Lectures on singularities of maps

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1 Introduction

Plan of the talk:

1. Lecture 1: Critical points of functions, Local algebra, Invariants of critical points, Classification of singularities
2. Lecture 2: Picard-Lefschetz theory, Milnor fibration, Monodromy and variation map, Intersection matrix of a singularity, Dynkin diagrams
3. Lecture 3: Derived category of sheaves, Constructible sheaves, Perverse sheaves
4. Lecture 4: Algebraic Picard-Lefschetz theory: nearby and vanishing cycles, Applications to the geometry of singular spaces
5. Lecture 5: Mixed Hodge Structure of an isolated critical point
6. Lecture 6: Singularities of maps, equivalences of maps, Thom polynomials, Giambelli-Thom-Porteous formula and generalisations via equivariant localisation.

2 Lecture 1

2.1 Critical points of functions

The function $f : \mathbb{C}^n \to \mathbb{C}$ is *critical* at the point p if $f'(p) = 0$. If p is a critical point, $f(p)$ is the critical value. A critical point is *nondegenerate* or *Morse critical point* if the second differential $f''(p)$ is a nondegenerate quadratic form. The *corank* of the critical point is the dimension of the kernel of $f''(p)$.

Let $\mathcal{O}_n$ denote the ring of function germs at $0 \in \mathbb{C}^n$, $\mathcal{D}_n$ the ring of biholomorphic maps $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. Then $\mathcal{D}_n$ acts on $\mathcal{O}_n$ by $g(f) = f \circ g^{-1}$. Two funtion germs are *equivalent* if they belong to the same orbit. Two critical points are said to be equivalent if their function germs are equivalent.

Definition 1. *The equivalence class of a function germ at a critical point is called* singularity

The same definitions hold for the d-germs $O_{n,d}$ and d-jets of biholomorphic maps at the origin $\mathcal{D}_{n,d}$.

Theorem 2.1 (Morse lemma)**.** *If* $0 \in \mathbb{C}^n$ *is a nondegenerate critical point, then locally* f *has the following form:*

$$f(x) = x_1^2 + \ldots + x_n^2.$$

For a smooth real function $f : \mathbb{R}^n \to \mathbb{R}$ the canonical form is

$$f(x) = -x_1^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_n^2$$

where λ is the index of the Morse critival point.

Theorem 2.2 (Morse lemma for degenerate critical points)**.** *Let* f *have a critical point of corank* k *at* $0 \in \mathbb{C}^n$*. Then* f *is equivalent to a function of the form*

$$\varphi(x_1, \ldots, x_n) + x_{k+1}^2 + \ldots + x_n^2$$

where $\varphi''(0) = 0$*, i.e* $\varphi \in \mathfrak{m}^3 \subset O_n$*.*

Definition 2 (Stable equivalence)**.** $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *and* $g : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ *are* stably equivalent *if they become equivalent after the addition of nondegenerate quadratic forms in supplementary variables.*

$$f(x_1, \ldots x_n) + x_{n+1}^2 + \ldots + x_k^2 \sim fg(y_1, \ldots y_m) + y_{m+1}^2 + \ldots + y_k^2$$

Weinstein shows that two functions of the same number of variables are stably equivalent if and only if they are equivalent. This notion allows us to compare degeneracies of critical points of functions of different number of variables.

Example 1. $f(x) = x^3$ *and* $g(x, y, z) = x^3 + yz$ *are stably equivalent at zero. Indeed,*

$$x_1^3 + x_2^2 + x_3^2 \sim y_1^3 + y_2 y_3$$

via the coordinate change $y_1 = x_1, y_2 = x_2 + ix_3, y_3 = x_2 - ix_3$*.*

2.2 Invariants of singularities

We already know one invariant: the corank. A finer classification is given by

Definition 3 (Local algebra)**.** *The local algebra* Q_f *of the singularity of* f *at the origin is the quotient of the algebra of function-germs by the gradient ideal of* f*:*

$$Q_f = O_n / O_n \langle \partial f . \partial x_1, \ldots \partial f / \partial x_n \rangle$$

The multiplicity $\mu(f)$ *of the critical point is the dimension of its local algebra as a* $\mathbb{C}$*-module:*

$$\mu(f) = \dim_{\mathbb{C}} Q_f$$

A critical point is said to be isolated *if* $\mu(f) < \infty$*.*

Note that the local algebra does not depend on the local coordinate system. Example: for $f(x) = x^3$ $Q_f = \mathfrak{m}^2$, $\mu(f) = 2$. $g(x) = xy^2$ is nonisolated.

Theorem 2.3. *The multiplicity of an isolated critical point is equal to the number of Morse critical points into which it decomposes under a generic deformation of the function.*

Remark 1. *In the real case things are different: $f(x) = e^{-1/x^2}$ has a unique critical point at zero, but $\mu(f) = \infty$. Moreover, $f(x) = x^3$ has multipliciy 2, but its deformation $x^3 + \lambda x$ has no real critical point.*

For a function $\mathbb{C}^n \to \mathbb{C}$ let $j_n^k f \in O_k$ denote its k-jet. A k-jet is *sufficient* for a singularity, if any two functions with that k-jet are equivalent. For example, the k-jet of a nondegenerate critical point is sufficient.

Theorem 2.4. *(Tougeron) The $(\mu + 1)$-jet of a critical point of multiplicity μ is sufficient.*

As a consequence:

$$O_n - \mathcal{D}_n(f) = O_{n,\mu+1} - \mathcal{D}_{n,\mu+1}(j_n^{\mu+1} f),$$

i.r the complement of the orbit of f in O_n is the same as the complement of the orbit of its $\mu + 1$-jet in the finite dimensional vector space $O_{n,\mu+1}$ This property allows us to work with finite dimensional vector spaces later in Lecture 6.

Example 2. *(Whitney) The germ of $f(x, y, z) = xy(x + y)(x - zy)(x - e^z y)$ at zero is not equivalent to a polynomial germ. Indeed, $\{f = 0\} \cap \{z = \lambda\}$ consists of 5 lines through the origin. If f was equivalent to an algebraic germ, the cross-ration of any 4 of these would depend algebraically on λ, but we have exp(z) instead.*

2.3 Deformations of critical points

A *deformation* with base $\Lambda = \mathbb{C}^l$ of the germ $f \in O_n$ is the germ at zero of a smooth map $F : (\mathbb{C}^n \times \mathbb{C}^l, 0) \to \mathbb{C}$ such that $F(x, 0) \equiv f(x)$. A deformation F' is *equivalent* to F if $F'(x, \lambda) = F(g(x, \lambda), \lambda)$, where $g : (\mathbb{C}^n \times \mathbb{C}^l) \to (\mathbb{C}^n, 0)$, with $g(x, 0) \equiv x$, is a smooth germ. The deformation F' is induced from F if $F'(x, \lambda') = F(x, \theta(\lambda))$, where $\theta : (\mathbb{C}^{l'}, 0) \to (\mathbb{C}^l, 0)$ is a smooth germ of mapping of the bases.

Definition 4. *A deformation $F(x, \lambda)$ is* versal *if every deformation of $f(x)$ is equivalent to a deformation induced from F, i.e has the form*

$$F'(x, \lambda') = F(g(x, \lambda'), \theta(\lambda')), \ g(x, 0) \equiv x, \theta(0) = 0$$

Example 3. *Every deformation of $f(x) = x^2$ has the form*

$$G(x, \mu) = \alpha(x, \mu)x^2 + \beta(\mu)x + \gamma(\mu)$$

with $\alpha(x, 0) \equiv 1$ and $\beta(0) = \gamma(0) = 0$. Prove this, and that $F(x, \lambda) = x^2 + \lambda$ is a versal deformation.

How can we check that a deformation is versal? Necessary condition: the deformation should be transversal to the orbit of f under the action of $\mathcal{D}_n$ on $\mathbb{C}^n$:

Definition 5. *A deformation $F(x,\lambda)$ of the germ $f(x)$ is* infinitesimally versal *if every function germ $\alpha(x)$ can be represented in in the form*

$$\alpha(x) = \sum_{i=1}^{n} h_i f_i + \sum_{j=1}^{l} c_j \frac{\partial F}{\partial \lambda_j}|_{\lambda=0}$$

Remark 2. *Note that the tangent space to the orbit $\mathcal{D}_n \cdot f$ in O_n is the gradient ideal $T_f\mathcal{D}_n(f) = I_{\nabla_f} = O_n\langle f_i\rangle$. The first term is an element of this tangent space, the second term is in the transversal direction.*

Theorem 2.5. *(Mather) Infinitesimally versal deformations are versal.*

Back to the example before: $f(x) = x^2$, then $Q_f = O_n/\mathfrak{m}$, so $F(x,\lambda) = x^2 + \lambda$ is versal, since $\frac{\partial F}{\partial \lambda}|_{\lambda=0} = 1$.

A versal deformation with minimal dimensional deformation space $\mathbb{C}^l$ is called *miniversal*. According to Mather‘s theorem

$$F(x,\lambda) = f(x) + \sum_{j=1}^{\mu} \lambda_j \varphi_j(x)$$

where $\varphi_j(x), j = 1,\dots,\mu$ form a basis of Q_f is miniversal.

The last invariant we introduce is the modality of a germ

Definition 6. *The modality of a sufficient germ is the least number m such that a small neighorhood of the germ in the function space can be covered by finite number of m-parameter families of germ-orbits.*

For instance, the modality of a Morse critical point is 0 by the Morse-lemma. The dimension of the $\mu = const$ stratum in the base of a miniversal deformation is $m+1$.

In the study of the monodromy of singularities the level bifurcation set plays an important role. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ with isolated critical point at 0. Note that for sufficiently small neighbohood $U \subset \mathbb{C}^n$ of 0 the level set $\{x : F(x,\lambda) = 0\}$ is sonsingular on the boundary ∂U and transverse to ∂U for every $\lambda \in \Lambda \subset \mathbb{C}^l$, where Λ is a small neighbohood of 0. We say that λ is singular if $V_\lambda = \{x : F(x,\lambda) = 0\} \cap U$ is singular.

Definition 7. *The* level bifurcation set, or discriminant *of the singularity f is the germ of the surface formed in $\mathbb{C}^l$ by the singular values of the parameter λ, i.e $\Sigma_f = \{\lambda : V_\lambda$ is singular$\}$.*

Example 4. *Let $f(x) = x^3$, $F(x,\lambda) = x^3 + \lambda_1 x + \lambda_0$. The discriminant of f is the set of those (λ_1, λ_0) for which $x^3 + \lambda_1 x + \lambda_0$ has multiple roots. Hence $\Sigma_f = \{27\lambda_0^2 + 4\lambda_1^3 = 0\}$.*

2.4 Classification of critical points

Recall: singularity=$\mathcal{D}_n$-orbit in O_n. From the topological viewpoint the most important characteristic of a critical point is the multiplicity μ. This is the number of Morse critical points into which the given singularity splits under a small perturbation. For a given $\mu = const$ stratum, let $c(\mu)$ denote the codimension of this stratum in O_n. A generic function has only Morse critical points of codimension $c = 0$. The third invarient is the modality m with the equality $\mu = c + m + 1$.

Example 5. *Let $f(x) = x^3$ with $\mu = 2, c = 1, m = 0$. This means that in a generic one-parameter family of functions there are isolated values of the parameter where A_2 singularity occurs.*

A complete classification is available just for small values of c, μ, m. Here we list the simple singularities corresponding to $m = 0$, i.e $c = \mu - 1$.

Proposition 1. *The simple singularities are (with the index denoting the multiplicity)*

1. *A_k-singularities (Morin singularities): $f(x) = x^{k+1}$;*
2. *D_k-singularities, $k \geq 4$: $f(x, y) = x^2 y + y^{k-1}$;*
3. *Exceptional singularities: $E_6 : f(x, y) = x^3 + y^4$,$E_7 : f(x, y) = x^3 + xy^3$ and $e_8 : f(x, y) = x^3 + y^5$.*

Connection to Dynkin diagrams

- The finite subgroups of $SO(3)$ are 1) the cyclic group $\mathbb{Z}_n$ 2) the dihedral group $\mathbb{D}_n = \mathbb{Z}_n \bowtie \mathbb{Z}_2$ 3)The symmetry group of the tetrahedron, octahedron and icosahedron.
  If $\Gamma \subset SO(3)$ is a finite subgroup, let $\Gamma^* \subset SU(2)$ denote its preimage under the two-sheeted covering map $SU(2) \to SO(3)$. Γ^* acts on $\mathbb{C}^2$. Then
  $$\mathbb{C}^2/\Gamma^* = \mathrm{Spec})(\mathbb{C}[x, y, z]/R)$$
  where R is a relation equivalent to the functions given in the proposition. For example, for the cyclic group $R = x^n + yz = 0 \sim x^n$.

- *Resolution of singularities* Let $V = \{f = 0\}$ be the zero level set of $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ with simple singularity at 0. For the minimal resolution $\pi : \tilde{V} \to V$
  $$\pi^{-1} = C_1 \cup \ldots \cup C_\mu, \; C_i \simeq \mathbb{CP}^1.$$
  The self-intersection index is $\langle C_i, C_i \rangle = -2$ (see later) and $\langle C_i, C_j \rangle = 0$ or 1. The graph which enstores this information is the Dynkin diagram of the corresponding Coxeter groups.

  The minimal resolution can be constracted as the Γ-invariant part of the Hilbert scheme of N points on $\mathbb{C}^2$, where $N = |\Gamma|$.

- An other approach is the MacKay-correspondence. Let $R_0, \ldots, R_n$ be the irreducible representations of $\Gamma^* \subset SU(2)$, and $Q = \mathbb{C}^2$ be the standard representation. Then
$$Q \otimes R_k = \oplus_{l=1}^{n} a_{k,l} R_l$$
and $2Id - (a_{kl})$ is the Cartan matrix of an extended Dynkin diagram.

3 Lecture 2

3.1 Monodromy Groups of Critical Points

The Picard-Lefschetz theory is the complex Morse theory. Let $f : \mathbb{C}^n \to \mathbb{C}, z \to t$ be a holomrophic function with an isolated critical point at $z = a$ with $f(a) = \alpha$. Fix $U \subset \mathbb{C}^n$ and $T \subset \mathbb{C}$ a small neighborhood of a and $f(a)$ satisfying:

- $V_\alpha = f^{-1}(\alpha)$ is transverse to ∂U.
- For every $t \in T, t \neq \alpha$ the level manifold $V_t = f^{-1}(t)$ is nonsingular in U and tranverse to ∂U.

Let $T^* = T \backslash \alpha$ and $V_T = f^{-1}(T) \cap U, V_{T^*} = f^{-1}(T^*) \cap U, \partial V_T = V_T \cap \partial U$. Then

- $V_{T^*} \to T^*$ is locally trivial fribration and since T is contractible, $\partial V_T \to T$ is trivial fibration.
- V_t for $t \neq \alpha$ is homotopy equivalent to a wedge of μ $(n-1)$ dimensional spheres. Therefore $H_{n-1}(V_t) = \mathbb{Z}^\mu$

Example 6. *Take $f(z, w) = z^2 + w^2$, and study the topology of the singular and sonsingular level manifolds.*

Fix $\beta \in \partial T$ and consider a loop $\gamma : [0, 1] \to T^*, \gamma(0) = \gamma(1) = \beta$ which makes an anticlockwise circuit along ∂T. Going around γ generates $h_\theta : V_\beta \to V_{\gamma(\theta)}$, and we can assume that $h_\gamma = h_1 : V_\beta \to V_\beta$ is the identity on the boundary ∂V_β.

Definition 8. *The action of the map h_γ induces a map*

1. *$h_\gamma : H_{n-1}(V_\beta) \to H_{n-1}(V_\beta)$ the monodromy operator of the singularity f.*
2. *$h^r_\gamma : H_{n-1}(V_\beta, \partial V_\beta) \to H_{n-1}(V_\beta, \partial V_\beta)$ the relative monodromy operator of the singularity f.*
3. *$var_\gamma : H_{n-1}(V_\beta, \partial V_\beta) \to H_{n-1}(V_\beta), \delta \to h_\gamma(\delta) - \delta$ the variation operator.*

For $f(x) = z^2 + w^2$ $V_1 = \{z^2 + w^2 = 1\}$ and $H_1(V_1) = \mathbb{Z}$ generated by the homology class of the circle Δ, and $H_1(V_1, \partial V_1) = \mathbb{Z}$ generated by ∇. Following $\gamma = exp(2\pi i\theta$ we get
$$h\Delta = \Delta, \; h\nabla = \nabla - \Delta$$

and therefore

$$h_\gamma = id,\ Var : \delta \to (\Delta \circ \delta)\Delta.$$

We call Δ the *vanishing cycle*. More generally for a Morse singularity $f(z_1, \dots, z_n) = \alpha + z_1^2 + \dots + z_n^2$ fixing a path $\varphi(\theta)$ with $\varphi(0) = \beta, \varphi(1) = \alpha$

$$S_\theta = \{z : z_1^2 + \dots + \zeta_n^2 = \sqrt{(}\varphi(\theta) - \alpha), Imz_i = 0, i = 1, \dots, n\} \subset V_{\varphi(\theta)}$$

is a generator of $H_{n-1}(V_{\varphi(\theta)})$, which contracts to a point. $\Delta = S_0$ is called the vanishing cycle. More precisely, V_β is isomorphic to the tangent bundle TS^{n-1}, and the Δ =zero section, and the generator ∇ of $H_{n-1}(V_\beta, \partial V_\beta)$ is given by the $n-1$-dimensional ball, the fibre of the bundle.

Theorem 3.1. *Picard-Lefschetz Let $\delta \in H_{n-1}(V_\beta, \partial V_\beta)$ and $\sigma \in H_{n-1}(V_\beta)$, $i_* : H_{n-1}(V_\beta) \to H_{n-1}(V_\beta, \partial V_\beta)$.*

1. $var_\gamma(\nabla) = (-1)^{n(n+1)/2}\Delta$, *and more generally* $var_\gamma(\delta) = (-1)^{n(n+1)/2}(\delta \circ \Delta)\Delta$
2. $h^r_\gamma(\delta) = \delta + (-1)^{n(n+1)/2}(\delta \circ \Delta)i_*(\Delta)$
3. $h_\gamma(\sigma) = \sigma + (-1)^{n(n+1)/2}(\sigma \circ \Delta)\Delta$
4. $\nabla \circ \Delta = 1$, *and* $\Delta \circ \Delta = \begin{cases} 0 & n \equiv 0(mod\ 2) \\ 2 & n \equiv 1(mod\ 4) \\ -2 & n \equiv 3(mod\ 4) \end{cases}$

Monodromy of degenerate critical points If f has an isolated degenerate critical point, we need to consider the Morsification of f, which is a small deformation of f having μ distinct Morse critical points $a_1, \dots, a_\mu$ and critical values $\alpha_1, \dots, \alpha_\mu$. $f + \varepsilon g$ is a Morsification if $g(z)$ is a linear function in general position. For small ε $a_1 \in U$ and $\alpha_i \in T$, $i = 1, \dots \mu$, and $T^* = T \backslash \{\alpha_1, \dots \alpha_\mu\}$. We assume the same for the level sets V_t $t \in T$ as before. Fix $\beta \in \partial T$ again, and each loop $\gamma \in \pi_1(T^*, \beta)$ defines a monodromy, relative monodromy and variation operator. These behave as follows under the composition $\gamma = \gamma_1 \circ \gamma_2$:

$$var_\gamma = var_{\gamma_1} + var_{\gamma_2} + var_{\gamma_1} \circ i_* \circ var_{\gamma_2}$$

$$h_\gamma = h_{\gamma_1} \circ h_{\gamma_2},\ h^r_\gamma = h^r_{\gamma_1} \circ h^r_{\gamma_2}$$

inducing a homomorphism $\pi_1(T^*, \beta) \to \mathrm{Aut}H_{n-1}(V_\beta)$, whose image is the *monodromy group*.

Take not self-intersecting paths φ_i, $i = 1, \dots, \mu$ from $\varphi_i(0) = \beta$ to $\varphi(1) = \alpha_i$ such that they are indexed in the increasing order of $\arg\varphi_i'(0)$. This is called a distinguished set of paths. The set of vanishing cycles $\Delta_1, \dots, \Delta_\mu \in H_{n-1}(V_\beta)$ is called a *distinguished basis of cycles* due to the following

Theorem 3.2. *A distinguished basis of cycles forms a basis of $H_{n-1}(V_\beta) \simeq \mathbb{Z}^\mu$.*

Example 7. *Take $f(z) = z^4$ ($\mu = 3$) and its Morsification $w = z^4 - 14z^2 + 24z$, which has three Morse critical points at $a_1 = 2, a_2 = 1, a_3 = -3$ with critical values $\alpha_1 = 8, \alpha_2 = 11, \alpha_3 = -135$. Let $\beta = 10$ be the noncritical value. The nonsingular fiber is $V_\beta = \{z_1, z_2, z_3, z_3\}$, its reduced homology group $H_0(V_\beta) = \mathbb{Z}^3$. Take a distinguished set of paths $\varphi_1, \varphi_2, \varphi_3$ in the w-plane, then $\Delta_i = [z_i] - [z_{i+1}]$ $i = 1, 2, 3$. The monodromy along the simple loop corresponding to φ_i permutes the points z_i and z_{i+1} in the fibre V_β.*

3.2 The Intersection Matrix of a Singularity

The monodromy along the simple loop corresponding to φ_i is given by

$$h_i : \delta \mapsto (-1)^{n(n+1)/2}(\delta \circ \Delta_i)\Delta_i$$

Definition 9. *The matrix $[(\Delta_i \circ \Delta_j)]$ of a distinguished basis $\Delta_1, \ldots, \Delta_\mu$ is called the* intersection matrix *of the singularity f.*

The intersection matrix is symmetric for n odd and skew-symmetric for n even. The diagonal elements are equal $\Delta_i \circ \Delta_i = (-1)^{n(n+1)/2}[1 + (-1)^{n+1}]$. Clearly, the monodreomy group Γ preserves the intersection form.

Recall that for the original singularity of f we were interested the classical monodromy along the boundary ∂T. This is given by

$$h_f = h_1 \circ \ldots \circ h_\mu,\ h_f^r = h_1^r \circ \ldots \circ h_\mu^r$$

$$var_f = var_{\gamma_1 \circ \ldots \circ \gamma_\mu} = \sum_{r=1}^{\mu} \sum_{i_1 < \ldots < i_r} \prod_{j=1}^{r} var_{i_j}$$

where the composition of the variation operators takes the embedding $H_{n-1}(V_\beta) \hookrightarrow H_{n-1}(V_\beta, \partial V_\beta)$ into account.

Theorem 3.3. *The variation operator of the singularity $var_f : H_{n-1}(V_\beta, \partial V_\beta) \to H_{n-1}(V_\beta)$ is an iomorphism. The matrix of this map uniquely determines the intersection matrix and vice versa via the formula:*

$$(\cdot \circ \cdot) = -(var_f^{-1} \cdot \circ \cdot) - (\cdot \circ var_f^{-1} \cdot).$$

Example 8. *Back to $f(z) = z^4$. The intersection matrix in the distinguished basis $\Delta_1, \Delta_2, \Delta_3$ is*

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

and

$$h_1 = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, h_2 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, h_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}$$

so

$$h_f = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}.$$

Remark 3. *1. Under stabilisation $\tilde{f} = f + z_{n+1}^2 + \ldots + z_{n+k}^2$ the intersection matrix changes as follows:*

$$(\tilde{\Delta}_i \circ \tilde{\Delta}_j) = (\mathrm{sign}(j-i))^k (-1)^{kn+k(k-1)/2} (\Delta_i \circ \Delta_j).$$

2. *Since h_i are reflections, the monodromy group Γ is a reflection group, and the corresponding Dynkin diagram is the one defined earlier. In other words, if we assign a vertex to each distinguished basis $\Delta_1, \ldots, \Delta_\mu$ and connect Δ_i with Δ_j if $\Delta_i \circ \Delta_j \neq 0$ then we get the Dynkin diagram of the singularity.*

   Theorem 3.4. *The monodromy groups of the simple singularities are the classical Weyl groups corresponding to their Dynkin diagrams.*

3. *The distinguished basis of vanishing cycles is not uniquely defined: we can change the set of distinguished paths and also the orientation of the distinguished cycles Δ_i. One can this way define an action of the braid group $Br(\mu)$ on the set of distinguished basis of vanishing cycles.*

3.3 The level bifurcation set and the monodromy group

So fat we have used a Morsification of a degenerated critical point to describe its monodromy group. An other option is the following.

- Let $F(x, \lambda) : (\mathbb{C}^n \times \mathbb{C}^\mu) \to \mathbb{C}$ be a miniversal deformation of $f(x) = F(x, 0)$, and $\Sigma_f \subset \mathbb{C}^\mu$ the level bifurcation set.
- For a small ball $D_\varepsilon \subset \mathbb{C}^\mu$ the topological type $(D_\varepsilon, \Sigma_\varepsilon)$ does not depend on ε and F, and the fibration

  $$\{F(x, \lambda) = 0, \lambda \notin \Sigma_f\} \to D_\varepsilon \backslash \Sigma_f : \ (x, \lambda) \to \lambda$$

  is a locally trivial fibration. The fiber $V_\lambda = \{x : F(x, \lambda) = 0\}$ is diffeomorphic to $V_t = \{x : f(x) = t\}$.
- Therefore we have a monodromy $\pi_1(D_\varepsilon \backslash \Sigma_f) \to Aut(H_{n-1}(V_\lambda)$

Theorem 3.5. *The image of the automorphism $\pi_1(D_\varepsilon \backslash \Sigma_f) \to Aut(H_{n-1}(V_\lambda)$ coincides with the monodromy group Γ of the singularity f.*

Indeed, let $F(x, \lambda) = F_0(x, \lambda') - \lambda_0$, where $\lambda' \in \mathbb{C}^\mu, \lambda_0 \in \mathbb{C}, \lambda = (\lambda', \lambda_0)$. As the perturbation $f_t(x)$ of the singularity f we can take a perturbation of the form $F_0(x, \lambda'(t))$. Let $p : \mathbb{C}^\mu \to \mathbb{C}^{\mu-1} : \lambda = (\lambda', \lambda_0) \to \lambda'$ be the natural mapping. If $f_t(x)$

is a Morse function (t is sufficiently small) then the line $L = p^{-1}(\lambda'(t))$ intersects Σ_ε transversally in those points $(\lambda_0, \lambda'(t)) \in \mathbb{C}^\mu$ for which λ_0 is a critical value of $f_t(x)$. So the space $L\backslash\Sigma_\varepsilon$ coincides with $T\backslash\{\alpha_1, \dots, \alpha_\mu\}$, i.e the complement of the set of critical values of f_t, and the restriction of the fibration is the Milnor fibration.

Theorem 3.6. *The monodreomy group of a singularity acts transitively on the set of vanishing cycles* $\Delta_1, \dots, \Delta_\mu$.

Exercise: Prove this using the description of Γ with the level bifurcation set.

Corollary 1. *The Dynkin diagram of a singularity is always connected.*

Theorem 3.7. *The trace* $trh_* = (-1)^{n-1}$.

Level bifurcation sets of simple singularities

We already know that the monodromy groups of simple singularities coincide with the classical Weyl groups A_k, D_k, E_7, E_8, E_9. These classical Weyl groups act canonically on $\mathbb{C}^k$. Let $S \subset \mathbb{C}^k$ be the union of non-regular orbits (with non-trivial stabilizer)

Theorem 3.8. *For simple singularities* A_k, D_k, E_k

$$(\mathbb{C}^k/W, S/W) \simeq (D_\varepsilon, \Sigma_f)$$

in a neighbourhood of zero.

Example 9. *Let* $f(x) = x^{k+1}$. *(A_k singularity). Then* $W = S_{k+1}$, *acting on* $\mathbb{C}^k$ *as follows.* $\mathbb{C}^k \hookrightarrow \mathbb{C}^{k+1}$ *as the hyperplane* $\sum_{i=1}^{k+1} x_i = 0$, *and the action of* S_{k+1} *is the permutation of the coordinates. Then*

$$\mathbb{C}^{k+1}/W \simeq \mathbb{C}^{k+1} : (x_1, \dots x_{k+1}) \mapsto (\sigma_1, \dots, \sigma_{k+1})$$

where σ_i *is the ith elementary symmetric polynomial of* $x_1, \dots, x_{k+1}$. *Then* $\mathbb{C}^k/W$ *maps to* $\{\sigma_1 = 0\} \subset \mathbb{C}^{k+1}$. S *is represented by* $x_i = x_j$. *A miniversal deformation of* f *is*

$$F(x, t_0, \dots, t_{k-1}) = x^{k+1} + t_{k-1}x^{k-1} + \dots + t_1 x + t_0$$

The level bifurcation set consists of those values of the parameters $t = (t_0, \dots, t_{k-1})$ *for which the function* $F(\cdot, t)$ *has a critical point with critical value* 0, *that is has a multiple root* $x_i = x_j$, *corresponding to a wall in* S.

Corollary 2. *The space* $\mathbb{C}^k\backslash S$ *is a* $K(\pi, 1)$ *space with* $\pi = \hat{B}_W = \pi_1(\mathbb{C}^k\backslash S)$.
The space $(\mathbb{C}^k\backslash S)/W = (\mathbb{C}^k/W)\backslash\Sigma_f$ *is a* $K(\pi, 1)$ *space for the group* $\pi = B_W = \pi_1((\mathbb{C}^k\backslash S)/W)$.

For $W = S_k$

$B_W =$ *usual Braid group with* $k+1$ *strands=fundamental group of space of unordered sets of n distinct comp*

$\hat{B}_W =$ *fundamental group of space of ordered sets of n distinct complex numbers*

We have

$$1 \longrightarrow \hat{B}_W \longrightarrow W \longrightarrow 1\ .$$

4 Lecture 3

4.1 Derived categories

Let $\mathcal{A}$ be an abelian category. We can think of 1) the category of R-modules with R a ring 2) if X is a topological space, $\mathcal{A}$ can be the category of sheaves of O_X-modules.

Definition 10. *Let $C^+(\mathcal{A})$ be the category of bounded below cochain complexes of objects of $\mathcal{A}$ with morphisms of complexes as morphisms. A morphism $f : M^\bullet \to N^\bullet$ is called quasi-isomorphism if $H^i(f)$ is an isomorphism for all $i \in \mathbb{Z}$. Let S the class of quasi-isomorphisms in $C^+(\mathcal{A})$.*

Definition 11. *The* derived category *of $\mathcal{A}$ is defined as*

$$D^+(\mathcal{A}) = C^+(\mathcal{A})[S^{-1}],$$

i.e the objects of $D^+(\mathcal{A})$ agree with the objects of $C^+(\mathcal{A})$, but the morphisms between $M^\bullet, N^\bullet \in D^+(\mathcal{A})$ are defined as

$$\mathrm{Mor}_{D^+(\mathcal{A})} = \{\, M^\bullet \xrightarrow{f_1} X_1^\bullet \xleftarrow{s_1} X_2^\bullet \xrightarrow{f_2} X_3^\bullet \ldots \xleftarrow{s_{n-1}} X_n^\bullet \,\}/\sim$$

with $X_i^\bullet \in Ob(C^+(\mathcal{A}))$, f_i morphisms of chain complexes and s_i quasi-isomorphisms, and the equivalence relation is roughtly explained below.

The exuivalence relation in the definiton is generated by some natural expectations: 1) $M^\bullet \xrightarrow{f} X^\bullet \xrightarrow{g} N^\bullet \sim M^\bullet \xrightarrow{g\circ f} N^\bullet$ 2) $M^\bullet \xrightarrow{s} N^\bullet \xleftarrow[s]{} M^\bullet \sim M^\bullet \xrightarrow{Id} M^\bullet$ 3) In a commutative diagram

$$\begin{array}{ccc} M^\bullet & \xrightarrow{f} & X^\bullet \\ \uparrow{\scriptstyle s} & & \uparrow{\scriptstyle t} \\ Y^\bullet & \xrightarrow{g} & N^\bullet \end{array}$$

$t^{-1} \circ f$ and $g \circ s^{-1}$ represent the same morphism. This definition is rather useless for computations. A more useful definition/proposition is the following

Proposition 2. *An object $I \in \mathcal{A}$ is injective if* $\mathrm{Hom}_{\mathcal{A}}(\cdot, I)$ *is exact. The category has enough injectives, if every object can be embedded into an injective object. In this case every $N^\bullet \in D^+(\mathcal{A})$ is quasi-isomorphic to a complex of injectives $I^\bullet(N^\bullet) \in C^+(\mathcal{A})$, and*

$$\mathrm{Hom}_{D^+(\mathcal{A})}(M^\bullet, N^\bullet) = \mathrm{Hom}_{C^+(\mathcal{A})}(M^\bullet, I^\bullet)/\textit{chain homotopy}$$

Remark 4. *We have a natural embedding $\mathcal{A} \subset D^+(\mathcal{A})$ if we think of $M \in \mathcal{A}$ as a complex $\cdots \longrightarrow 0 \longrightarrow M \longrightarrow 0 \longrightarrow \cdots$. The injective representative is $I^\bullet(M) = \cdots \longrightarrow 0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$, the usual injective resolution.*

We can think of elements of $D^+(\mathcal{A})$ as complexes of injectives.

Definition 12. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Then the right derived functor*

$$RF = F \circ I : D^+(\mathcal{A}) \to D^+(\mathcal{B})$$

maps $M^\bullet \to F(I^\bullet(M^\bullet))$. We ith derived functor of F is

$$R^i F = H^i \circ RF : D^+(\mathcal{A}) \to \mathcal{B}.$$

Important examples are:

1. If $\mathcal{A} = \mathcal{S}\langle_X$ is the category of sheaves on X, $\mathcal{B}$ is the category of abelian groups and $F = \Gamma$ is the global section functor, then

$$R^i\Gamma(\mathcal{F}^\bullet) = \mathbb{H}^i(X, \mathcal{F}^\bullet)$$

   is the ith hypercohomology of $\mathcal{F}^\bullet \in \mathcal{S}\langle_X$.

2. If $M, N \in \mathcal{A}$, we get

$$R^i(\mathrm{Hom}_D(M, N) = \mathrm{Hom}_D(M, N[i]) = \mathrm{Ext}^i_{\mathcal{A}}(M, N).$$

The category $D^+(\mathcal{A})$ is not abelian but *triangulated* i.e we have the following structures:

1. Shift functor $[1] : D^+(\mathcal{A}) \to D^+(\mathcal{A})$. We define this as $(X^\bullet[1])^i = X^{i+1}$ and $d^i_{X^\bullet[1]} = (-1)^i d^{i+1}_{X^\bullet}$.

2. Distinguished triangles. Diagrams of the form

$$X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow X^\bullet[1]$$

   are called triangles. They are distinguished if they are isomorphic in $D^+(\mathcal{A})$ to a diagram of the form

$$X^\bullet \longrightarrow Y^\bullet \longrightarrow \mathrm{Cone}^\bullet(f) \longrightarrow X^\bullet[1] \; ,$$

   where $\mathrm{Cone}^i(f) \simeq Y^i \oplus X^{i+1}$ with differential $(x, y) \mapsto (f(x)+d(y), (-1)^{i+1}d(x))$.

3. These data satisfy a number of axioms

Theorem 4.1. *Let $X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow X^\bullet[1]$ be distinguished. There is a long exact sequence of cohomology sheaves:*

$$\cdots \longrightarrow \mathcal{H}^i(X^\bullet) \longrightarrow \mathcal{H}^i(Y^\bullet) \longrightarrow \mathcal{H}^i(Z^\bullet) \longrightarrow \mathcal{H}^{i+1}(X^\bullet) \longrightarrow \mathcal{H}^{i+1}(Y^\bullet) \longrightarrow \cdots$$

4.2 Constructible sheaves

Let $X/\mathbb{C}$ be an algebraic variety. Locally, X can be embedded into $\mathbb{C}^n$, and this also induces the standard metric topology on X. We denote by X^{an} this topological space.

Definition 13. *A local system on X is a locally constant sheaf on X^{an}, with finite dimensional stalks.*

Alternatively, a local system is equivalent to a representation of the fundamental group $\pi_1(X)$ on a finite dimensional vecotr space (the stalk at a base point), and it is also equivalent to a vector bundle with a flat connection.

Let $\mathcal{S}(X)$ be the category of sheaves of $\mathbb{Q}$-vector spaces on X^{an}.

Definition 14. *A sheaf $\mathcal{F}^\bullet \in \mathcal{S}(X)$ is called constructible, if there is a sequence of closed subvarieties*

$$X_0 \subset X_1 \subset \ldots \subset X_{n-1} \subset X_n = X$$

such that $\mathcal{F}^\bullet|_{X_i - X_{i-1}}$ is locally constant with finite dimensional fibres, i.e local system. Let $D^b_c(X)$ be the full subcategory of $D^+(\mathcal{S}(X))$ of complexes $\mathcal{F}^\bullet$ with $\mathcal{H}^i(\mathcal{F})$ constructible for all i, and equal to zero for all but finite numbr of $i \in \mathbb{Z}$.

The category of constructible sheaves has not enough injectives, so we do not want to work with its derived category. And $D^b_c(X)$ is the right replacement.

4.3 The functors $f_*, f^*, f_!, f^!$

Let $f : X \to Y$ be a morphism of topological spaces, $\mathcal{F} \in Sh(X), \mathcal{G} \in Sh(Y)$. Then

1. $f_* : Sh(X) \to Sh(Y)$ is *the direct image* functor defined as $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V)$. f^* is exact. Its derived functor is $f_* = Rf_* : D^+(Sh(X)) \to D^+(Sh(Y))$.

2. $f^{-1} : Sh(Y) \to Sh(X)$ is the *inverse image* functor, defined as the sheafication of the presheaf dfined by $f^{-1}\mathcal{G}(V) = \varinjlim \mathcal{G}(V)$ where V ranges through the open subsets of Y containing $f(U)$. It is left exact. Its derived functor is $f^* = Rf^* : D^+(Sh(Y)) \to D^+(Sh(X))$.

   If moreover $\mathcal{G} \in \mathcal{O}_Y - mod$ is not just a sheaf of abelian groups but sheaf of $\mathcal{O}_Y$-modules, then $f^*\mathcal{G} = \mathcal{O}_X \otimes_{\mathcal{O}_Y} f^*\mathcal{G}$ is a $\mathcal{O}_X$-module, whose derives functor $f^* : D^b_c(\mathcal{O}_Y - mod) \to D^b_c(\mathcal{O}_X - mod)$ has the same notaton.

Proposition 3. *f_*, f^* restrict to the subcategory $D^b_c(X)$:*

$$f_* : D^b_c(X) \to D^b_c(Y), \ \ f^* : D^b_c(Y) \to D^b_c(X)$$

If X is a smooth, not necessarily compact variety, then we have Poincarè duality

$$H^i_c(X, \mathcal{F})^\vee \simeq H^{2d-i}(X, \mathcal{F}^\vee)$$

where on the lhs we have comohology with compact support. Let $\Gamma_c(\mathcal{F}) \subset \Gamma(\mathcal{F})$ be the subgroup of sections whose support is compact, and $R\Gamma_c$ its derived functor.

Definition 15. *Let $f : X \to Y$ be a map of topological spaces, and $\mathcal{F} \in Sh(X)$. Then $f_!(\mathcal{F})$ is the sheafification of $U \mapsto \Gamma_c(f^{-1}U, \mathcal{F})$, and $f_! = Rf_! : D^+(Sh(X)) \to D^+(Sh(Y))$ is its derived functor.*

Example 10. *If f is proper then $f_! = f_*$. If $j : X \to Y$ is an open immersion then $j_!$ is the extension by zero.*

Proposition 4. *If $f : X \to Y$ is a morphism of varieties then*

$$Rf_! : D^b_c(X) \to D^b_c(Y).$$

Theorem 4.2. *(Verdier duality) Let $f : X \to Y$ be a morphism of varieties. Then $Rf_!$ has a right adjoint*

$$Rf^! : D^b_c(Y) \to D^b_c(X),$$

which means that for $\mathcal{F}^\bullet \in D^b_c(Y), \mathcal{G}^\bullet \in D^b_c(X)$

$$\mathrm{Hom}_{D(Y)}(Rf_!\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \mathrm{Hom}_{D(X)}(\mathcal{F}^\bullet, Rf^!\mathcal{G}^\bullet)$$

It is important to point out that there is no functor $f^!$ between sheaves which we derive, and $f^! = Rf^!$ is defined only on the level of derived categories.

Example 11. *1. If f is smooth (flat) of relative dimension d, then $f^! = f^*[2d]$.*

2.

Theorem 4.3. *(Verdier duality, sheafified version*

$$R\mathcal{H}om_{Sh(Y)}(Rf_!\mathcal{F}^\bullet, \mathcal{G}^\bullet) = Rf_* R\mathcal{H}om_{Sh(X)}(\mathcal{F}^\bullet, Rf^!\mathcal{G}^\bullet)$$

where

$$\mathcal{H}om^i_{Sh(X)}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)(U) = \mathrm{Hom}_{Sh(X)}(\mathcal{F}^i(U), \mathcal{G}^i(U))$$

is the sheaf of homomorphisms of the complexes $\mathcal{F}^\bullet, \mathcal{G}^\bullet$.

Definition 16. *Let $a : X \to pt$. We call $\mathbb{D}_X = Ra^!\mathbb{Q}$ the dualizing complex and*

$$\mathbb{D}_X(\mathcal{F}^\bullet) = R\mathcal{H}om(\mathcal{F}^\bullet, \mathbb{D}_X)$$

is the Verdier dual of $\mathcal{F}^\bullet \in D^b_c(X)$.

Example 12. • *For $X = pt$ the Verdier dual is the $\mathbb{Q}$-dual of complexes, i.e $\mathbb{D}(\mathcal{F}^\bullet)^i = (\mathcal{F}^{-i})^\vee$.*

Apply the sheafified version of the Verdier duality to $a : X \to pt, \mathcal{G}^\bullet = \mathbb{Q}$. We have elements of $D^b_c(pt)$, i.e complexes of vector spaces on both sides. Since $Ra_* = \Gamma, Ra_! = \Gamma_c$ we get

$$\Gamma_c(X, \mathcal{F}^\bullet)^\vee = \Gamma(X, \mathbb{D}_X(\mathcal{F}^\bullet)),$$

so taking the ith cohomology for $\mathcal{F} \in Sh(X) \subset D^b_c(X)$ we arrive at

$$H^i_c(X, \mathcal{F}^\bullet) =$$

4.4 Perverse sheaves

Definition 17. *A complex of sheaves $\mathcal{F}^\bullet \in D^b_c(X)$ is perverse sheaf if*

$$\dim\{H^{-i}(j_x^*\mathcal{F}^\bullet) \neq 0\} \leq 2i \text{ and } \dim\{H^i(j_x^!\mathcal{F}^\bullet) \neq 0\} \leq 2i$$

Proposition 5. *(BBD)*
The perverse sheaves form an Abelian subcategory of $D^b_c(X)$, denoted by $P(X)$. A sequence

$$0 \longrightarrow \mathcal{E}^\bullet \longrightarrow \mathcal{F}^\bullet \longrightarrow \mathcal{G}^\bullet \longrightarrow 0$$

of perverse sheaves is exact if and only if there is a map ϕ such that

$$\mathcal{E}^\bullet \longrightarrow \mathcal{F}^\bullet \longrightarrow \mathcal{G}^\bullet \longrightarrow \mathcal{E}^\bullet[1]$$

is a distinguished triangle in $D^b_c(X)$.

2. *The Verdier dual $\mathbb{D}_X$ preserves sheaves, and it restricts to an exact contravariant functor from $P(X)$ to itself.*

3. *$P(X)$ is Artinian, i.e every perverse sheaf $\mathcal{F}^\bullet$ has a finite lengh decomposition series*

$$0 \longrightarrow \mathcal{F}^\bullet_0 \longrightarrow \mathcal{F}^\bullet_1 \longrightarrow \cdots \qquad \longrightarrow \mathcal{F}^\bullet_n = \mathcal{F}^\bullet$$

for which the quotients $\mathcal{F}^\bullet_i/\mathcal{F}^\bullet_{i-1}$ are simple perverse sheaves. The simple perverse sheaves are (shifted) intersection cohomology sheaves.

5 Lecture 4

In this lecture we use the categorical language to describe the vanishing cycles, the monodromy and variation operators. We try to justify the efficiency of this language by showing two important applications. The nearby and vanishing cycles were introduced by Deligne [4].

5.1 Nearby cycles

Let X be a complex algebraic variety (not necessarily compact) and $f : (X, x) \to (\mathbb{C}, 0)$ be a holomorphic map. For $t \in \mathbb{C}$ we let $X_t = f^{-1}(X)$ be the fibre over $t \in \mathbb{C}$. Let $T \subset \mathbb{C}$ and $U_x \subset X$ be a small open ball around 0 resp x, such that the local Milnor fibers $V_{x,t} = X_t \cap U_x$ around $x \in X_0$ satisfy the conditions given in Lecture 2 for $t \in T$.

Notice that in this lecture we indicate the point $x \in X_0$ in the local Milnor fibers, as x is not fixed.

We will construct two functors:

$$\phi_f, \varphi_f : D_c^b X \to D_c^b(X_0).$$

Take the following cartesian diagram:

$$\begin{array}{ccccccc}
X_0 & \xrightarrow{\iota} & X & \xleftarrow{j} & X\backslash X_0 & \xleftarrow{\pi} & \widetilde{X\backslash X_0} \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle u.cover} \\
\{0\} & \longrightarrow & \mathbb{C} & \longleftarrow & \mathbb{C}\backslash\{0\} & \xleftarrow[\pi]{} & \widetilde{\mathbb{C}\backslash\{0\}}
\end{array}$$

where $\widetilde{\mathbb{C}\backslash\{0\}}$ is the universal cover of $\mathbb{C}\backslash\{0\}$, and $\widetilde{X\backslash X_0} = \{(x,t) \in (X\backslash X_0) \times \widetilde{\mathbb{C}\backslash\{0\}} | f(x) = \pi(t)\}$ is the pull-back.

Definition 18. *Definition/Proposition Let $\mathcal{F}^\bullet \in D^b(X)$ be a complex. We define the nearby cycles functor*

$$\psi_f : D_c^b(X) \to D_c^b(X_0)$$

to be $\psi_f(\mathcal{F}^\bullet) = i^* R(j \circ \pi)_*(j \circ \pi)^* \mathcal{F}^\bullet$.

Remark 5. *In spite of the fact that $j \circ \pi$ is not proper on $supp((j \circ \pi)^{-1}(\mathcal{F}^\bullet))$, one can show that $\psi_f(\mathcal{F}^\bullet) \in D_c^b(X_0)$.*

There is a deck transformation $h : \widetilde{X\backslash X_0} \to \widetilde{X\backslash X_0}$ coming from the action of the generator of $\mathbb{Z} = \pi_1(\mathbb{C}\backslash\{0\})$ on $\widetilde{\mathbb{C}\backslash\{0\}}$ via the pull-back, satisfying $\pi \circ h = \pi$. This homeomorphism induces an isomorphism of complexes $M : \psi_f(\mathcal{F}^\bullet) \to \psi_f(\mathcal{F}^\bullet)$. The following proposition explains the term 'nearby cycles'.

Proposition 6. *Let $f : (X, x) \to (\mathbb{C}, 0)$ has a critical point at x, and T, U_x are the Milnor-neighborhoods of 0 and x respectively, as given in the previous lecture, $V_{t,x} = X_t \cap U_x$ the local Milnor fibre.*

1. *For $\mathcal{F}^\bullet \in D_c^b(X)$ and $0 \neq t \in T$*

   $$\mathcal{H}^i(\psi_f \mathcal{F}^\bullet)_x \simeq \mathbb{H}^i(V_{x,t}, \mathcal{F}^\bullet)$$

   Applying this with $\mathcal{F}^\bullet = \mathbb{C}_X$ we get

   $$H^i(V_{x,0}, \psi_f \mathbb{C}_X) \simeq H^i(V_{x,t}).$$

2. *The monodromy morphism M_x on the l.h.s corresponds to the classical monodromy operator h_f (studied in Lecture 2) on the r.h.s.*

Remark 6. *By Poincaré duality $H^i(V_{x,t}) \simeq H^c_{2n-2-i}(V_{x,t}) \simeq H_{2n-2-i}(V_{x,t}, \partial V_{x,t})$, so the monodromy operator M corresponds to h_f^r.*

The action of the classical monodromy operator on the r.h.s for general $\mathcal{F}^\bullet$ can be described as follows. We can work with proper Milnor fibration $U_x \cap f^{-1}(U\backslash\{0\}) \to U\backslash\{0\}$. Then $R^k f_*(\mathcal{F}^\bullet)$ is a local system on $U\backslash\{0\}$, and this local system corresponds to a representation $\rho : \pi_1(U\backslash\{0\}) \to Aut(E)$ where

$$E = R^k f_*(\mathcal{F}^\bullet)_t = \mathbb{H}^k(\overline{V}_{x,t}, \mathcal{F}^\bullet) = \mathbb{H}^k(V_{x,t}, \mathcal{F}^\bullet).$$

The monodromy operator M is $\rho(\gamma)$ where γ is the generator $\pi_1(U\backslash\{0\})$.

5.2 Vanishing cycles

Consider the adjunction morphism

$$\mathcal{F}^\bullet \to R(j\circ\pi)_*(j\circ\pi)^{-1}(\mathcal{F}^\bullet)$$

and apply the functor i^* to get the *comparison map*

$$c : D^b_c(X_0) \to D^b_c(X_0) : \; i^{-1}\mathcal{F}^\bullet \mapsto \varphi_f\mathcal{F}^\bullet.$$

Definition 19. *Let $\mathcal{F}^\bullet \in D^b_c(X)$. Extend the comparison map c to the unique distinguished triangle in $D^b_c(X_0)$*

$$i^*\mathcal{F}^\bullet \xrightarrow{c} \psi_f\mathcal{F}^\bullet \xrightarrow{can} \varphi_f\mathcal{F}^\bullet \xrightarrow{[+1]}$$

and define $\varphi_f\mathcal{F}^\bullet \in D^b_c(X_0)$ to be the vanishing cycle *and the mapping cone φ_f : $D^b_c(X) \to D^b_c(X_0)$ the em vanishing cycle functor.*

We can define a monodromy operator $M_v : \varphi_f\mathcal{F}^\bullet \to \varphi_f\mathcal{F}^\bullet$ by extending the equality $M\circ c = c$ to the following automorphism of the distinguished triangle:

$$\begin{array}{ccccccc}
i^*\mathcal{F}^\bullet & \xrightarrow{c} & \psi_f\mathcal{F}^\bullet & \xrightarrow{can} & \varphi_f\mathcal{F}^\bullet & \xrightarrow{[+1]} & i^*\mathcal{F}^\bullet[+1] \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle M_v} & & \downarrow{\scriptstyle id} \\
i^*\mathcal{F}^\bullet & \xrightarrow{c} & \psi_f\mathcal{F}^\bullet & \xrightarrow{can} & \varphi_f\mathcal{F}^\bullet & \xrightarrow{[+1]} & i^*\mathcal{F}^\bullet[+1]
\end{array}$$

This diagram does not define M_v uniquely, but with more care it can be fixed. What is the geometric meaning of the vanishing cycle?

Let $\mathcal{F}^\bullet = \mathbb{C}_X$ be the constant sheaf, applying the global section functor to the long exact sequence of sheaves arising from the distinguished triangle is

$$\longrightarrow H^i(V_{x,0}) \longrightarrow H^i(V_{x,t}) \longrightarrow H^i(V_{x,0},\varphi_f\mathbb{C}_X) \longrightarrow H^{i+1}(V_{x,0}) \longrightarrow$$

The dual sequence is

$$\longrightarrow H_{i+1}(V_{x,0}) \longrightarrow H^i(V_{x,0},\varphi_f\mathbb{C}_X)^\vee \xrightarrow{spec} H_i(V_{x,t}) \longrightarrow H_i(V_{x,0}) \longrightarrow$$

The map here is the *specialisation map*, which typycally involves collapsing certain cycles in the fibre $V_{x,t}$, as we have seen before. These vanishing cycles are given by the image of $H^i(V_{x,0},\varphi_f\mathbb{C}_X)^\vee$ in $H_i(V_{x,t})$.

If we take the stalk at x of the same long exact sequence of cohomology sheaves (and using the Proposition) we get

$$\longrightarrow \mathbb{H}^i(V_{x,0},\mathcal{F}^\bullet) \longrightarrow \mathbb{H}^i(V_{x,t},\varphi_f\mathcal{F}^\bullet) \longrightarrow \mathcal{H}^i(\varphi_f\mathcal{F}^\bullet)_x \longrightarrow \mathbb{H}^{i+1}(V_{x,0},\mathcal{F}^\bullet) \longrightarrow$$

If $X$is smooth then $V_{x,0} = X_0 \cap U_x$ is contractible (we don‘t prove this here) the first term here is 0 unless $i = 0$, and we get the analog of Proposition 6

Proposition 7. *Let $f : (X, x) \to (\mathbb{C}, 0)$ has a critical point at x, and U, U_x are the Milnor-neighborhoods of* 0 *and x respectively, as given in the previous lecture.*

1. *For $\mathcal{F}^\bullet \in D^b_c(X)$ and $0 \neq t \in T$*

$$\mathcal{H}^i(\varphi_f \mathcal{F}^\bullet)_x \simeq \mathbb{H}^{k+1}(B^0_\delta(x), B^0_\delta(x) \cap X_t; \mathcal{F}^\bullet)$$

   Applying this with $\mathcal{F}^\bullet = \mathbb{C}_X$ and assuming that X is smooth

$$H^i(V_{x,0}, \varphi_f \mathbb{C}_X) \simeq \tilde{H}^i(V_{x,t}).$$

2. *The monodromy morphism M_v on the l.h.s corresponds to the classical relative monodromy operator h^r_f (studied in Lecture 2) on the reduced homology groups.(see the remark after Proposition 6).*

5.3 The variation morphism

The analog of the variation map of Lecture 2 is a natural transformation

$$var : \varphi_f \mathcal{F}^\bullet \to \psi_f \mathcal{F}^\bullet$$

called the *variation morphism* obtained heuristically by completing the diagram

$$\begin{array}{ccc} i^*\mathcal{F}^\bullet & \longrightarrow & \psi_f \mathcal{F}^\bullet \\ \downarrow & & \downarrow {\scriptstyle M-Id} \\ 0 & \longrightarrow & \psi_f \mathcal{F}^\bullet \end{array}$$

to a morphism of distinguished triangles

$$\begin{array}{ccccccc} i^*\mathcal{F}^\bullet & \longrightarrow & \psi_f \mathcal{F}^\bullet & \xrightarrow{can} & \varphi_f \mathcal{F}^\bullet & \xrightarrow{[+1]} & \\ \downarrow & & \downarrow {\scriptstyle M-Id} & & \downarrow {\scriptstyle var} & & \\ 0 & \longrightarrow & \psi_f \mathcal{F}^\bullet & \xrightarrow{id} & \psi_f \mathcal{F}^\bullet & \xrightarrow{[+1]} & \end{array}$$

A precise definition can be found in [7]. The variation operator is by definition coincides with the classical variation map.

Example 13. *Take again the Morse singularity $X = \mathbb{C}^2$, $f(z, w) = z^2 + w^2$, with $\mu = 1$. We have already seen the local Milnor fibres at $x = (0, 0)$ and in general, the nearby and vanishing cycles at a point $(z, w) \in X_0 = \{z^2 + w^2 = 0\}$ are*

$$\mathcal{H}^i(\psi_f \mathbb{C}_X)_x = \begin{cases} \mathbb{C} \text{ if } i = 0 \\ \mathbb{C} \text{ if } i = 1, x = (0,0) \\ 0 \text{ otherwise} \end{cases}, \quad \mathcal{H}^i(\varphi_f \mathbb{C}_X)_x = \begin{cases} \mathbb{C} \text{ if } i = 1,\ x = (0,0) \\ 0 \text{ otherwise} \end{cases}$$

This $\mathbb{C}$ for $i =$ corresponds to the waist of the Milnor fibre which contracts to a point in the singular fibre.

5.4 Applications of nearby and vanishing cycles

The following theorem is a special case of the general principle that perverse sheaves are preserved under specialisation.

Theorem 5.1. *(Goresky-MacPherson, [9], Kashiwara-Shapira [7]) The right shifts* $\psi_f[-1]$ *and* $\varphi_f[-1]$ *of the nearby and vanishing cycle functors preserve perverse sheaves, i.e they induce functors*

$$\psi_f[-1], \varphi_f[-1] : \mathrm{Perv}(X) \to \mathrm{Perv}(X_0).$$

Lemma 1. *For any complex* $\mathcal{F}^\bullet \in D^b_c(X)$ *we have*

$$\mathrm{supp}\mathcal{H}^i(\varphi_f \mathcal{F}^\bullet) \subset X_0 \cap \mathrm{Sing}(f),$$

where

$$\mathrm{Sing}(f) = \{x \in \mathbb{C}^{n+1} : df(x) = 0\}$$

is the singular locus.

As an illustration for the efficiency of perverse sheaves we prove the following theorem.

Theorem 5.2. *(Connectivity of Milnor fibers, D.B Massey Appendix B in [8],A. Dimca §6.1 in [5])*

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ *be a non-constant analytic function germ with an isolated singularity at* 0 *and let* $s = \dim \mathrm{Sing}(f)$*. Let* $X_t = f^{-1}(t)$ *denote the Milnor fiber at* 0*, then*

$$\tilde{H}^i(X_t, \mathbb{C}) = 0$$

for $i \notin [n - s, n]$.

Proof. Since $X = \mathbb{C}^{n+1}$ is a smooth manifold, $\mathbb{C}_X[n+1] \in \mathrm{Perv}(X)$ and therefore $\varphi_f[-1](\mathbb{C}_X[n+1]) \in \mathrm{Perv}(X_0)$. By Lemma 1

$$\mathrm{supp}\mathcal{H}^i(\varphi_f[-1]\mathbb{C}_X[n+1]) \subset X_0 \cap \mathrm{Sing}(f),$$

and therefore

$$\varphi_f[-1]\mathbb{C}_X[n+1]|_{\mathrm{Sing}(f)} \in \mathrm{Perv}(\mathrm{Sing}(f)).$$

Recall that any $\mathcal{F}^\bullet \in \mathrm{Perv}(X)$ can be represented with a complex satisfying

$$\mathcal{F}^m = 0 \text{ for } m < -\dim(X), m > 0$$

Therefore

$$\mathcal{H}^i(\varphi_f[-1]\mathbb{C}_X[n+1])_0 = \mathcal{H}^i((\varphi_f[-1]\mathbb{C}_X[n+1])|_{\mathrm{Sing}(f)})_0 = 0$$

for $i < -s$ and $i > 0$. This gives us

$$\mathcal{H}^i(\varphi_f[-1]\mathbb{C}_X[n+1])_0 = \mathcal{H}^{n+i}(\varphi_f \mathbb{C}_X)_0 = \tilde{H}^{i+n}(X_t, A) = 0$$

for $i < -s$ and $i > 0$. □

6 Lecture 5, Mixed Hodge Structure on the vanishing cohomology fibre

The goal of this lecture is to describe the Mixed Hodge Structure on the cohomology of the Milnor fibre following Varchenko and Scherk-Steenbrink.

6.1 Gauss-Manin connection and the cohomology bundle

- Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of holomorphic map with isolated singularity at $0 \in \mathbb{C}^n$. Let $X_t = f^{-1}(t) \cap B_\varepsilon$ denote the local Milnor fibre.

- The sheaf (of vector spaces) $\mathcal{H}^i_f = \mathcal{H}^i(f_*(\mathbb{C}_{X'}) = \mathcal{R}^i f_*(\mathbb{C}_{X'})$ is the ith vanishing cohomology bundle associated to the singularity. Its stalk at $t \in \mathbb{C}$ is $H^i(X_t, \mathbb{C})$. The sheaf of sections

$$\mathcal{H}^i_\pi \otimes_{\mathbb{C}_{T'}} O_{T'}$$

  is a sheaf of $O_{T'}$-modules. We use the same notation for the $\mathbb{C}_{T'}$-module (vector bundle) and $O_{T'}$-module (sheaf).

- $\mathcal{H}^i_f$ is a local system on T'. The corresponding canonical flat connection

$$\nabla : \mathcal{H} \to \Omega_{S'} \otimes_{O_{S'}} \mathcal{H}$$

  is called the Gauss-Manin connection.

- The cohomology of the Milnor fibres can be computed using De Rham complexes. By Poincare lemma

$$\cdots \qquad 0 \longrightarrow \mathbb{C}_X \longrightarrow 0 \ldots \simeq \cdots \longrightarrow 0 \longrightarrow \mathbb{C}_X \longrightarrow O_X \longrightarrow \Omega^1_X \longrightarrow \cdots,$$

  and therefore $\mathbb{H}^i(\Omega^\bullet_X) = H^i(X, \mathbb{C})$. The relative De Rham complex $\Omega^\bullet_{X/T} = \Omega^\bullet_X/(f^*\Omega^1_T \wedge \Omega^{\bullet-1}_X$ is a resolution of $f^{-1}(O_S)$. We define the relative De Rham cohomology as

$$\mathcal{H}^i_{DR}(X/T) = \mathcal{H}^i(f_*\Omega^\bullet_{X/T})$$

  .

  Lemma 2. *We have an exact sequence of sheaves on T:*

$$0 \longrightarrow \mathcal{R}^i f_*\mathbb{C}_X \otimes_{\mathbb{C}_T} O_T \longrightarrow \mathcal{H}^i_{DR}(X/T) \longrightarrow f_*\mathcal{H}^i(\Omega^\bullet_{X/T}) \longrightarrow 0,$$

  where the right hand sheaf has support $0 \in \mathbb{C}$. *Therefore* $\mathcal{H}^i_f = \mathcal{H}^i_{DR}(X/T)|_{T'}$.

6.2 The period map

- Let $\mathcal{H}_\bullet = \mathrm{Hom}_{O_{T'}}(\mathcal{H}_f^\bullet, O_{T'})$ be the vanishing homology sheaf. This is a local system on S' with stalk $\mathcal{H}_\bullet|_t = H_\bullet(X_t, \mathbb{C})$. Then we have a pairing using integration:

$$\mathcal{H}^\bullet = \mathcal{H}_{DR}^\bullet(X'/T') \times \mathcal{H}_\bullet \to O_{S'} : (\omega, \sigma) \mapsto \int_\sigma \omega$$

  The point is that the result is a holomorphic section of $O_{S'}$.

- Let $\omega \in \Omega^{n-1}(B)$ be a holomorphic $n-1$-form on the neighborhood of $0 \in \mathbb{C}^n \times \mathbb{C}$. Then $\omega|_{X_t}$ is a closed form, since $\dim(X_t) = n-1$. So ω defines a cohomology class $[\omega] \in H^{n-1}(V_t)$. The global section $[\omega] \in Gamma(\mathcal{H}_f^{n-1})$ is a *period* of the form ω.

- **The residue form** If ω is a holomorphic n-form on $\mathbb{C}^n$ then since

$$0 = H^n(X_t, \mathbb{C}) = \mathcal{H}_f^n|_t = \mathcal{H}_D R^n(X/T)|_t = H^n(\Omega^\bullet_{X/T}|_t),$$

  $\omega = df \wedge \psi$ in a small neighbourhood U of t, where $\psi \in \Gamma(U, \mathcal{H}_f^{n-1})$ is uniquely determined. $\psi|_{X_t}$ is the *residue form* of ω, denoted by ω/df_t, its defines a class $[\omega/df]_t \in H^{n-1}(X_t, \mathbb{C})$ and a section $[\omega/df] \in \Gamma(T', \mathcal{H}_f^{n-1})$. Global sections of this form are called *geometric sections*. Theorem: Global sections are holomorphic sections. The period map is a special case: for an $n-1$-form ω, $[\omega] = [\omega \wedge df/df]$.

Example 14. *If $f = z_1^2 + \ldots + z_n^2$ is a Morse singularity, $V_t \sim \{z : z_1^2 + \ldots + z_n^2 = t; Imz_i = 0\} = \sqrt{t}S^{n-1}$, and therefore*

$$[\omega/df]|S_t = \int_{S_t} \omega_t/dt = \frac{d}{dt}\int_{B_t} \omega = \frac{d}{dt}(ct^{n/2} + \ldots) = ct^{n/2-1} + \ldots$$

.

Note that $\mathcal{H}_f^{n-1}$ and $\mathcal{H}_{DR}^{n-1}(X/T)$ are coherent, rank μ sheaves of $O_{T'}$ (resp O_T) -modules.

Theorem 6.1. *For generic n-forms $\omega_1, \ldots, \omega_\mu \in \Omega^n(\mathbb{C}^n)$ the geometric sections $[\omega_1/df], \ldots, [\omega_\mu/df]$ give a trivialization of $\mathcal{H}_f^{n-1}$.*

- The Gauss-Manin connection on $\mathcal{H}_f^\bullet$ is specified by the covariant derivation ∇_t along the vector field $\frac{\partial}{\partial t}$ on T'.

Theorem 6.2. *1. Let $\omega \in \Omega^p_{X/T}$ be a closed form representing the class $[\omega] \in \mathcal{H}^p_{DR}(X/T)$. Then $d\omega = 0$, so $d\omega = df \wedge \eta$, and*

$$\nabla_t[\omega] = \eta = [d\omega/df]$$

2. *∇_t is a singular differential operator and its monodromy can be canonically identified with the classical monodromy h.*

Example 15. *Let $f(z)$ be a quasihomogeneous function of degree one with weight vector $(v_1, \dots, v_n$. We can give an explicit description of the trivialization of $\mathcal{H}_f^{n-1}$ as follows. Let $\omega_{\mathbf{k}} = \mathbf{z}^{\mathbf{k}} dz/df$ be a quaihomogeneous differential form of degree $l_{\mathbf{k}} = (\mathbf{k}+1, \mathbf{v}) - 1$. Then $\nabla_t[\omega_{\mathbf{k}}] = l_{\mathbf{k}}[\omega_{\mathbf{k}}]/t$. That is, $[\omega_{\mathbf{k}}]$ is an eigenvector, and the corresponding eigenvalue of the monodreomy operator is $e^{2\pi i l_{\mathbf{k}}}$.*

If $\mathbf{z}^{\mathbf{k}_1}, \dots, \mathbf{z}^{\mathbf{k}_\mu}$ is a basis of the local algebra Q_f, then the corresponding forms $\omega_{\mathbf{k}_1}, \dots, \omega_{\mathbf{k}_\mu}$ give a trivialization of $\mathcal{H}_f^{n-1}$, and in this basis the monodromy operator is diagonal with eigenvalues $e^{2\pi i l_s}$.

- Let $\omega = (\omega_1/df, \dots, \omega_\mu/df)$ be a trivialization of $\mathcal{H}_f^{n-1} \to T'$. Then

$$\nabla_t[\omega_i/df] = \sum p_{i,j}(t)[\omega_j/df], t \in T' \tag{1}$$

  where $p_{i,j}(t)$ is holomorphic on T', but meromorphic on T with regular singularity at 0. Introduce the matrix $P(t) = [p_{i,j}(t)]$ and the vector $I(t) = (\int_{\delta(t)} \omega_1/df, \dots, \int_{\delta(t)} \omega_\mu/df)$, where $\delta(t)$ is a covariantly constant section of $\mathcal{H}_{n-1,f}$. Then the system (1) can be rewritten as

$$dI/dt = PI,$$

  and is called the *Picard-Fuchs equation* of the trivialization ω. If $(s_1, \dots, s_\mu)$ is the dual frame of ω in $\mathcal{H}_{n-1,f}$, then $Is = \sum I_i s_i$ is a covariantly constant section of $\mathcal{H}_{n-1,f}$.

- The fundamental matrix of solutions has the form $\Phi = Qe^R$, where R is a constant matrix and Q is holomorphic on T', meromorphic with regular singularity at 0. (in other words, the Picard-Fuchs equation has a regular singularity) The change of variables $I = QI'$ takes $dI/dt = PI$ into

$$\frac{dI'}{dt} = t^{-1}RI'$$

  with a simple pole at $t = 0$. The fundamental matrix of this system is

$$\Phi = e^{R \ln t}$$

  Consequently, its solutions (coordinates of I') admit series expansion of the form

$$\int_{\delta(t)} \omega = \sum a_{k,\alpha} t^\alpha (\ln t)^k,$$

  where α are the eigenvalues of R. More precisely:

Theorem 6.3. *Let ω be a holomprphic n-form. Then*

1. *In each sector $a \leq arg(t) \leq b$ there is a series expansion $\int_{\delta(t)}[\omega/df] = \sum a_{k,l}t^{\alpha}(\ln t)^k$*
2. *α id rational and $\alpha > -1$, and $e^{2\pi i\alpha}$ is an eigenvalue of the monodromy operator h_*.*
3. *$a_{k,l} = 0$ whenever h_* has no Jordan blocks corresponding to eigenvalue $e^{2\pi i\alpha}$ of size $\geq k+1$.*

Corollary 3. *The operator $h_*^N - E$ is nilpotent for some N.*

- $a_{k,\alpha}$ depends linearly on δ, and therefore it defines a covariantly constant (multi valued) section $A^{\omega}_{k,\alpha} \in \Gamma(T', \mathcal{H}^*_f)$ such that $\int_\delta A^{\omega}_{k,\alpha} = a_{k,\alpha}(\omega, \delta)$, and therefore

$$[\omega/df] = \sum A^{\omega}_{k,\alpha} t^{\alpha}(\ln t)^k \tag{2}$$

6.3 The Mixed Hodge Structure on $\mathcal{H}_f$

Let $H_{\mathbb{Z}} \subset H_{\mathbb{R}}$ be a f.d. real vector space with an integer lattice, and $H = H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification.

Definition 20. *A pure Hodge structure of weight k on H is a decomposition of H into direct sum of subspaces $H = \oplus_{p+q=k} H^{p,q}$ with the property $H^{p,q} = \overline{H^{q,p}}$.*

Example 16. *If X is a complex Kahler manifold then $H^k(X, \mathbb{C})$ has a pure HS of weight k defined by $H^{p,q}(X) = H^q(X, \Omega^p)$, which is the subspace of closed differential forms of type (p, q).*

A pure HS of weight k defines a decreasing filtration

$$\{0\} = F^{k+1} \subset \ldots \subset F^0 = H$$

where $F^p = \oplus_{i \geq p} H^{i,k-i}$. By definition

$$F^p \cap \overline{F^q} = H^{p,q}, F^p \oplus \overline{F^{q+1}} = H.$$

Definition 21.

1. *A decreasing filtration F^p on H with the property that $F^p \oplus \overline{F^{k+1-p}} = H$ is called the a Hodge filtration.*
2. *A complex linear map $\phi : H \to H'$ which sends the lattice $H_{\mathbb{Z}}$ to $H'_{\mathbb{Z}}$ is called a morphism of type (r, r) of pure HS's if $\phi(H^{p,q}) \subset H'^{p+r,q+r}$ for all p, q.*

Definition 22. *A Mixed Hodge Structure on H is a pair of filtrations*

1. *the weight filtratin W_k, which is an increasing filtration on H with subspaces in $H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$.*
2. *the Hodge filtration F^p, which is a decreasing filtration on H*

such that F defines a pure Hodge structure of weight k on W_k/W_{k-1}, more precisely

$$(F^p \cap W_k) + (F^{q+1} \cap W_k) + W_{k-1} = W_k \text{ and } F^p \cap \overline{F^{q+1}} + W_{k-1} = W_{k-1}.$$

The map $\phi : H \to H'$ is called a morphism of type (r, r) of MHS's if

$$\phi(W_k) \subset W'_{k+2r}, \ \ \phi(F^p) \subset F'^{p+r}$$

In other words, if

$$H^{(k)} = gr_k W = W_k/W_{k-1} \text{ and } F^p_{(k)} = (F^p \cap W_k)/W_{k-1}$$

then $H^{p,q} = F^p_{(k)} \cap \overline{F^q_{(k)}}$ defines a pure HS on $H^{(k)}$:

$$H^{(k)} = oplus_{p+q=k} H^{p,q}.$$

Example 17. *MHS on $H^*(X\backslash U, \mathbb{C})$ where $U \subset X$ is nonsingular divisor in the projective variety X.*

Let $\alpha(\omega) = \min\{\alpha : \exists i, A^{\omega}_{i,\alpha} \neq 0\}$ be the *order* of the geometric section $[\omega/df]$, and

$$[\omega/df]_max = \sum_k A^{\omega}_{k,\alpha(\omega)} t^{\alpha(\omega)} (\ln t)^k$$

the *proncipal part* of $[\omega/df]$.

Theorem 6.4. *Asymptotic Hodge filtration on $\mathcal{H}^*_f$. Let*

$$F^p_t = \mathbb{C}\langle [\omega/df]_{max}(t) : \alpha(\omega) \leq n - p - 1 \rangle \subset H^{n-1}(X_t, \mathbb{C}).$$

Then

1. *F^p_t is the fibre of an analytic subbundle F^p of the $\mathcal{H}^*_f$.*
2. *$F^p_t \subset H^{n-1}(X_t, \mathbb{C})$ is invariant under the semisimple part of the monodromy operator.*
3. *$F^n = \{0\}$, $F^0 = \mathcal{H}^{n-1}_f$.*

The weight filtration

The weight filtration in $\mathcal{H}^{n-1}_f$ is defined using the monodromy operator h_*. **The weight filtration of a nilpotent operator**

Theorem 6.5. *Schmid Let N be a nilpotent operator acting on the finite dimensional vector space H. There exists unique increasing filtration*

$$0 \subset \ldots \subset W_k \subset W_{k+1} \subset \ldots \subset H$$

for which

1. $N(W_k) \subset W_{k-2}$*;*

2. $N^k : W_{s+k}/W_{s+k-1} \to W_{s-k}/W_{s-k-1}$ *is an isomorphism for any integer* k*.*

This is called the weight filtration of index s *corresponding to* N*.*

Let $M = h_*$ be the monodromy operator, and

$$M = M_u M_s$$

be the decomposition into a unipotent and semisimple matrix. (That is, the eigenvalues of M_u are equal to 1 and M_s has an eigenbasis.) Let $N = \log(M_u - E) = \sum_i (M_u - E)^i / i$, and let

$$H_\lambda^{n-1}(V_t, \mathbb{C}) = \cup W_{k,t}^\lambda$$

be the weight filtration corresponding to N and index

$$s = \begin{cases} n \text{if } \lambda = 1 \\ n - 1 \text{if } \lambda \neq 1 \end{cases} .$$

Define the weigfht filtration as

$$W_{k,t} = \oplus_\lambda W_{k,t}^\lambda.$$

Theorem 6.6. *1.* W_k *is an analytic subbundle of* $\mathcal{H}_f^{n-1}$*.*

2. W_k *is invariant under the Gauss-Manin connection*

3. W_k *is invariant under the action of* M_s*.*

4. (Steenbrink) $W_[2n - 2] = \mathcal{H}_f^{n-1}$*.*

Theorem 6.7. *The defined weight and Hodge filtrations define a Mixed Hodge Structure on* $\mathcal{H}_f^{n-1}$*. The construction os functorial: let* $g : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ *be a map of finite multiplicity at the origin, and assume that* $f \circ g$ *has an isolated singularity at* 0*. Then* $g^* : \mathcal{H}_f \to \mathcal{H}_{f \circ g}$ *is a* $(0, 0)$*-morphism of MHS's.*

Corollary 4. *Steenbrink The dimension of any Jordan block of* M *is len, and the dimension of the block with eigenvalue* 1 *is* $\leq n - 1$*.*

Example 18. *For* $f = z_1^2 + \ldots + z_n^2$ *the bundle* $\mathcal{H}_f^{n-1} \to T'$ *is one dimensional, and*

$$0 = F^{[n/2]+1} \subset F^{[n/2]} = \mathcal{H}_f^{n-1}$$

where $F^{[n/2]}$ *is the principal part of a generic n-form* ω*.*

$$0 = W_{2[n/2]-1} \subset W_{2[n/2]} = \mathcal{H}_f^{n-1}$$

and so

$$H^{n-1}(X_t, \mathbb{C}) = H_t^{[n/2],[n/2]} = F_t^{[n/2]} \cap \overline{F_t^{[n/2]}}.$$

7 Lecture 6: Singularities of maps and Thom polynomials

So far we have studies singularities of functions $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, and local behaviour of the fibers close to the critical point. A holomorphic map between complex manifolds $f : M^n \to K^k$ locally at $p \in M$ is given by a map-jet $(\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$.

7.1 The setup

Let $(e_1, \ldots, e_n)$ be the basis of $\mathbb{C}^n$, and denote the corresponding coordinates by $(x_1, \ldots, x_n)$. Introduce the notation $\mathcal{J}(n) = \{h \in \mathbb{C}[[x_1 \ldots x_n]];\ h(0) = 0\}$ for the algebra of power series without a constant term, and let $\mathcal{J}_d(n)$ be the space of d-jets of holomorphic functions on $\mathbb{C}^n$ near the origin, i.e. the quotient of $\mathcal{J}(n)$ by the ideal of those power series whose lowest order term is of degree at least $d + 1$. As a linear space, $\mathcal{J}_d(n)$ may be identified with polynomials on $\mathbb{C}^n$ of degree at most d without a constant term.

Our basic object is $\mathcal{J}_d(n, k)$, the space of d-jets of holomorphic maps $(\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$. This is a finite-dimensional complex vector space, which one can identify $\mathcal{J}_d(n) \otimes \mathbb{C}^k$; hence $\dim \mathcal{J}_d(n, k) = k\binom{n+d}{d} - k$. We will call the elements of $\mathcal{J}_d(k, n)$ *map-jets of order d*, or simply map-jets.

One can compose map-jets via substitution and elimination of terms of degree greater than d; this leads to the composition maps

$$\mathcal{J}_d(n, k) \times \mathcal{J}_d(m, n) \to \mathcal{J}_d(m, k), \quad (\Psi_2, \Psi_1) \mapsto \Psi_2 \circ \Psi_1. \tag{3}$$

When $d = 1$, $\mathcal{J}_1(m, n)$ may be identified with n-by-m matrices, and (3) reduces to multiplication of matrices. By taking the linear parts of jets, we obtain a map

$$\mathrm{Lin} : \mathcal{J}_d(n, k) \to \hom(\mathbb{C}^n, \mathbb{C}^k),$$

which is compatible with the compositions (3) and matrix multiplication.

Consider now the set

$$\mathrm{Diff}_d(n) = \{\Delta \in \mathcal{J}_d(n, n);\ \mathrm{Lin}(\Delta) \text{ invertible}\}.$$

The composition map (3) endows this set with the structure of an algebraic group, which has a faithful representation on $\mathcal{J}_d(n)$. Using the compositions (3) again, we obtain the so-called *left-right* action of the group $\mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$ on $\mathcal{J}_d(k, n)$:

$$[(\Delta_L, \Delta_R), \Psi] \mapsto \Delta_L \circ \Psi \circ \Delta_R^{-1}$$

Note that the action of $\mathrm{Diff}_d(n)$ is linear, while the action of $\mathrm{Diff}_d(k)$ is not. *Singularity theory*, in the sense that we are considering here, studies the left-right-invariant algebraic subsets of $\mathcal{J}_d(n, k)$.

A natural way to form such subsets is as follows. Observe that to each element $\Psi = (P_1, \ldots, P_k) \in \mathcal{J}_d(n, k)$, where $P_i \in \mathcal{J}_d(n)$ for $i = 1 \ldots k$, we can associate the quotient algebra $A_\Psi = \mathcal{J}_d(n)/I\langle P_1, \ldots, P_k\rangle$: the algebra $\mathcal{J}_d(n)$ modulo the ideal generated by the elements of the sequence. Since $\mathcal{J}_d(n)^{d+1} = 0$, we also have A_Ψ^{d+1}=0. We will call A_Ψ the *nilpotent algebra*[1] of the map-jet Ψ. For $\Psi = 0$ this nilpotent algebra is $\mathcal{J}_d(n)$, while for a generic Ψ (in fact, as soon as rank[Lin(Ψ)] = n) we have $A_\Psi = 0$.

Now let A be a nilpotent algebra, as defined above. Consider the subset

$$\Theta_A^{n\to k} = \{(P_1, \ldots, P_k) \in \mathcal{J}_d(n,k);\ \mathcal{J}_d(n)/I\langle P_1, \ldots, P_n\rangle \cong A\} \tag{4}$$

of the map-jets of order d. Again, the dependence on the parameters d, n and k will be usually omitted.

It is easy to show that Θ_A is $\mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$-invariant. A key observation is that although two map-jets with the same nilpotent algebra may be in different $\mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$-orbits, there is a group acting on $\mathcal{J}_d(n,k)$ whose orbits are exactly the sets $\Theta_A^{n\to k}$ for various nilpotent algebras A. This group is defined as the semidirect product

$$\mathcal{K}_d(n,k) = GL_k(\mathbb{C} \oplus \mathcal{J}_d(n)) \rtimes \mathrm{Diff}_d(n), \tag{5}$$

using the natural action of $\mathrm{Diff}_d(n)$ on $\mathcal{J}_d(n)$; the algebra $\mathbb{C} \oplus \mathcal{J}_d(n)$ is the augmentation of $\mathcal{J}_d(n)$ by constants. The vector space $\mathcal{J}_d(n)$ is naturally a module over $\mathbb{C} \oplus \mathcal{J}_d(n)$, and hence $\mathcal{K}_d(n,k)$ acts on $\mathcal{J}_d(n,k)$ via

$$[(M, \Delta), \Psi] \mapsto (M \cdot \Psi) \circ \Delta^{-1}, \tag{6}$$

where "·" stands for matrix multiplication.

Proposition 8. *Two map-jets in $\mathcal{J}_d(k,n)$ have the same nilpotent algebra if and only if they are in the same $\mathcal{K}_d$-orbit.*

Remark 7. *Two jets in the same $\mathcal{K}_d$-orbit are called* contact equivalent*, or $\mathcal{K}$-equivalent (cf. [1]). The term V-equivalence is also used (e.g. [**?**]). The varieties Θ_A are called* contact singularity classes *or simply* contact singularities.

Using the fact that $\mathcal{K}_d$ is connected, it is not difficult to derive the following properties of Θ_A.

Proposition 9. *Let A be a nilpotent algebra such that $A^{d+1} = 0$ and $n \geq \dim(A/A^2)$. Then for k sufficiently large, $\Theta_A^{n\to k}$ is a nonempty,* $\mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$*-invariant, irreducible quasiprojective algebraic variety of codimension* $(k - n + 1)\dim(A)$ *in $\mathcal{J}_d(n,k)$.*

Note that the codimension of Θ_A depends only on the difference $k - n$ and does not depend on d.

[1]Instead of this algebra, it is customary to use the so-called *local algebra* of Ψ, which is simply the augmentation of A_Ψ by the constants.

In the present paper, we will study certain rough topological invariants of contact singularities; these invariants depend only on the closure of the singularity locus in $\mathcal{J}_d(n,k)$. As it turns out, in an asymptotic sense, the closures of contact orbits are also closures of left-right orbits, hence, from our point of view, these two types of singularity classes are closely related.

While we will not need this statement, we describe it in some details for reference. Roughly, we claim that for fixed A and r, and sufficiently large n, there is a dense left-right orbit in $\Theta_A^{n\to n+r}$.

Let r be a nonnegative integer. An *unfolding* of a map-jet $\Psi \in \mathcal{J}_d(n,k)$ is a map-jet $\widehat{\Psi} \in \mathcal{J}_d(k+r, n+r)$ of the form

$$(x_1,\ldots,x_n,y_1,\ldots,y_r) \mapsto (F(x_1,\ldots,x_n,y_1,\ldots,y_r),y_1,\ldots,y_r)$$

where $F \in \mathcal{J}_d(n+r,k)$ satisfies

$$F(x_1\ldots,x_n,0,\ldots,0) = \Psi(x_1,\ldots,x_n).$$

The *trivial unfolding* is the map-jet

$$(x_1,\ldots,x_n,y_1,\ldots,y_r) \to (\Psi(x_1,\ldots,x_n),y_1,\ldots,y_r).$$

Definition 23. *A map-jet $\Psi \in \mathcal{J}_d(n,k)$ is* stable *if all unfoldings of Ψ are left-right equivalent to the trivial unfolding.*

Informally, a germ of a holomorphic map $f : N \to K$ of complex manifolds at a point $x \in N$ is stable if for any small deformation $\tilde{f}$ of f, there is a point in the vicinity of x at which the germ of $\tilde{f}$ is left-right equivalent to the germ of f at x.

Now we can formulate the relationship between contact and left-right orbits precisely.

Proposition 10.
1. *If $\widehat{\Psi}$ is an unfolding of Ψ, then $A_{\widehat{\Psi}} \cong A_\Psi$.*
2. *Every map germ has a stable unfolding.*
3. *If a map germ is stable, then its left-right orbit is dense in its contact orbit.*

7.2 Thom polynomials

Consider a holomorphic map $f : N^n \to K^k$ between two complex manifolds. For a singularity $O \subset \mathcal{J}_d(n,k)$ (i.e O is a left-right orbit) and holomorphic $f : N \to K$, we can define the set

$$Z_O[f] = \{p \in N;\ \tilde{f}_p \in O\},$$

which is independent of any coordinate choices. Under some additional technical assumptions, for N compact, appropriate closed O, and f sufficiently generic, $Z_O[f]$ is an analytic subvariety of N. The computation of the Poincaré dual class $\alpha_O[f] \in H^*(N,\mathbb{Z}$ of this set is one of the fundamental problems of global singularity theory.

Theorem 7.1 (Thom's principle). *There exists a polynomial* $Th_{O,n,k}(\mathbf{x}, \mathbf{y}) \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_k]$ *such that for generic* $f : M \to K$

$$[Z_O[f]] = Th_{O,n,k}(TM, f^*TK) \in H^*(M, \mathbb{Z}),$$

where on the l.h.s. we substitute the Chern roots of the corresponding bundles.

The Thom polynomial is given as folloes. Recall that

$$\mathbb{C}[x_1, \dots, x_n, y_1 \dots, y_k] = H_{\mathbb{C}^n \times \mathbb{C}^k}(pt) = H_{\mathbb{C}^n \times \mathbb{C}^k}(\mathcal{J}_d(n, k)).$$

Moreover, if V is a complex vector space with a group action and $\Sigma \subset V$ is a G-invariant (analytic) subvariety, then the equivariant Poincaré dual is defined as

$$ePD(Sigma, V) = PD[EG \times_G \Sigma, EG \times_G V] \in H_G(V) = H_G(pt)$$

is a polynomial when G is a torus and a symmetric polynomial when $G = GL(n)$. Using this identification we arrive at

Theorem 7.2 (Thom,Damon). *The polynomial is given as the equvivariant Poincarè dual*

$$Th_{O,n,k} = ePD(O, \mathcal{J}_d(n, k)).$$

7.3 An example

Let

$$O = \Theta_1 = \{f \in J_1(n, k) : Q_{\nabla(f)} \simeq \mathbb{C}[t]/t^2\} = \{A \in \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n) \dim\ker A = 1\} =$$
$$= \{A \in \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n) \exists! v \in \mathbb{C}^n,\ v \neq 0 :\ Av = 0\}$$

These maps may be identified with k-by-n matrices, and the weight of the action on the entry e_{ji} is equal to $\theta_j - \lambda_i$ under the left-right action of $GL(n) \times GL(k)$.

We have a natural equivariant fibration $\pi : \overline{\Theta}_1 \to \mathbb{P}^{n-1}$. Fiber over a point $[v] \in \mathbb{P}^{n-1}$ is the linear subspace $\{A;\ Av = 0\} \subset \Theta_1$.

Fixed-point data: n fixed points on $\mathbb{P}^{n-1}$: $p_1 \dots p_n$, corresponding to the coordinate axes. The weights of $T_{p_i}\mathbb{P}^{n-1}$ are $\{\lambda_s - \lambda_i;\ s \neq i\}$. The fiber at p_i is the set of matrices A with all entries in the ith column vanishing. Normalization axiom $\Rightarrow$ mdeg of the fiber at p_i is $\prod_{j=1}^k (\eta_j - \lambda_i)$, so:

$$Tp(\Theta_1) = \sum_{i=1}^n \frac{\prod_{j=1}^k (\theta_j - \lambda_i)}{\prod_{s \neq i} (\lambda_s - \lambda_i)} = \operatorname*{Res}_{q=0} \frac{\prod_{j=1}^k (1 + q\theta_j)}{\prod_{i=1}^n (1 + q\lambda_i)} \frac{dq}{q^{k-n+2}} = c_{k-n+1}(TM - f^*TK)$$

(Consider the rational differential form

$$-\frac{\prod_{j=1}^k (\theta_j - z)}{\prod_{i=1}^n (\lambda_i - z)} dz.$$

The residues of this form at finite poles: $\{z = \lambda_i;\ i = 1 \dots n\}$ exactly recover the terms of the sum. Apply the residue theorem, and change variables: $z = -1/q$.)

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