem, and change variables: $z = -1/q$, and we get

$$\mathrm{mdeg}[\Sigma_1, \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^k)] = \mathrm{res}_{q=0} \frac{\prod_{j=1}^{k}(1 + q\theta_j)}{\prod_{i=1}^{n}(1 + q\lambda_i)} \frac{dq}{q^{k-n+2}} = c_{k-n+1}.$$

Theorem (B-Szenes, 2007)

$$Tp_k^{m-n} = \mathrm{Res}_{\mathbf{z}=\infty} \frac{\prod_{i<j}(z_i - z_j)\, \mathcal{Q}_k(z_1 \dots z_k)}{\prod_{i+j\le l\le k}(z_i + z_j - z_l)} \cdot \prod_{l=1}^{k} c\left(\frac{1}{z_l}\right) z_l^{m-n}\, dz_l$$

- We integrate on the cycle $|z_1| > |z_2| > \dots |z_k|$, which determines the Laurent expansion.
- $c(q) = 1 + c_1 q + c_2 q^2 + \dots$
- $Q_k(z_1, \dots, z_k)$ is the multidegree of a Borel-orbit in a representation of $GL(k)$.

$$Q_1 = Q_2 = Q_3 = 1,\, Q_4 = 2z_1 + z_2 - z_4$$

Conjecture (Rimanyi, 1998)

$$Tp_k^{m-n} \in \mathbb{N}[c_1, \dots, c_{k(m-n+1)}] \;\; i.e \;\; \frac{\prod_{i<j}(z_i - z_j)\, \mathcal{Q}_k(z_1 \dots z_k)}{\prod_{i+j\le l\le k}(z_i + z_j - z_l)} > 0.$$