The Green-Griffiths conjecture and equivariant localization

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The Green-Griffiths conjecture

- Let X be a complex manifold, $n = \dim_{\mathbb{C}}(X)$
- X is said to be hyperbolic in the sense of Brody if there are no non-constant entire holomorphic curves $f : \mathbb{C} \to X$.
- Brody has shown that for X compact
  Brody hyperbolicity $\iff$ Kobayashi hyperbolicity
  (degeneracy of the Kobayashi pseudo-metric)

Conjecture (Green-Griffith-Lang, 1979)

Let X be a projective variety of general type. Then there exists an algebraic variety $Y \subsetneq X$ such that for all non-constant holomorphic $f : \mathbb{C} \to X$ one has $f(\mathbb{C}) \subset Y$.

- McQuillen (1998): Positive answer for surfaces if $c_1^2 - c_2 > 0$.
- Demailly (1990)/Siu (1996): Strategy for projective hypersurfaces
- Siu (1996) Positive answer for hypersurfaces of high degree.
- Diverio, Merker, Rousseau (2009): Effective lower bound,
  $deg(X) > 2^{n^5} \Rightarrow GGL$.

Strategy(Demailly '95, Siu '96, DMR '09)

- $$f : \mathbb{C} \to X, \quad t \to f(t) = (f_1(t), f_2(t), \ldots, f_n(t))$$

  a curve written in some local holomorphic coordinates $(z_1, \ldots, z_n)$ on X. Define the bundle

  $$(J_k X)_x = \{\hat{f}_{[k]} : f : (\mathbb{C}, 0) \to (X, x)\} \to X$$

  sending $f_{[k]}$ to $f(0)$. Fibre is $J_k(1, n)$
- The group of reparametrizations $\mathbf{G}_k = J_k^{reg}(1, 1)$ acts fiberwise on $J_k X$. The fibres of $J_k X$ can be identified with $J_k(1, n)$, the action is linearised as before.
- $\mathbf{G}_k = \mathbb{C}^* \ltimes U_d$, and for $\lambda \in \mathbb{C}^*$

  $$(\lambda \cdot f)(t) = f(\lambda \cdot t), \text{ so } \lambda \cdot (f', f'', \ldots f^{(k)}) = (\lambda f', \lambda^2 f'', \ldots \lambda^k f^{(k)}).$$
- Consider algebraic differential operators = polynomial functions on $J_k X$. Locally in multi-index notation

  $$Q(f', f'', \ldots, f^{(k)}) = \sum_{\alpha_i \in \mathbb{N}^n} a_{\alpha_1, \alpha_2, \ldots \alpha_k}(f(t))(f'(t)^{\alpha_1} f''(t)^{\alpha_2} \cdots f^{(k)}(t)^{\alpha_k},$$

  where $a_{\alpha_1, \alpha_2, \ldots \alpha_k}(z)$ are holomorphic coefficients on X and $t \to z = f(t)$ is a curve.

- Q is homogeneous of weighted degree m under the $\mathbb{C}^*$ action iff

$$Q(\lambda f', \lambda^2 f'', \dots \lambda^k f^{(k)}) = \lambda^m Q(f', f'', \dots f^{(k)}).$$

Definition

- *(Green-Griffiths '78)*
  $E_{k,m}^{GG}$ *is the sheaf of algebraic differential operators of order* k *and weighted degree* m*.*
- *(Demailly, '95)*
  The bundle of invariant jet differentials of order k *and weighted degree* m *is the subbundle* $E_{k,m} \subset E_{k,m}^{GG}$*, whose elements are invariant under arbitrary changes of parametrization, i.e. for* $\phi \in \mathbf{G}_k$

$$Q((f\circ\phi)', (f\circ\phi)'', \dots, (f\circ\phi)^{(k)}) = \phi'(0)^m Q(f', f'', \dots, f^{(k)}).$$

Therefore:

$$\oplus_m (E_{k,m})_x = \oplus_m (E_{k,m}^{GG})_x^{\mathbb{U}} = \mathcal{O}((J_k X)_x)^{\mathbf{G}_k} = \mathcal{O}(J_k(1,n)/\mathbf{G}_k)$$

Applying our previous contstruction fibrewise we get

Proposition

1. *The quotient $J_kX/\mathbf{G}_k$ has the structure of a locally trivial bundle over X, and there is a holomorphic embedding*

$$\phi^{\mathbb{P}} : J_kX/\mathbf{G}_k \hookrightarrow \mathbb{P}(\wedge^k(T_X^* \oplus \mathrm{Sym}^2(T_X^*) \oplus \ldots \oplus \mathrm{Sym}^k(T_X^*))$$

The fibrewise closure of the image $\mathcal{X}_k = \overline{\mathrm{im}\phi^{\mathbb{P}}}$ is a relative compactification of $J_k(T_X^*)/\mathbf{G}_k$ over X.

2. $$(\pi_k)_*\mathcal{O}_{\mathcal{X}_k}(m) = \mathcal{O}(E_{k,m\binom{k+1}{2}})$$

where $\pi_k : \mathbb{P}(\wedge^k(T_X^* \oplus \mathrm{Sym}^2(T_X^*) \oplus \ldots \oplus \mathrm{Sym}^k(T_X^*))) \to X$ is the projection.

Theorem (Fundamental vanishing theorem)

(Green-Griffiths '78, Demailly '95, Siu '96) Let $P \in H^0(X, E_{k,m} \otimes \mathcal{O}(-A))$ *be a global algebraic differential operator whose coefficients vanish on some ample divisor* A*. Then for any* $f : \mathbb{C} \to X$, $P(f_{[k]}(\mathbb{C})) \equiv 0$*.(note* $f_{[k]}(\mathbb{C}) \subset J_k X$*)*

Corollary

1. *Let* σ *be a nonzero element of*

$$H^0(\mathcal{X}_k, \mathcal{O}_{\mathcal{X}_k}(m) \otimes \pi^* \mathcal{O}(-A)) \simeq H^0(X, E_{k,m\binom{k+1}{2}} \otimes \mathcal{O}(-A)),$$

   then $f_{[k]}(\mathbb{C}) \subset Z_\sigma$*, where* $Z_\sigma \subset \mathcal{X}_d$ *is the zero divisor.*

2. *If* σ_j *is a basis of of sections then the image* $f(\mathbb{C})$ *lies in* $Y = \pi_k(\bigcap Z_{P_j})$*, hence GGL holds if there are enough independent differential equations so that* $Y = \pi_k(\bigcap(Z_{P_j})) \subsetneq X$*.*

It is crucial to control in a more precise way the order of vanishing of these differential operators along the ample divisor. Thus, we need here a slightly different theorem.

Theorem (DMR, 2009)

Assume that $n = k$, and there exist a $\delta = \delta(n) > 0$ and $D = D(n, \delta)$ such that

$$H^0(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n}(m) \otimes \pi^* K_X^{-\delta m}) \simeq H^0(X, E_{n,m\binom{n+1}{2}} T_X^* \otimes K_X^{-\delta m}) \neq 0$$

whenever $\deg(X) > D(n, \delta)$ provided that $m > m_{D,\delta,n}$ is large enough. Then GGL holds for

$$\deg(X) \geq \max(D(n, \delta), \frac{n^2 + 2n}{\delta} + n + 2).$$

We use the algebraic Morse inequalities of Demailly/Trapani. Let $L \to X$ be a holomorphic line bundle given as

$$L = F \otimes G^{-1}, F, G \text{ nef bundles.}$$

Then

$$\sum_{j=0}^{q} (-1)^{q-j} h^j(X, L^{\otimes m} \otimes E) \leq r \frac{m^n}{n!} \sum_{j=0}^{q} (-1)^{q-j} \binom{n}{j} F^{n-j} \cdot G^j + o(m^n).$$

$q = 2$ asserts

$$F^n - nF^{n-1}G > 0 \Rightarrow H^0(L^{\otimes m}) \neq 0 \text{ for } m \gg 0.$$

Proposition (B-Kirwan)

F and G are nef bundles in the following equality:

$$\underbrace{\mathcal{O}_{\mathcal{X}_n}(1) \otimes \pi^* K_X^{-\delta\binom{n+1}{2}}}_{L} = \underbrace{(\mathcal{O}_{\mathcal{X}_n}(1) \otimes \pi^* \mathcal{O}_X(2n^2))}_{F} \otimes$$

$$\otimes \underbrace{(\pi^* \mathcal{O}_X(2n^2) \otimes \pi^* K_X^{\delta\binom{n+1}{2}})^{-1}}_{G}.$$

- $h = c_1(\mathcal{O}_X(1)), c_1(K_X) = -c_1(X) = (d-n-2)h$, and $\mathcal{O}_{\mathcal{X}_n}(1) = \det \tau$, where $\tau \to \mathcal{X}$ is the tautological n-bundle.
- $\dim(\mathcal{X}_n) = n^2$, and we want to compute the integral

$$\int_{\mathcal{X}_n} (c_1(\det \tau) + 2n^2\pi^* h)^{n^2} -$$

$$- n^2 (c_1(\det \tau) + 2n^2\pi^* h)^{n^2-1} (2n^2\pi^* h + \delta \binom{n+1}{2} (d-n-2)h)$$

Applying the double fibration model, and after proving the stronger vanishing property we get

Residue formula for the Demailly intersection number

$$I = \int_X \operatorname{Res}_{\mathbf{z}=\infty} \frac{\prod_{i<j}(z_i - z_j)\, \mathcal{Q}_d(z_1 \dots z_n) R(\mathbf{z}, h, d, \delta)}{\prod_{1 \le i+j \le l \le n}(z_i + z_j - z_l)(z_1 \dots z_n)^n} \cdot \prod_{l=1}^{n}\left(1 + \frac{dh}{z_l}\right) \prod_{l=1}^{n}\left(1 - \frac{h}{z_l} + \frac{h^2}{z_l^2} - \dots\right)^{n+2}$$

where

$$R(\mathbf{z}, h, d, \delta) = (-z_1 - \dots - z_n + 2n^2 h)^{n^2} - n^2(-z_1 - \dots - z_n + 2n^2 h)^{n^2-1}\left(2n^2 h + \delta \binom{n+1}{2}(d-n-2)h\right)$$

Analysis of the formula

- The iterated residue is the coefficient of $\frac{1}{z_1 \dots z_n}$, and has the form $h^n p(d, n, \delta)$.
- Integration on X is the substitution $h^n = d$, so the result is $dp(d, n, \delta)$
- $p(n, d, \delta) = a_n(n, \delta)d^n + \ldots + a_0(n, \delta)$ is a degree-n polynomial in $d = \deg(X)$.
- The leading coefficient is

$$a_n(n, \delta) = \left(1 - n^2 \binom{n+1}{2} \delta\right) \Theta(n),$$

where

$$\Theta(n) = \mathrm{coeff}_1 \frac{Q(\mathbf{z}) \prod_{i<j}(z_i - z_j)(z_1 + \ldots + z_n)^{n^2}}{\prod_{i+j \le l \le n} (z_i + z_j - z_l)(z_1 \ldots z_n)^n}$$

Corollary

For $\delta < \frac{2}{n^3(n+1)}$ the leading coefficient of the Demailly intersection number is positive.

More information about $Q(\mathbf{z})$ is needed!

Conjecture

Define

$$Tp_k(z_1,\ldots,z_k)=\frac{\prod_{m<l}(z_m-z_l)\,Q_k(z_1\ldots z_k)}{\prod_{m+r\leq l\leq k}(z_m+z_r-z_l)}$$

Then

$$\frac{coeff_{z_1^{i_1}\ldots z_k^{i_k}}Tp_k}{coeff_{z_1^{i_1}\ldots z_l^{i_l+1}\ldots z_m^{i_m-1}\ldots z_k^{i_k}}Tp_k}<k^2$$

Theorem (B, 2009)

Conjecture for Thom polynomials of A_n singularities $\Rightarrow$ GGL is true for $d=\deg(X)>n^6$.

The given iterated residue formula is suitable to compute intersection numbers. Define

$$\chi(X, E_{k,m}T_X^*) = \sum_{i=0}^{n}(-1)^i \dim H^i(X, E_{k,m}T_X^*)$$

Well-known:

$$\chi(X, E_{k,m}) = \int_X [ch(E_{k,m}) \cdot Td(T_X)]_n$$

Theorem (Iterated residue formula for the Euler-characteristics)

$$\chi(X, \pi_*\mathcal{O}_{X_n}(m)) =$$

$$\int_X \mathrm{Res}_{\mathbf{z}=\infty} \frac{\prod_{i<j}(z_i - z_j)\, \mathcal{Q}_n(z_1 \ldots z_n) ch(\mathcal{O}_{X_n}(m)) Td(T_X)}{\prod_{1 \le i+j \le l \le n}(z_i + z_j - z_l)(z_1 \ldots z_n)^n} \cdot$$

$$\cdot \prod_{l=1}^{n}\left(1 + \frac{dh}{z_l}\right) \prod_{l=1}^{n}\left(1 - \frac{h}{z_l} + \frac{h^2}{z_l^2} - \ldots\right)^{n+2}$$

where

$$ch(\mathcal{O}_{X_n}(1)) = e^{m(z_1 + \ldots + z_n)}, \;\; Td(T_X) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \ldots$$

Appendix

Localization on the Demailly-tower gives an other iterated residue formula:

Theorem

$$\chi(X, \pi_* \mathcal{O}_{X_k}(\mathbf{a})) =$$

$$= (-1)^k \int_X \mathrm{Res}_{\mathbf{z}=\infty} \frac{\prod_{2 \le t_1 \le t_2 \le k} (z_{t_1} + z_{t_1+1} + \ldots + z_{t_2}) ch(\mathcal{O}_{X_k}(\mathbf{a})) \cdot Td(T_X)}{\prod_{1 \le s_1 < s_2 \le k} (-z_{s_1} + z_{s_1+1} + \ldots + z_{s_2}) \prod_{j=1}^{k} (z_1 + \ldots + z_j)^n} \prod_{j=1}^{k} \frac{1 + \frac{dh}{z_1 + \ldots + z_j}}{\left(1 + \frac{h}{z_1 + \ldots + z_j}\right)^{n+2}} d\mathbf{z}$$

where

$$ch(\mathcal{O}_{X_k}(\mathbf{a})) = e^{a_1 z_1 + \ldots + a_k z_k)}, \;\; Td(T_X) = 1 + \frac{1}{2} c_1 + \frac{1}{12}(c_1^2 + c_2) + \ldots$$