Thom polynomials of singularities via equivariant localization
Joint work with Andras Szenes and Frances C. Kirwan

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11th International Workshop on Real and Complex Singularities, Sao Carlos

July 26, 2010.

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- $\mathcal{D} = \mathrm{Diff}(\mathbb{C}^n, 0) \times \mathrm{Diff}(\mathbb{C}^k, 0) \curvearrowright J_d(n,k)$ with

$$(A, B)p = BpA^{-1}$$

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$$MD_A^{n \to k} \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_k]^{S_n \times S_k}$$

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Haefliger and Kosinski:

$$c(q) = c_0 + c_1 q + c_2 q^2 + \dots = \frac{c(f^*(TM))}{c(TN)} = \frac{\prod_{m=1}^k (1 + \theta_m q)}{\prod_{i=1}^n (1 + \lambda_i q)}$$

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3. $H_G^*(V) = H_G^*(pt)$ is the G-equivariant cohomology ring of V.
   $H_{GL(d)}^*(pt) = \mathbb{C}[x_1, \ldots, x_d]^{S_d}$.

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Vergne: there is an equivariant Thom class:

$$Thom_G(V) \in H_G^{\dim V}(V)$$

s.t

$$\mathrm{mdeg}[\Sigma, V] = \int_\Sigma Thom_G(V).$$

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Example: $(\mathbb{C}^*)^3$ acts on $\mathbb{C}^4$ with weights $\eta_1, \ldots \eta_4$. Let $\eta_1 + \eta_2 = \eta_3 + \eta_4$, and

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$$\Sigma_t = \mathrm{Spec}(\mathbb{C}[y_1, y_2, y_3, y_4]/(y_1 y_2 - t y_3 y_4)),$$

For $t = 0$ $\Sigma_0 = \{y_1 y_2 = 0\}$, so normalisation says

$$\mathrm{mdeg}[\Sigma, \mathbb{C}^4] = \eta_1 + \eta_2 = \eta_3 + \eta_4.$$

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Theorem (The test curve model of Porteous and Gaffney)

$$\Sigma_d(n,k) \doteq \{\Psi \in J_d(n,k) | \exists \gamma \in J_d^{reg}(1,n) \text{ such that } \Psi \circ \gamma = 0\}.$$

$$(\mathbb{C},0) \xrightarrow{\gamma} (\mathbb{C}^n,0) \xrightarrow{\Psi} (\mathbb{C}^k,0) \tag{1}$$

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Proposition

$\Sigma_d^0 = \{\Psi : \dim\ker\Psi = 1\} \doteq \Sigma_d$ *fibers with linear fibres over* $J_d^{reg}(1,n)/\mathbf{G}_d$.

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- Then $\mathbf{G}_d$ acts on $J_d(1,n)$ on the right by

$$\left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_d \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \ldots & 2\alpha_1\alpha_{d-1} + \ldots \\ 0 & 0 & \alpha_1^3 & \ldots & 3\alpha_1^2\alpha_{d-2} + \ldots \\ 0 & 0 & 0 & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \alpha_1^d \end{pmatrix} : \alpha_1 \in \mathbb{C}^* \alpha_i \in \mathbb{C} \right\};$$

where the polynomial in the (i,j) entry is

$$p_{i,j}(\bar{\alpha}) = \sum_{a_1+a_2+\ldots+a_i=j} \alpha_{a_1}\alpha_{a_2}\ldots\alpha_{a_i}.$$

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$$\rho(v_1, \ldots, v_d) = (v_1, v_2 + v_1^2, \ldots, \sum_{a_1+a_2+\ldots+a_i=j} v_{a_1} v_{a_2} \ldots v_{a_i}, \ldots),$$

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4. *For* $d = n$ *the boundary orbits of* $\overline{\mathrm{im}(\varrho)} \subset A_d$ *have codimension at least* 2, *so* $\overline{\mathrm{im}(\rho)} = \mathrm{Spec}(\mathbb{C}[A_d]^{G_d})$.

Principle: NON-REDUCTIVE QUOTIENTS ARE REPRESENTED AS REDUCTIVE ORBITS IN PARTIAL FLAG MANIFOLDS

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$$Tp_d^{k-n} = \int_{\Sigma_{\mathbf{d}}} \mathrm{Thom}_d(n,k) = \int_{\mathrm{Flag}_{\mathbf{d}}(\mathbb{C}^{\mathbf{n}})} \int_{F \in \mathrm{Flag}_{\mathbf{d}}(\mathrm{Sym}_{\leq \mathbf{d}}\mathbb{C}^{\mathbf{n}})} \mathrm{Thom}|_F$$

Cartan model:

$$\Omega_G(M) = \{\alpha : \mathfrak{g} \to \Omega(M) : \alpha(gX) = g\alpha(X), g \in G\} = \\ = (\mathbb{C}[\mathfrak{g}] \otimes \Omega(M))^G$$

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where $(g \cdot \alpha)(X) = g \cdot (\alpha(g^{-1} \cdot X))$. Define $(d_G\alpha)(X) = (d - \iota(X_M))\alpha(X)$

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Theorem (ABBV Localization in equivariant cohomology)

$G = T$ acts on M with isolated fixed points.

$$\int_M \alpha = (2\pi)^l \sum_{p \in M^T} \frac{\alpha_0(p)}{\mathrm{Euler}^T(T_pM)}$$

In other words:

$$\int_M \alpha(X) = (2\pi)^l \sum_{p \in M^T} \frac{\alpha(X)_0(p)}{\prod_i \lambda_i}$$

where λ_i are the weights of the Lie action

$$X : \xi \in T_pM \to [X_M(p), \xi] \in T_pM.$$

Toy example: $d = 1$

$$\Sigma_1 \doteq \{A \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^k);\ \dim\ker A = 1\} =$$
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Consider the rational differential form

$$-\frac{\prod_{j=1}^{k}(\theta_j - z)}{\prod_{i=1}^{n}(\lambda_i - z)}\, dz.$$

The residues of this form at finite poles: $\{z = \lambda_i;\ i = 1 \dots n\}$ exactly recover the terms of the sum. Apply the residue theorem, and change variables: $z = -1/q$, and we get

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Theorem (B-Szenes, 2007)

$$Tp_d^{k-n} = \mathrm{Res}_{\mathbf{z}=\infty} \frac{\prod_{m<l}(z_m - z_l)\, \mathcal{Q}_d(z_1 \dots z_d)}{\prod_{m+r \le l \le d}(z_m + z_r - z_l)} \cdot \prod_{l=1}^{d} c\left(\frac{1}{z_l}\right) z_l^{k-n}\, dz_l$$

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This can be proved for $d \leq 4$ from the formula. Short history:

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- Giambelli-Thom-Porteous formula '50s: $d = 1$ case
- F. Ronga 1980 $d = 2$
- R. Rimanyi 1998 for $n = k$, $d \le 8$, positivity conjecture
- R. Rimanyi, G. Berczi formula for $d = 3$, proved by P. Pragacz in 2004

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THANK YOU FOR YOUR ATTENTION