Thom polynomials of singularities and the Green-Griffiths conjecture
Joint work with Frances Kirwan

Gergely Bérczi – Oxford

Freiburg-Nancy-Strasbourg Algebraic Geometry Seminar, Strasbourg

June 14, 2010.

Singularities of maps

Set up: $k \leq n \leq m$ fixed

Singularities of maps

Set up: $k \leq n \leq m$ fixed

- Let A be a nilpotent algebra, $\dim A/\mathbb{C} = k$. We will take $A_k = z\mathbb{C}[z]/z^{k+1}$

Singularities of maps

Set up: $k \leq n \leq m$ fixed

- Let A be a nilpotent algebra, $\dim A/\mathbb{C} = k$. We will take $A_k = z\mathbb{C}[z]/z^{k+1}$
- $V = J_k(n,m) =$
  $\{p = (p_1, \ldots, p_m) \in \mathrm{Poly}(\mathbb{C}^n, \mathbb{C}^m) : \deg p_i \leq k, p_i(0) = 0\}$

Singularities of maps

Set up: $k \leq n \leq m$ fixed

- Let A be a nilpotent algebra, $\dim A/\mathbb{C} = k$. We will take $A_k = z\mathbb{C}[z]/z^{k+1}$
- $V = J_k(n,m) =$
  $\{p = (p_1, \ldots, p_m) \in \mathrm{Poly}(\mathbb{C}^n, \mathbb{C}^m) : \deg p_i \leq k, p_i(0) = 0\}$
- $\Sigma = \{p \in J_k(n,m) : \mathbb{C}[x_1, \ldots, x_n]/\langle p_1, \ldots, p_m \rangle = A\}$

Singularities of maps

Set up: $k \leq n \leq m$ fixed

- Let A be a nilpotent algebra, $\dim A/\mathbb{C} = k$. We will take $A_k = z\mathbb{C}[z]/z^{k+1}$
- $V = J_k(n, m) =$
  $\{p = (p_1, \ldots, p_m) \in \mathrm{Poly}(\mathbb{C}^n, \mathbb{C}^m) : \deg p_i \leq k, p_i(0) = 0\}$
- $\Sigma = \{p \in J_k(n, m) : \mathbb{C}[x_1, \ldots, x_n]/\langle p_1, \ldots, p_m\rangle = A\}$
- $\mathcal{D} = \mathrm{Diff}(\mathbb{C}^n, 0) \times \mathrm{Diff}(\mathbb{C}^m, 0) \curvearrowright J_d(n, m)$ with

$$(A, B)p = BpA^{-1}$$

Motivation: $f : N^n \to M^m$ (N^n, M^m complex manifolds)

$$Z(f) = \left\{p \in N | \widehat{f_p} \in \Sigma\right\}$$

Singularities of maps

Set up: $k \leq n \leq m$ fixed

- Let A be a nilpotent algebra, $\dim A/\mathbb{C} = k$. We will take $A_k = z\mathbb{C}[z]/z^{k+1}$
- $V = J_k(n,m) =$
  $\{p = (p_1, \ldots, p_m) \in \mathrm{Poly}(\mathbb{C}^n, \mathbb{C}^m) : \deg p_i \leq k, p_i(0) = 0\}$
- $\Sigma = \{p \in J_k(n,m) : \mathbb{C}[x_1, \ldots, x_n]/\langle p_1, \ldots, p_m \rangle = A\}$
- $\mathcal{D} = \mathrm{Diff}(\mathbb{C}^n, 0) \times \mathrm{Diff}(\mathbb{C}^m, 0) \curvearrowright J_d(n,m)$ with

$$(A,B)p = BpA^{-1}$$

Motivation: $f : N^n \to M^m$ (N^n, M^m complex manifolds)

$$Z(f) = \left\{ p \in N | \widehat{f}_p \in \Sigma \right\}$$

Thom's principle: there is a well-defined polynomial

$$MD_A^{n \to m} \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_m]^{S_n \times S_m}$$

such that

$$[Z(f)] = MD_A(TN, f^*(TM)) \in H^*(N, \mathbb{C}).$$

Singularities of maps

Set up: $k \leq n \leq m$ fixed

- Let A be a nilpotent algebra, $\dim A/\mathbb{C} = k$. We will take $A_k = z\mathbb{C}[z]/z^{k+1}$
- $V = J_k(n,m) =$
  $\{p = (p_1, \ldots, p_m) \in \mathrm{Poly}(\mathbb{C}^n, \mathbb{C}^m) : \deg p_i \leq k, p_i(0) = 0\}$
- $\Sigma = \{p \in J_k(n,m) : \mathbb{C}[x_1, \ldots, x_n]/\langle p_1, \ldots, p_m\rangle = A\}$
- $\mathcal{D} = \mathrm{Diff}(\mathbb{C}^n, 0) \times \mathrm{Diff}(\mathbb{C}^m, 0) \curvearrowright J_d(n,m)$ with

$$(A, B)p = BpA^{-1}$$

Motivation: $f : N^n \to M^m$ (N^n, M^m complex manifolds)

$$Z(f) = \left\{p \in N | \widehat{f}_p \in \Sigma\right\}$$

Thom's principle: there is a well-defined polynomial

$$MD_A^{n \to m} \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_m]^{S_n \times S_m}$$

such that

$$[Z(f)] = MD_A(TN, f^*(TM)) \in H^*(N, \mathbb{C}).$$

Haefliger and Kosinski:

$$c(q) = c_0 + c_1 q + c_2 q^2 + \ldots = \frac{c(f^*(TM))}{c(TN)} = \frac{\prod_{m=1}^{k}(1 + \theta_m q)}{\prod_{i=1}^{n}(1 + \lambda_i q)}$$

Then

$$MD_A(TN, f^*(TM)) = Tp_A^{k-n}(c_1, c_2, \ldots)$$

MD stands for: Multidegree, Equivariant Hilbert polynomial, Equivariant Poincaré dual

Then

$$MD_A(TN, f^*(TM)) = Tp_A^{k-n}(c_1, c_2, \ldots)$$

MD stands for: Multidegree, Equivariant Hilbert polynomial, Equivariant Poincaré dual

What is this? Algebraic invariant of G-invariant subvarieties of a complex vector space.

Then

$$MD_A(TN, f^*(TM)) = Tp_A^{k-n}(c_1, c_2, \ldots)$$

MD stands for: Multidegree, Equivariant Hilbert polynomial, Equivariant Poincaré dual
What is this? Algebraic invariant of G-invariant subvarieties of a complex vector space.
Set up:

Then

$$MD_A(TN, f^*(TM)) = Tp_A^{k-n}(c_1, c_2, \ldots)$$

MD stands for: Multidegree, Equivariant Hilbert polynomial, Equivariant Poincaré dual

What is this? Algebraic invariant of G-invariant subvarieties of a complex vector space.

Set up:

1. $V = \mathbb{C}^N$ complex vector space, with a G-action.

Then

$$MD_A(TN, f^*(TM)) = Tp_A^{k-n}(c_1, c_2, \dots)$$

MD stands for: Multidegree, Equivariant Hilbert polynomial, Equivariant Poincaré dual
What is this? Algebraic invariant of G-invariant subvarieties of a complex vector space.
Set up:

1. $V = \mathbb{C}^N$ complex vector space, with a G-action.
2. $\Sigma \subset V$ is a G-invariant closed subvariety.

Then

$$MD_A(TN, f^*(TM)) = Tp_A^{k-n}(c_1, c_2, \ldots)$$

MD stands for: Multidegree, Equivariant Hilbert polynomial, Equivariant Poincaré dual

What is this? Algebraic invariant of G-invariant subvarieties of a complex vector space.

Set up:

1. $V = \mathbb{C}^N$ complex vector space, with a G-action.
2. $\Sigma \subset V$ is a G-invariant closed subvariety.
3. $H_G^*(V) = H_G^*(pt)$ is the G-equivariant cohomology ring of V. $H_{GL(d)}^*(pt) = \mathbb{C}[x_1, \ldots, x_d]^{S_d}$.

Definition/Description

$$\mathrm{mdeg}[\Sigma, V] \in H_G^{\mathrm{codim}(\Sigma \in V)}(pt)$$

Definition/Description

$$\mathrm{mdeg}[\Sigma, V] \in H_G^{\mathrm{codim}(\Sigma \in V)}(pt)$$

Vergne's integral definition-topology

Definition/Description

$$\mathrm{mdeg}[\Sigma, V] \in H_G^{\mathrm{codim}(\Sigma \in V)}(pt)$$

Vergne's integral definition-topology
$EG \times_G \Sigma \subset EG \times_G V$ represents a homology cycle. Then

$$\mathrm{mdeg}[\Sigma, V] = PD(EG \times_G \Sigma \subset EG \times_G V)$$

So multidegree=equivariant Poincaré dual.

Definition/Description

$$\mathrm{mdeg}[\Sigma, V] \in H_G^{\mathrm{codim}(\Sigma \in V)}(pt)$$

Vergne's integral definition-topology
$EG \times_G \Sigma \subset EG \times_G V$ represents a homology cycle. Then

$$\mathrm{mdeg}[\Sigma, V] = PD(EG \times_G \Sigma \subset EG \times_G V)$$

So multidegree=equivariant Poincaré dual.
Vergne: there is an equivariant Thom class:

$$Thom_G(V) \in H_G^{\dim V}(V)$$

s.t

$$\mathrm{mdeg}[\Sigma, V] = \int_\Sigma Thom_G(V).$$

Sturmfeld's axiomatic definition

Sturmfeld's axiomatic definition

$\mathrm{mdeg}[\Sigma, V]$ is characterized by the following axioms:

Sturmfeld's axiomatic definition

$\mathrm{mdeg}[\Sigma, V]$ is characterized by the following axioms:

additive If $\Sigma \doteq \cup \Sigma_i$

$$\mathrm{mdeg}[\Sigma, V] = \sum_{i=1}^{c} \mathrm{mult}(\Sigma_i) \cdot \mathrm{mdeg}[\Sigma_i, W],$$

Sturmfeld's axiomatic definition

$\mathrm{mdeg}[\Sigma, V]$ is characterized by the following axioms:

additive If $\Sigma \doteq \cup\Sigma_i$

$$\mathrm{mdeg}[\Sigma, V] = \sum_{i=1}^{c} \mathrm{mult}(\Sigma_i) \cdot \mathrm{mdeg}[\Sigma_i, W],$$

degenerative $\mathrm{mdeg}[\Sigma, V] = \mathrm{mdeg}[\mathrm{Groeb}(\Sigma), V]$

Sturmfeld's axiomatic definition

$\mathrm{mdeg}[\Sigma, V]$ is characterized by the following axioms:

additive If $\Sigma \doteq \cup\Sigma_i$

$$\mathrm{mdeg}[\Sigma, V] = \sum_{i=1}^{c} \mathrm{mult}(\Sigma_i) \cdot \mathrm{mdeg}[\Sigma_i, W],$$

degenerative $\mathrm{mdeg}[\Sigma, V] = \mathrm{mdeg}[\mathrm{Groeb}(\Sigma), V]$

normalized For T-invariant linear subspaces of V the invariant is defined to be equal to the product of weights in the normal direction.

Sturmfeld's axiomatic definition

$\mathrm{mdeg}[\Sigma, V]$ is characterized by the following axioms:

additive If $\Sigma \doteq \cup\Sigma_i$

$$\mathrm{mdeg}[\Sigma, V] = \sum_{i=1}^{c} \mathrm{mult}(\Sigma_i) \cdot \mathrm{mdeg}[\Sigma_i, W],$$

degenerative $\mathrm{mdeg}[\Sigma, V] = \mathrm{mdeg}[\mathrm{Groeb}(\Sigma), V]$

normalized For T-invariant linear subspaces of V the invariant is defined to be equal to the product of weights in the normal direction.

Example: $(\mathbb{C}^*)^3$ acts on $\mathbb{C}^4$ with weights $\eta_1, \ldots \eta_4$. Let $\eta_1 + \eta_2 = \eta_3 + \eta_4$, and

$$\Sigma = \mathrm{Spec}(\mathbb{C}[y_1, y_2, y_3, y_4]/(y_1 y_2 - y_3 y_4)).$$

Sturmfeld's axiomatic definition

$\mathrm{mdeg}[\Sigma, V]$ is characterized by the following axioms:

additive If $\Sigma \doteq \cup \Sigma_i$

$$\mathrm{mdeg}[\Sigma, V] = \sum_{i=1}^{c} \mathrm{mult}(\Sigma_i) \cdot \mathrm{mdeg}[\Sigma_i, W],$$

degenerative $\mathrm{mdeg}[\Sigma, V] = \mathrm{mdeg}[\mathrm{Groeb}(\Sigma), V]$

normalized For T-invariant linear subspaces of V the invariant is defined to be equal to the product of weights in the normal direction.

Example: $(\mathbb{C}^*)^3$ acts on $\mathbb{C}^4$ with weights $\eta_1, \ldots \eta_4$. Let $\eta_1 + \eta_2 = \eta_3 + \eta_4$, and

$$\Sigma = \mathrm{Spec}(\mathbb{C}[y_1, y_2, y_3, y_4]/(y_1 y_2 - y_3 y_4)).$$

$$\Sigma_t = \mathrm{Spec}(\mathbb{C}[y_1, y_2, y_3, y_4]/(y_1 y_2 - t y_3 y_4)),$$

For $t = 0$ $\Sigma_0 = \{y_1 y_2 = 0\}$, so normalisation says

$$\mathrm{mdeg}[\Sigma, \mathbb{C}^4] = \eta_1 + \eta_2 = \eta_3 + \eta_4.$$

Thom's principle again:

$$MD_d^{n\to m} = \mathrm{mdeg}^{\mathrm{GL}_{\boldsymbol{n}}\times \mathrm{GL}_{\boldsymbol{m}}}[\Sigma_k, J_k(n,m)].$$

Theorem (The test curve model of Porteous and Gaffney)

$$\Sigma_k(n,m) \doteq \{\Psi \in J_k(n,m) | \exists \gamma \in J_k^{reg}(1,n) \text{ such that } \Psi \circ \gamma = 0\}.$$

$$(\mathbb{C},0) \xrightarrow{\gamma} (\mathbb{C}^n,0) \xrightarrow{\Psi} (\mathbb{C}^m,0) \tag{1}$$

Thom's principle again:

$$MD_d^{n\to m} = \mathrm{mdeg}^{\mathrm{GL}_n \times \mathrm{GL}_m}[\Sigma_k, J_k(n,m)].$$

Theorem (The test curve model of Porteous and Gaffney)

$$\Sigma_k(n,m) \doteq \{\Psi \in J_k(n,m) | \exists \gamma \in J_k^{reg}(1,n) \text{ such that } \Psi \circ \gamma = 0\}.$$

$$(\mathbb{C},0) \xrightarrow{\gamma} (\mathbb{C}^n,0) \xrightarrow{\Psi} (\mathbb{C}^m,0) \tag{1}$$

Observation: If $\varphi \in J_k^{reg}(1,1) = \mathbf{G}_k$, then

$$\Psi \circ \gamma = 0 \quad \Rightarrow \quad \Psi \circ (\gamma \circ \varphi) = 0$$

$$(\mathbb{C},0) \xrightarrow{\varphi} (\mathbb{C},0) \xrightarrow{\gamma} (\mathbb{C}^n,0) \xrightarrow{\Psi} (\mathbb{C}^m,0) \tag{2}$$

Thom's principle again:

$$MD_d^{n\to m} = \mathrm{mdeg}^{\mathrm{GL}_{\boldsymbol{n}}\times\mathrm{GL}_{\boldsymbol{m}}}[\Sigma_k, J_k(n,m)].$$

Theorem (The test curve model of Porteous and Gaffney)

$$\Sigma_k(n,m) \doteq \{\Psi \in J_k(n,m) | \exists\gamma \in J_k^{reg}(1,n) \text{ such that } \Psi\circ\gamma = 0\}.$$

$$(\mathbb{C},0) \xrightarrow{\gamma} (\mathbb{C}^n,0) \xrightarrow{\Psi} (\mathbb{C}^m,0) \tag{1}$$

Observation: If $\varphi \in J_k^{reg}(1,1) = \mathbf{G}_k$, then

$$\Psi\circ\gamma = 0 \quad \Rightarrow \quad \Psi\circ(\gamma\circ\varphi) = 0$$

$$(\mathbb{C},0) \xrightarrow{\varphi} (\mathbb{C},0) \xrightarrow{\gamma} (\mathbb{C}^n,0) \xrightarrow{\Psi} (\mathbb{C}^m,0) \tag{2}$$

Proposition

$\Sigma_k^0 = \{\Psi : \dim\ker\Psi = 1\} \doteq \Sigma_k$ *fibers with linear fibres over* $J_k^{reg}(1,n)/\mathbf{G}_k$.

- identify $J_k(1,n)$ with $\mathrm{Hom}(\mathbb{C}^k,\mathbb{C}^n)$;

- identify $J_k(1,n)$ with $\mathrm{Hom}(\mathbb{C}^k,\mathbb{C}^n)$;
- Then $\mathbf{G}_k$ acts on $J_k(1,n)$ on the right by

$$\left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \ldots & 2\alpha_1\alpha_{k-1}+\ldots \\ 0 & 0 & \alpha_1^3 & \ldots & 3\alpha_1^2\alpha_{k-2}+\ldots \\ 0 & 0 & 0 & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \alpha_1^k \end{pmatrix} : \alpha_1 \in \mathbb{C}^* \alpha_i \in \mathbb{C} \right\};$$

where the polynomial in the (i,j) entry is

$$p_{i,j}(\bar{\alpha}) = \sum_{a_1+a_2+\ldots+a_i=j} \alpha_{a_1}\alpha_{a_2}\ldots\alpha_{a_i}.$$

- Let $\mathrm{Sym}_{\leq k}\mathbb{C}^n = \oplus_{i=1}^{k}\mathrm{Sym}^i\mathbb{C}^n$.

- Let $\mathrm{Sym}_{\le k}\mathbb{C}^n = \oplus_{i=1}^k \mathrm{Sym}^i\mathbb{C}^n$. Define a map

$$\rho : \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n) \to \mathrm{Hom}(\mathbb{C}^k, \mathrm{Sym}_{\le k}\mathbb{C}^n)$$

$$\rho(v_1, \ldots, v_k) = (v_1, v_2 + v_1^2, \ldots, \sum_{a_1+a_2+\ldots+a_i=j} v_{a_1} v_{a_2} \ldots v_{a_i}, \ldots),$$

- Let $\mathrm{Sym}_{\leq k}\mathbb{C}^n = \oplus_{i=1}^k \mathrm{Sym}^i\mathbb{C}^n$. Define a map

$$\rho : \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n) \to \mathrm{Hom}(\mathbb{C}^k, \mathrm{Sym}_{\leq k}\mathbb{C}^n)$$

$$\rho(v_1, \ldots, v_k) = (v_1, v_2 + v_1^2, \ldots, \sum_{a_1+a_2+\ldots+a_i=j} v_{a_1} v_{a_2} \ldots v_{a_i}, \ldots),$$

It descends to an injective maps

$$\rho^{flag} : \mathrm{Hom}^0(\mathbb{C}^k, \mathbb{C}^n)/\mathbf{G}_k \hookrightarrow \mathrm{Flag}_k(\mathrm{Sym}_{\leq k}\mathbb{C}^n)$$

$$\rho^{grass} : \mathrm{Hom}^0(\mathbb{C}^k, \mathbb{C}^n)/\mathbf{G}_k \hookrightarrow \mathrm{Grass}_k(\mathrm{Sym}_{\leq k}\mathbb{C}^n)$$

$$\rho^{proj} = Pluck \circ \rho^{grass} : \mathrm{Hom}^0(\mathbb{C}^k, \mathbb{C}^n)/\mathbf{G}_k \hookrightarrow \mathbb{P}(\wedge^k(\mathrm{Sym}_{\leq k}\mathbb{C}^n))$$

- A little stronger statement still holds: the truncated map

$$\rho : \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n) \to \mathrm{Hom}(\mathbb{C}^k, \mathrm{Sym}_{\leq 2}\mathbb{C}^n)$$

induces an embedding

$$\rho^{grass} : \mathrm{Hom}^{reg}(\mathbb{C}^k, \mathbb{C}^n)/\mathbf{G}_k \hookrightarrow \mathrm{Grass}_k(\mathrm{Sym}_{\leq 2}\mathbb{C}^n)$$

Theorem (B-Kirwan, 2009)

1. *For* $k \leq n$ $\mathrm{im}(\varrho) \subset \mathrm{Grass}$ *is a* GL_k *orbit.*

Theorem (B-Kirwan, 2009)

1. *For* $k \leq n$ $\mathrm{im}(\varrho) \subset \mathrm{Grass}$ *is a* GL_k *orbit.*
2. *For* $k \leq n$ $\overline{\mathrm{im}(\varrho)} \subset \mathrm{Grass}$ *is the finite union of* $GL(k)$ *orbits.*

Theorem (B-Kirwan, 2009)

1. *For* $k \leq n$ $\mathrm{im}(\varrho) \subset \mathrm{Grass}$ *is a* GL_k *orbit.*
2. *For* $k \leq n$ $\overline{\mathrm{im}(\varrho)} \subset \mathrm{Grass}$ *is the finite union of* $GL(k)$ *orbits.*
3. *For any* k, n*, the boundary orbits of* $\varrho(Hom^0) \subset \mathbb{P}(\wedge^k(\mathrm{Sym}_{\leq k}\mathbb{C}^n))$ *have codimension at least* 2*, so* $\overline{\mathrm{im}(\rho)} = \mathrm{Spec}(\mathbb{C}[A_k]^{G_k})$.

Theorem (B-Kirwan, 2009)

1. *For* $k \leq n$ $\mathrm{im}(\varrho) \subset \mathrm{Grass}$ *is a* GL_k *orbit.*
2. *For* $k \leq n$ $\overline{\mathrm{im}(\varrho)} \subset \mathrm{Grass}$ *is the finite union of* $GL(k)$ *orbits.*
3. *For any* k, n*, the boundary orbits of* $\varrho(Hom^0) \subset \mathbb{P}(\wedge^k(\mathrm{Sym}_{\leq k}\mathbb{C}^n))$ *have codimension at least* 2*, so* $\overline{\mathrm{im}(\rho)} = \mathrm{Spec}(\mathbb{C}[A_k]^{G_k})$*.*

Theorem (B-Kirwan, 2009)

Theorem (B-Kirwan, 2009)

1. *For* $k \leq n$ $\mathrm{im}(\varrho) \subset \mathrm{Grass}$ *is a* GL_k *orbit.*
2. *For* $k \leq n$ $\overline{\mathrm{im}(\varrho)} \subset \mathrm{Grass}$ *is the finite union of* $GL(k)$ *orbits.*
3. *For any* k, n*, the boundary orbits of* $\varrho(Hom^0) \subset \mathbb{P}(\wedge^k(\mathrm{Sym}_{\leq k}\mathbb{C}^n))$ *have codimension at least* 2*, so* $\overline{\mathrm{im}(\rho)} = \mathrm{Spec}(\mathbb{C}[A_k]^{\mathbf{G}_k})$*.*

Theorem (B-Kirwan, 2009)

1. $E_k^n = (\mathcal{O}(J_k(1,n)))^{G_k}$ *(the Demailly-Semple ring) is finitely generated.*

Theorem (B-Kirwan, 2009)

1. *For* $k \leq n$ $\mathrm{im}(\varrho) \subset \mathrm{Grass}$ *is a* GL_k *orbit.*
2. *For* $k \leq n$ $\overline{\mathrm{im}(\varrho)} \subset \mathrm{Grass}$ *is the finite union of* $GL(k)$ *orbits.*
3. *For any* k, n*, the boundary orbits of* $\varrho(Hom^0) \subset \mathbb{P}(\wedge^k(\mathrm{Sym}_{\leq k}\mathbb{C}^n))$ *have codimension at least* 2*, so* $\overline{\mathrm{im}(\rho)} = \mathrm{Spec}(\mathbb{C}[A_k]^{\mathbf{G_k}})$*.*

Theorem (B-Kirwan, 2009)

1. $E_k^n = (\mathcal{O}(J_k(1,n)))^{G_k}$ *(the Demailly-Semple ring) is finitely generated.*
2. *The generators are given by elements of the radical of the ideal generated by* $\Delta_{\mathbf{i_1},\ldots,\mathbf{i}_k}$*, where* $\Delta_{\mathbf{i_1},\ldots,\mathbf{i}_k}$ *is the minor of* $\rho(f' \ldots, f^{(k)}) \in \mathrm{Hom}(\mathbb{C}^k, \mathrm{Sym}_{\leq k}\mathbb{C}^n)$*.*

Theorem (B-Kirwan, 2009)

1. *For* $k \leq n$ $\mathrm{im}(\varrho) \subset \mathrm{Grass}$ *is a* GL_k *orbit.*
2. *For* $k \leq n$ $\overline{\mathrm{im}(\varrho)} \subset \mathrm{Grass}$ *is the finite union of* $GL(k)$ *orbits.*
3. *For any* k, n*, the boundary orbits of* $\varrho(Hom^0) \subset \mathbb{P}(\wedge^k(\mathrm{Sym}_{\leq k}\mathbb{C}^n))$ *have codimension at least* 2*, so* $\overline{\mathrm{im}(\rho)} = \mathrm{Spec}(\mathbb{C}[A_k]^{\mathbf{G}_k})$*.*

Theorem (B-Kirwan, 2009)

1. $E_k^n = (\mathcal{O}(J_k(1,n)))^{G_k}$ *(the Demailly-Semple ring) is finitely generated.*
2. *The generators are given by elements of the radical of the ideal generated by* $\Delta_{\mathbf{i}_1,\ldots,\mathbf{i}_k}$*, where* $\Delta_{\mathbf{i}_1,\ldots,\mathbf{i}_k}$ *is the minor of* $\rho(f' \ldots, f^{(k)}) \in \mathrm{Hom}(\mathbb{C}^k, \mathrm{Sym}_{\leq k}\mathbb{C}^n)$*.*
3. *The biinvariants* $E^{GL(n)} \subset E_k^n$ *are generated by the radical of the ideal generated by the minors* $\{\Delta_{\mathbf{i}_1,\ldots,\mathbf{i}_n} : \mathbf{i}_1, \ldots, \mathbf{i}_n$ *is descendent*$\}$*,*

Example

$n=3, k=3$. E_3^3 is generated by 16 independent invariants.

$$J_3^{\mathrm{reg}}(1,2)=\{(f_i',f_i'',f_i''':i=1,2,3)\in(\mathbb{C}^3)^3;(f_1',f_2',f_3')\neq(0,0,0)\},$$

Then

$$(f_1',f_2',f_3',f_1'',f_2'',f_3'',f_1''',f_2''',f_3''')\mapsto$$
$$\begin{pmatrix} f_1' & f_2' & f_3' & 0 & 0 & 0 & 0 & 0 & 0 \\ f_1'' & f_2'' & f_3'' & (f_1')^2 & 2f_1'f_2' & (f_2')^2 & 2f_1'f_3' & 2f_2'f_3' & (f_3')^2 \\ f_1''' & f_2''' & f_3''' & 2f_1'f_1'' & 2f_1'f_2''+2f_1''f_2' & 2f_2'f_2'' & 2f_1'f_3''+2f_3'f_1'' & 2f_2'f_3''+2f_2''f_3' & 2f_3'f_3'' \end{pmatrix}$$

The computation: double localization+vanishing theorem:

$$\Sigma_k(n,m) \qquad\qquad \subset J_k(n,m)$$

The computation: double localization+vanishing theorem:

$$\begin{array}{ccc} \Sigma_k(n,m) & & \subset J_k(n,m) \\ \downarrow & & \\ J_k^{reg}(1,n)/\mathbf{G}_k & & \subset \mathrm{Flag}_k(\mathrm{Sym}_{\leq k}\mathbb{C}^n) \end{array}$$

The computation: double localization+vanishing theorem:

$$\begin{array}{ccc} \Sigma_k(n,m) & & \subset J_k(n,m) \\ \downarrow & & \\ J_k^{reg}(1,n)/\mathbf{G}_k & & \subset \mathrm{Flag}_k(\mathrm{Sym}_{\leq k}\mathbb{C}^n) \\ \downarrow & & \\ J_k^{reg}(1,n)/B_k = \mathrm{Flag}_k(\mathbb{C}^n) & & \end{array} \tag{3}$$

The computation: double localization+vanishing theorem:

$$\begin{array}{ccc}
\Sigma_k(n,m) & & \subset J_k(n,m) \\
\downarrow & & \\
J_k^{reg}(1,n)/\mathbf{G}_k & & \subset \mathrm{Flag}_k(\mathrm{Sym}_{\le k}\mathbb{C}^n) \\
\downarrow & & \\
J_k^{reg}(1,n)/B_k = \mathrm{Flag}_k(\mathbb{C}^n) & &
\end{array} \tag{3}$$

$$Tp_k^{m-n} = \int_{\Sigma_k} \mathrm{Thom}_k(n,m) = \int_{\mathrm{Flag}_k(\mathbb{C}^n)} \int_{F \in \mathrm{Flag}_k(\mathrm{Sym}_{\le k}\mathbb{C}^n)} \mathrm{Thom}|_F$$

Theorem (B-Szenes, 2007)

$$Tp_k^{m-n} = \mathrm{Res}_{\mathbf{z}=\infty} \frac{\prod_{i<j}(z_i - z_j)\, \mathcal{Q}_k(z_1 \dots z_k)}{\prod_{i+j \le l \le k}(z_i + z_j - z_l)} \cdot \prod_{l=1}^{k} c\left(\frac{1}{z_l}\right) z_l^{m-n}\, dz_l$$

Theorem (B-Szenes, 2007)

$$Tp_k^{m-n} = \mathrm{Res}_{\mathbf{z}=\infty} \frac{\prod_{i<j}(z_i - z_j)\, \mathcal{Q}_k(z_1 \dots z_k)}{\prod_{i+j \le l \le k}(z_i + z_j - z_l)} \cdot \prod_{l=1}^{k} c\left(\frac{1}{z_l}\right) z_l^{m-n}\, dz_l$$

- We integrate on the cycle $|z_1| > |z_2| > \dots |z_k|$, which determines the Laurent expansion.

Theorem (B-Szenes, 2007)

$$Tp_k^{m-n} = \mathrm{Res}_{\mathbf{z}=\infty} \frac{\prod_{i<j}(z_i - z_j)\, \mathcal{Q}_k(z_1 \ldots z_k)}{\prod_{i+j\le l\le k}(z_i + z_j - z_l)} \cdot \prod_{l=1}^{k} c\left(\frac{1}{z_l}\right) z_l^{m-n}\, dz_l$$

- We integrate on the cycle $|z_1| > |z_2| > \ldots |z_k|$, which determines the Laurent expansion.
- $c(q) = 1 + c_1 q + c_2 q^2 + \ldots$

Theorem (B-Szenes, 2007)

$$Tp_k^{m-n} = \mathrm{Res}_{\mathbf{z}=\infty} \frac{\prod_{i<j}(z_i - z_j)\, \mathcal{Q}_k(z_1 \dots z_k)}{\prod_{i+j\leq l\leq k}(z_i + z_j - z_l)} \cdot \prod_{l=1}^{k} c\left(\frac{1}{z_l}\right) z_l^{m-n}\, dz_l$$

- We integrate on the cycle $|z_1| > |z_2| > \dots |z_k|$, which determines the Laurent expansion.
- $c(q) = 1 + c_1 q + c_2 q^2 + \dots$
- $Q_k(z_1, \dots, z_k)$ is the multidegree of a Borel-orbit in a representation of $GL(k)$.

$$Q_1 = Q_2 = Q_3 = 1,\, Q_4 = 2z_1 + z_2 - z_4$$

Theorem (B-Szenes, 2007)

$$Tp_k^{m-n} = \mathrm{Res}_{\mathbf{z}=\infty} \frac{\prod_{i<j}(z_i - z_j)\, \mathcal{Q}_k(z_1 \dots z_k)}{\prod_{i+j \leq l \leq k}(z_i + z_j - z_l)} \cdot \prod_{l=1}^{k} c\left(\frac{1}{z_l}\right) z_l^{m-n}\, dz_l$$

- We integrate on the cycle $|z_1| > |z_2| > \dots |z_k|$, which determines the Laurent expansion.
- $c(q) = 1 + c_1 q + c_2 q^2 + \dots$
- $Q_k(z_1, \dots, z_k)$ is the multidegree of a Borel-orbit in a representation of $GL(k)$.

$$Q_1 = Q_2 = Q_3 = 1, Q_4 = 2z_1 + z_2 - z_4$$

Conjecture (Rimanyi, 1998)

$$Tp_k^{m-n} \in \mathbb{N}[c_1, \dots, c_{k(m-n+1)}] \;\; i.e\; \frac{\prod_{i<j}(z_i - z_j)\, \mathcal{Q}_k(z_1 \dots z_k)}{\prod_{i+j \leq l \leq k}(z_i + z_j - z_l)} > 0.$$

The Green-Griffiths conjecture

- Let X be a complex manifold, $n = \dim_{\mathbb{C}}(X)$

The Green-Griffiths conjecture

- Let X be a complex manifold, $n = \dim_{\mathbb{C}}(X)$
- X is said to be hyperbolic in the sense of Brody if there are no non-constant entire holomorphic curves $f : \mathbb{C} \to X$.

The Green-Griffiths conjecture

- Let X be a complex manifold, $n = \dim_{\mathbb{C}}(X)$
- X is said to be hyperbolic in the sense of Brody if there are no non-constant entire holomorphic curves $f : \mathbb{C} \to X$.
- Brody has shown that for X compact
  Brody hyperbolicity $\iff$ Kobayashi hyperbolicity
  (degeneracy of the Kobayashi pseudo-metric)

Conjecture (Green-Griffith-Lang, 1979)

Let X be a projective variety of general type. Then there exists an algebraic variety $Y \subsetneq X$ such that for all non-constant holomorphic $f : \mathbb{C} \to X$ one has $f(\mathbb{C}) \subset Y$.

The Green-Griffiths conjecture

- Let X be a complex manifold, $n = \dim_{\mathbb{C}}(X)$
- X is said to be hyperbolic in the sense of Brody if there are no non-constant entire holomorphic curves $f : \mathbb{C} \to X$.
- Brody has shown that for X compact
  Brody hyperbolicity $\iff$ Kobayashi hyperbolicity
  (degeneracy of the Kobayashi pseudo-metric)

Conjecture (Green-Griffith-Lang, 1979)

Let X be a projective variety of general type. Then there exists an algebraic variety $Y \subsetneq X$ such that for all non-constant holomorphic $f : \mathbb{C} \to X$ one has $f(\mathbb{C}) \subset Y$.

- McQuillen (1998): Positive answer for surfaces if $c_1^2 - c_2 > 0$.

The Green-Griffiths conjecture

- Let X be a complex manifold, $n = \dim_{\mathbb{C}}(X)$
- X is said to be hyperbolic in the sense of Brody if there are no non-constant entire holomorphic curves $f : \mathbb{C} \to X$.
- Brody has shown that for X compact
  Brody hyperbolicity $\iff$ Kobayashi hyperbolicity
  (degeneracy of the Kobayashi pseudo-metric)

Conjecture (Green-Griffith-Lang, 1979)

Let X be a projective variety of general type. Then there exists an algebraic variety $Y \subsetneq X$ such that for all non-constant holomorphic $f : \mathbb{C} \to X$ one has $f(\mathbb{C}) \subset Y$.

- McQuillen (1998): Positive answer for surfaces if $c_1^2 - c_2 > 0$.
- Demailly (1990)/Siu (1996): Strategy for projective hypersurfaces

The Green-Griffiths conjecture

- Let X be a complex manifold, $n = \dim_{\mathbb{C}}(X)$
- X is said to be hyperbolic in the sense of Brody if there are no non-constant entire holomorphic curves $f : \mathbb{C} \to X$.
- Brody has shown that for X compact
  Brody hyperbolicity $\iff$ Kobayashi hyperbolicity
  (degeneracy of the Kobayashi pseudo-metric)

Conjecture (Green-Griffith-Lang, 1979)

Let X be a projective variety of general type. Then there exists an algebraic variety $Y \subsetneq X$ such that for all non-constant holomorphic $f : \mathbb{C} \to X$ one has $f(\mathbb{C}) \subset Y$.

- McQuillen (1998): Positive answer for surfaces if $c_1^2 - c_2 > 0$.
- Demailly (1990)/Siu (1996): Strategy for projective hypersurfaces
- Siu (1996) Positive answer for hypersurfaces of high degree.

The Green-Griffiths conjecture

- Let X be a complex manifold, $n = \dim_{\mathbb{C}}(X)$
- X is said to be hyperbolic in the sense of Brody if there are no non-constant entire holomorphic curves $f : \mathbb{C} \to X$.
- Brody has shown that for X compact
  Brody hyperbolicity $\iff$ Kobayashi hyperbolicity
  (degeneracy of the Kobayashi pseudo-metric)

Conjecture (Green-Griffith-Lang, 1979)

Let X be a projective variety of general type. Then there exists an algebraic variety $Y \subsetneq X$ such that for all non-constant holomorphic $f : \mathbb{C} \to X$ one has $f(\mathbb{C}) \subset Y$.

- McQuillen (1998): Positive answer for surfaces if $c_1^2 - c_2 > 0$.
- Demailly (1990)/Siu (1996): Strategy for projective hypersurfaces
- Siu (1996) Positive answer for hypersurfaces of high degree.
- Diverio, Merker, Rousseau (2009): Effective lower bound, $deg(X) > 2^{n^5} \Rightarrow GGL$.

Strategy(Demailly '95, Siu '96, DMR '09)

- $$f : \mathbb{C} \to X, \quad t \to f(t) = (f_1(t), f_2(t), \ldots, f_n(t))$$
  a curve written in some local holomorphic coordinates $(z_1, \ldots, z_n)$ on X. Define the bundle
  $$(J_k X)_x = \{\hat{f}_{[k]} : f : (\mathbb{C}, 0) \to (X, x)\} \to X$$
  sending $f_{[k]}$ to $f(0)$. Fibre is $J_k(1, n)$

Strategy(Demailly '95, Siu '96, DMR '09)

- $$f : \mathbb{C} \to X, \quad t \to f(t) = (f_1(t), f_2(t), \ldots, f_n(t))$$
  a curve written in some local holomorphic coordinates $(z_1, \ldots, z_n)$ on X. Define the bundle
  $$(J_k X)_x = \{\hat{f}_{[k]} : f : (\mathbb{C}, 0) \to (X, x)\} \to X$$
  sending $f_{[k]}$ to $f(0)$. Fibre is $J_k(1, n)$
- The group of reparametrizations $\mathbf{G}_k = J_k^{reg}(1, 1)$ acts fiberwise on $J_k X$. The fibres of $J_k X$ can be identified with $J_k(1, n)$, the action is linearised as before.

Strategy(Demailly '95, Siu '96, DMR '09)

- $$f : \mathbb{C} \to X, \quad t \to f(t) = (f_1(t), f_2(t), \dots, f_n(t))$$
  a curve written in some local holomorphic coordinates $(z_1, \dots, z_n)$ on X. Define the bundle
  $$(J_k X)_x = \{\hat{f}_{[k]} : f : (\mathbb{C}, 0) \to (X, x)\} \to X$$
  sending $f_{[k]}$ to $f(0)$. Fibre is $J_k(1, n)$
- The group of reparametrizations $\mathbf{G}_k = J_k^{reg}(1, 1)$ acts fiberwise on $J_k X$. The fibres of $J_k X$ can be identified with $J_k(1, n)$, the action is linearised as before.
- $\mathbf{G}_k = \mathbb{C}^* \ltimes U_d$, and for $\lambda \in \mathbb{C}^*$
  $$(\lambda \cdot f)(t) = f(\lambda \cdot t), \text{ so } \lambda \cdot (f', f'', \dots f^{(k)}) = (\lambda f', \lambda^2 f'', \dots \lambda^k f^{(k)}).$$

Strategy(Demailly '95, Siu '96, DMR '09)

- $$f:\mathbb{C}\to X, \quad t\to f(t)=(f_1(t),f_2(t),\dots,f_n(t))$$
  a curve written in some local holomorphic coordinates $(z_1,\dots,z_n)$ on X. Define the bundle
  $$(J_kX)_x=\{\hat{f}_{[k]}:f:(\mathbb{C},0)\to(X,x)\}\to X$$
  sending $f_{[k]}$ to $f(0)$. Fibre is $J_k(1,n)$
- The group of reparametrizations $\mathbf{G}_k=J_k^{reg}(1,1)$ acts fiberwise on J_kX. The fibres of J_kX can be identified with $J_k(1,n)$, the action is linearised as before.
- $\mathbf{G}_k=\mathbb{C}^*\ltimes U_d$, and for $\lambda\in\mathbb{C}^*$
  $$(\lambda\cdot f)(t)=f(\lambda\cdot t), \text{ so } \lambda\cdot(f',f'',\dots f^{(k)})=(\lambda f',\lambda^2f'',\dots\lambda^kf^{(k)}).$$
- Consider algebraic differential operators = polynomial functions on J_kX. Locally in multi-index notation
  $$Q(f',f'',\dots,f^{(k)})=\sum_{\alpha_i\in\mathbb{N}^n}a_{\alpha_1,\alpha_2,\dots\alpha_k}(f(t))(f'(t)^{\alpha_1}f''(t)^{\alpha_2}\cdots f^{(k)}(t)^{\alpha_k},$$
  where $a_{\alpha_1,\alpha_2,\dots\alpha_k}(z)$ are holomorphic coefficients on X and $t\to z=f(t)$ is a curve.

- Q is homogeneous of weighted degree m under the $\mathbb{C}^*$ action iff

$$Q(\lambda f', \lambda^2 f'', \ldots \lambda^k f^{(k)}) = \lambda^m Q(f', f'', \ldots f^{(k)}).$$

- Q is homogeneous of weighted degree m under the $\mathbb{C}^*$ action iff

$$Q(\lambda f', \lambda^2 f'', \dots \lambda^k f^{(k)}) = \lambda^m Q(f', f'', \dots f^{(k)}).$$

Definition

- *(Green-Griffiths '78)* $E_{k,m}^{GG}$ *is the sheaf of algebraic differential operators of order* k *and weighted degree* m.

- Q is homogeneous of weighted degree m under the $\mathbb{C}^*$ action iff

$$Q(\lambda f', \lambda^2 f'', \dots \lambda^k f^{(k)}) = \lambda^m Q(f', f'', \dots f^{(k)}).$$

Definition

- *(Green-Griffiths '78)*
  $E_{k,m}^{GG}$ *is the sheaf of algebraic differential operators of order k and weighted degree m.*
- *(Demailly, '95)*
  The bundle of invariant jet differentials of order k and weighted degree m is the subbundle $E_{k,m} \subset E_{k,m}^{GG}$*, whose elements are invariant under arbitrary changes of parametrization, i.e. for* $\phi \in \mathbf{G}_k$

$$Q((f \circ \phi)', (f \circ \phi)'', \dots, (f \circ \phi)^{(k)}) = \phi'(0)^m Q(f', f'', \dots, f^{(k)}).$$

- Q is homogeneous of weighted degree m under the $\mathbb{C}^*$ action iff

$$Q(\lambda f', \lambda^2 f'', \ldots \lambda^k f^{(k)}) = \lambda^m Q(f', f'', \ldots f^{(k)}).$$

Definition

- *(Green-Griffiths '78)*
  $E_{k,m}^{GG}$ *is the sheaf of algebraic differential operators of order k and weighted degree m.*
- *(Demailly, '95)*
  The bundle of invariant jet differentials of order k and weighted degree m is the subbundle $E_{k,m} \subset E_{k,m}^{GG}$, whose elements are invariant under arbitrary changes of parametrization, i.e. for $\phi \in \mathbf{G}_k$

$$Q((f \circ \phi)', (f \circ \phi)'', \ldots, (f \circ \phi)^{(k)}) = \phi'(0)^m Q(f', f'', \ldots, f^{(k)}).$$

Therefore:

$$\oplus_m (E_{k,m})_x = \oplus_m (E_{k,m}^{GG})_x^{\mathbb{U}} = \mathcal{O}((J_k X)_x)^{\mathbf{G}_k} = \mathcal{O}(J_k(1,n)/\mathbf{G}_k)$$

Applying our previous contstruction fibrewise we get

Proposition

1. *The quotient* $J_kX/\mathbf{G}_k$ *has the structure of a locally trivial bundle over* X*, and there is a holomorphic embedding*

$$\phi^{\mathbb{P}} : J_kX/\mathbf{G}_k \hookrightarrow \mathbb{P}(\wedge^k(T_X^* \oplus \mathrm{Sym}^2(T_X^*) \oplus \ldots \oplus \mathrm{Sym}^k(T_X^*))$$

Applying our previous contstruction fibrewise we get

Proposition

1. *The quotient* $J_kX/\mathbf{G}_k$ *has the structure of a locally trivial bundle over* X*, and there is a holomorphic embedding*

$$\phi^{\mathbb{P}} : J_kX/\mathbf{G}_k \hookrightarrow \mathbb{P}(\wedge^k(T_X^* \oplus \mathrm{Sym}^2(T_X^*) \oplus \ldots \oplus \mathrm{Sym}^k(T_X^*))$$

The fibrewise closure of the image $\mathcal{X}_k = \overline{\mathrm{im}\phi^{\mathbb{P}}}$ *is a relative compactification of* $J_k(T_X^*)/\mathbf{G}_k$ *over* X*.*

Applying our previous contstruction fibrewise we get

Proposition

1. *The quotient $J_kX/\mathbf{G}_k$ has the structure of a locally trivial bundle over X, and there is a holomorphic embedding*

$$\phi^{\mathbb{P}} : J_kX/\mathbf{G}_k \hookrightarrow \mathbb{P}(\wedge^k(T_X^* \oplus \mathrm{Sym}^2(T_X^*) \oplus \ldots \oplus \mathrm{Sym}^k(T_X^*))$$

The fibrewise closure of the image $\mathcal{X}_k = \overline{\mathrm{im}\phi^{\mathbb{P}}}$ is a relative compactification of $J_k(T_X^*)/\mathbf{G}_k$ over X.

2. $$(\pi_k)_*\mathcal{O}_{\mathcal{X}_k}(m) = \mathcal{O}(E_{k,m\binom{k+1}{2}})$$

where $\pi_k : \mathbb{P}(\wedge^k(T_X^* \oplus \mathrm{Sym}^2(T_X^*) \oplus \ldots \oplus \mathrm{Sym}^k(T_X^*))) \to X$ is the projection.

Theorem (Fundamental vanishing theorem)

(Green-Griffiths '78, Demailly '95, Siu '96) Let $P \in H^0(X, E_{k,m} \otimes \mathcal{O}(-A))$ *be a global algebraic differential operator whose coefficients vanish on some ample divisor A. Then for any* $f : \mathbb{C} \to X$, $P(f_{[k]}(\mathbb{C})) \equiv 0$.*(note* $f_{[k]}(\mathbb{C}) \subset J_k X$*)*

Theorem (Fundamental vanishing theorem)

(Green-Griffiths '78, Demailly '95, Siu '96) Let $P \in H^0(X, E_{k,m} \otimes \mathcal{O}(-A))$ *be a global algebraic differential operator whose coefficients vanish on some ample divisor A. Then for any* $f : \mathbb{C} \to X$, $P(f_{[k]}(\mathbb{C})) \equiv 0$*.(note* $f_{[k]}(\mathbb{C}) \subset J_k X$*)*

Corollary

1. *Let* σ *be a nonzero element of*

$$H^0(\mathcal{X}_k, \mathcal{O}_{\mathcal{X}_k}(m) \otimes \pi^* \mathcal{O}(-A)) \simeq H^0(X, E_{k,m\binom{k+1}{2}} \otimes \mathcal{O}(-A)),$$

then $f_{[k]}(\mathbb{C}) \subset Z_\sigma$*, where* $Z_\sigma \subset \mathcal{X}_d$ *is the zero divisor.*

Theorem (Fundamental vanishing theorem)

(Green-Griffiths '78, Demailly '95, Siu '96) Let $P \in H^0(X, E_{k,m} \otimes \mathcal{O}(-A))$ *be a global algebraic differential operator whose coefficients vanish on some ample divisor A. Then for any* $f : \mathbb{C} \to X$, $P(f_{[k]}(\mathbb{C})) \equiv 0$. *(note* $f_{[k]}(\mathbb{C}) \subset J_k X$ *)*

Corollary

1. *Let* σ *be a nonzero element of*

$$H^0(\mathcal{X}_k, \mathcal{O}_{\mathcal{X}_k}(m) \otimes \pi^* \mathcal{O}(-A)) \simeq H^0(X, E_{k,m\binom{k+1}{2}} \otimes \mathcal{O}(-A)),$$

   then $f_{[k]}(\mathbb{C}) \subset Z_\sigma$, *where* $Z_\sigma \subset \mathcal{X}_d$ *is the zero divisor.*

2. *If* σ_j *is a basis of of sections then the image* $f(\mathbb{C})$ *lies in* $Y = \pi_k(\bigcap Z_{P_j})$, *hence GGL holds if there are enough independent differential equations so that* $Y = \pi_k(\bigcap(Z_{P_j})) \subsetneq X$.

Theorem (Fundamental vanishing theorem)

(Green-Griffiths '78, Demailly '95, Siu '96) Let $P \in H^0(X, E_{k,m} \otimes \mathcal{O}(-A))$ *be a global algebraic differential operator whose coefficients vanish on some ample divisor A. Then for any* $f : \mathbb{C} \to X$, $P(f_{[k]}(\mathbb{C})) \equiv 0$.*(note* $f_{[k]}(\mathbb{C}) \subset J_k X$*)*

Corollary

1. *Let* σ *be a nonzero element of*

$$H^0(\mathcal{X}_k, \mathcal{O}_{\mathcal{X}_k}(m) \otimes \pi^*\mathcal{O}(-A)) \simeq H^0(X, E_{k,m\binom{k+1}{2}} \otimes \mathcal{O}(-A)),$$

then $f_{[k]}(\mathbb{C}) \subset Z_\sigma$, *where* $Z_\sigma \subset \mathcal{X}_d$ *is the zero divisor.*

2. *If* σ_j *is a basis of of sections then the image* $f(\mathbb{C})$ *lies in* $Y = \pi_k(\bigcap Z_{P_j})$, *hence GGL holds if there are enough independent differential equations so that* $Y = \pi_k(\bigcap(Z_{P_j})) \subsetneq X$.

It is crucial to control in a more precise way the order of vanishing of these differential operators along the ample divisor. Thus, we need here a slightly different theorem.

Theorem (DMR, 2009)

Assume that $n = k$, *and there exist a* $\delta = \delta(n) > 0$ *and* $D = D(n, \delta)$ *such that*

$$H^0(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n}(m) \otimes \pi^* K_X^{-\delta m}) \simeq H^0(X, E_{n, m\binom{n+1}{2}} T_X^* \otimes K_X^{-\delta m}) \neq 0$$

Theorem (DMR, 2009)

Assume that $n = k$, *and there exist a* $\delta = \delta(n) > 0$ *and* $D = D(n, \delta)$ *such that*

$$H^0(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n}(m) \otimes \pi^* K_X^{-\delta m}) \simeq H^0(X, E_{n,m\binom{n+1}{2}} T_X^* \otimes K_X^{-\delta m}) \neq 0$$

whenever $\deg(X) > D(n, \delta)$ *provided that* $m > m_{D,\delta,n}$ *is large enough. Then GGL holds for*

$$\deg(X) \geq \max(D(n, \delta), \frac{n^2 + 2n}{\delta} + n + 2).$$

Theorem (DMR, 2009)

Assume that $n = k$, and there exist a $\delta = \delta(n) > 0$ and $D = D(n, \delta)$ such that

$$H^0(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n}(m) \otimes \pi^* K_X^{-\delta m}) \simeq H^0(X, E_{n,m\binom{n+1}{2}} T_X^* \otimes K_X^{-\delta m}) \neq 0$$

whenever $\deg(X) > D(n, \delta)$ provided that $m > m_{D,\delta,n}$ is large enough. Then GGL holds for

$$\deg(X) \geq \max(D(n, \delta), \frac{n^2 + 2n}{\delta} + n + 2).$$

We use the algebraic Morse inequalities of Demailly/Trapani. Let $L \to X$ be a holomorphic line bundle given as

$$L = F \otimes G^{-1}, F, G \text{ nef bundles.}$$

Theorem (DMR, 2009)

Assume that $n = k$, *and there exist a* $\delta = \delta(n) > 0$ *and* $D = D(n, \delta)$ *such that*

$$H^0(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n}(m) \otimes \pi^* K_X^{-\delta m}) \simeq H^0(X, E_{n,m\binom{n+1}{2}} T_X^* \otimes K_X^{-\delta m}) \neq 0$$

whenever $\deg(X) > D(n, \delta)$ *provided that* $m > m_{D,\delta,n}$ *is large enough. Then GGL holds for*

$$\deg(X) \geq \max(D(n, \delta), \frac{n^2 + 2n}{\delta} + n + 2).$$

We use the algebraic Morse inequalities of Demailly/Trapani. Let $L \to X$ be a holomorphic line bundle given as

$$L = F \otimes G^{-1}, F, G \text{ nef bundles.}$$

Then

$$\sum_{j=0}^{q} (-1)^{q-j} h^j(X, L^{\otimes m} \otimes E) \leq r \frac{m^n}{n!} \sum_{j=0}^{q} (-1)^{q-j} \binom{n}{j} F^{n-j} \cdot G^j + o(m^n).$$

Theorem (DMR, 2009)

Assume that $n = k$, and there exist a $\delta = \delta(n) > 0$ and $D = D(n, \delta)$ such that

$$H^0(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n}(m) \otimes \pi^* K_X^{-\delta m}) \simeq H^0(X, E_{n,m\binom{n+1}{2}} T_X^* \otimes K_X^{-\delta m}) \neq 0$$

whenever $\deg(X) > D(n, \delta)$ *provided that* $m > m_{D,\delta,n}$ *is large enough. Then GGL holds for*

$$\deg(X) \geq \max(D(n, \delta), \frac{n^2 + 2n}{\delta} + n + 2).$$

We use the algebraic Morse inequalities of Demailly/Trapani. Let $L \to X$ be a holomorphic line bundle given as

$$L = F \otimes G^{-1}, F, G \text{ nef bundles.}$$

Then

$$\sum_{j=0}^{q} (-1)^{q-j} h^j(X, L^{\otimes m} \otimes E) \leq r \frac{m^n}{n!} \sum_{j=0}^{q} (-1)^{q-j} \binom{n}{j} F^{n-j} \cdot G^j + o(m^n).$$

$q = 2$ asserts

$$F^n - nF^{n-1}G > 0 \Rightarrow H^0(L^{\otimes m}) \neq 0 \text{ for } m \gg 0.$$

Proposition (B-Kirwan)

F and G are nef bundles in the following equality:

$$\underbrace{\mathcal{O}_{\mathcal{X}_n}(1) \otimes \pi^* K_X^{-\delta\binom{n+1}{2}}}_{L} = \underbrace{(\mathcal{O}_{\mathcal{X}_n}(1) \otimes \pi^* \mathcal{O}_X(2n^2))}_{F} \otimes$$

$$\otimes \underbrace{(\pi^* \mathcal{O}_X(2n^2) \otimes \pi^* K_X^{\delta\binom{n+1}{2}})^{-1}}_{G}.$$

Proposition (B-Kirwan)

F and G are nef bundles in the following equality:

$$\underbrace{\mathcal{O}_{\mathcal{X}_n}(1) \otimes \pi^* K_X^{-\delta\binom{n+1}{2}}}_{L} = \underbrace{(\mathcal{O}_{\mathcal{X}_n}(1) \otimes \pi^* \mathcal{O}_X(2n^2))}_{F} \otimes$$

$$\otimes \underbrace{(\pi^* \mathcal{O}_X(2n^2) \otimes \pi^* K_X^{\delta\binom{n+1}{2}})^{-1}}_{G}.$$

- $h = c_1(\mathcal{O}_X(1)), c_1(K_X) = -c_1(X) = (d-n-2)h$, and $\mathcal{O}_{\mathcal{X}_n}(1) = \det \tau$, where $\tau \to \mathcal{X}$ is the tautological n-bundle.

Proposition (B-Kirwan)

F and G are nef bundles in the following equality:

$$\underbrace{\mathcal{O}_{\mathcal{X}_n}(1) \otimes \pi^* K_X^{-\delta\binom{n+1}{2}}}_{L} = \underbrace{(\mathcal{O}_{\mathcal{X}_n}(1) \otimes \pi^* \mathcal{O}_X(2n^2))}_{F} \otimes$$

$$\otimes \underbrace{(\pi^* \mathcal{O}_X(2n^2) \otimes \pi^* K_X^{\delta\binom{n+1}{2}})^{-1}}_{G}.$$

- $h = c_1(\mathcal{O}_X(1)), c_1(K_X) = -c_1(X) = (d-n-2)h$, and $\mathcal{O}_{\mathcal{X}_n}(1) = \det \tau$, where $\tau \to \mathcal{X}$ is the tautological n-bundle.
- $\dim(\mathcal{X}_n) = n^2$, and we want to compute the integral

$$\int_{\mathcal{X}_n} (c_1(\det \tau) + 2n^2 \pi^* h)^{n^2} -$$

$$- n^2 (c_1(\det \tau) + 2n^2 \pi^* h)^{n^2-1} (2n^2 \pi^* h + \delta \binom{n+1}{2} (d-n-2)h)$$

Applying the double fibration model, and after proving the stronger vanishing property we get

Residue formula for the Demailly intersection number

$$I = \int_X \mathrm{Res}_{\mathbf{z}=\infty} \frac{\prod_{i<j}(z_i - z_j)\, \mathcal{Q}_d(z_1 \dots z_n) R(\mathbf{z}, h, d, \delta)}{\prod_{1\leq i+j\leq l\leq n}(z_i + z_j - z_l)(z_1 \dots z_n)^n} \cdot \prod_{l=1}^{n}\left(1 + \frac{dh}{z_l}\right) \prod_{l=1}^{n}\left(1 - \frac{h}{z_l} + \frac{h^2}{z_l^2} - \dots\right)^{n+2}$$

where

Applying the double fibration model, and after proving the stronger vanishing property we get

Residue formula for the Demailly intersection number

$$I = \int_X \operatorname{Res}_{\mathbf{z}=\infty} \frac{\prod_{i<j}(z_i - z_j)\, \mathcal{Q}_d(z_1 \dots z_n) R(\mathbf{z}, h, d, \delta)}{\prod_{1 \le i+j \le l \le n}(z_i + z_j - z_l)(z_1 \dots z_n)^n} \cdot \\ \cdot \prod_{l=1}^{n}\left(1 + \frac{dh}{z_l}\right) \prod_{l=1}^{n}\left(1 - \frac{h}{z_l} + \frac{h^2}{z_l^2} - \dots\right)^{n+2}$$

where

$$R(\mathbf{z}, h, d, \delta) = (-z_1 - \dots - z_n + 2n^2 h)^{n^2} - \\ - n^2(-z_1 - \dots - z_n + 2n^2 h)^{n^2-1}\left(2n^2 h + \delta \binom{n+1}{2}(d-n-2)h\right)$$

Analysis of the formula

Analysis of the formula

- The iterated residue is the coefficient of $\frac{1}{z_1 \dots z_n}$, and has the form $h^n p(d, n, \delta)$.

Analysis of the formula

- The iterated residue is the coefficient of $\frac{1}{z_1 \dots z_n}$, and has the form $h^n p(d, n, \delta)$.
- Integration on X is the substitution $h^n = d$, so the result is $dp(d, n, \delta)$

Analysis of the formula

- The iterated residue is the coefficient of $\frac{1}{z_1 \dots z_n}$, and has the form $h^n p(d, n, \delta)$.
- Integration on X is the substitution $h^n = d$, so the result is $dp(d, n, \delta)$
- $p(n, d, \delta) = a_n(n, \delta) d^n + \ldots + a_0(n, \delta)$ is a degree-n polynomial in $d = \deg(X)$.

Analysis of the formula

- The iterated residue is the coefficient of $\frac{1}{z_1 \ldots z_n}$, and has the form $h^n p(d, n, \delta)$.
- Integration on X is the substitution $h^n = d$, so the result is $dp(d, n, \delta)$
- $p(n, d, \delta) = a_n(n, \delta)d^n + \ldots + a_0(n, \delta)$ is a degree-n polynomial in $d = \deg(X)$.
- The leading coefficient is

$$a_n(n, \delta) = \left(1 - n^2 \binom{n+1}{2} \delta\right) \Theta(n),$$

where

$$\Theta(n) = \mathrm{coeff}_1 \frac{Q(\mathbf{z}) \prod_{i<j} (z_i - z_j)(z_1 + \ldots + z_n)^{n^2}}{\prod_{i+j \le l \le n} (z_i + z_j - z_l)(z_1 \ldots z_n)^n}$$

Corollary

For $\delta < \frac{2}{n^3(n+1)}$ the leading coefficient of the Demailly intersection number is positive.

More information about $Q(\mathbf{z})$ is needed!

Theorem (B, 2009)

Rimanyi conjecture for Thom polynomials of A_n singularities $\Rightarrow$ GGL is true for $d = \deg(X) > n^6$.

The given iterated residue formula is suitable to compute intersection numbers. Define

$$\chi(X, E_{k,m}T_X^*) = \sum_{i=0}^{n} (-1)^i \dim H^i(X, E_{k,m}T_X^*)$$

The given iterated residue formula is suitable to compute intersection numbers. Define

$$\chi(X, E_{k,m}T_X^*) = \sum_{i=0}^{n} (-1)^i \dim H^i(X, E_{k,m}T_X^*)$$

Well-known:

$$\chi(X, E_{k,m}) = \int_X [ch(E_{k,m}) \cdot Td(T_X)]_n$$

The given iterated residue formula is suitable to compute intersection numbers. Define

$$\chi(X, E_{k,m}T_X^*) = \sum_{i=0}^{n}(-1)^i \dim H^i(X, E_{k,m}T_X^*)$$

Well-known:

$$\chi(X, E_{k,m}) = \int_X [ch(E_{k,m}) \cdot Td(T_X)]_n$$

Theorem (Iterated residue formula for the Euler-characteristics)

$$\chi(X, \pi_* \mathcal{O}_{X_n}(m)) =$$
$$\int_X \operatorname{Res}_{\mathbf{z}=\infty} \frac{\prod_{i<j}(z_i - z_j)\, \mathcal{Q}_n(z_1 \ldots z_n) ch(\mathcal{O}_{X_n}(m)) Td(T_X)}{\prod_{1 \le i+j \le l \le n}(z_i + z_j - z_l)(z_1 \ldots z_n)^n} \cdot$$
$$\cdot \prod_{l=1}^{n}\left(1 + \frac{dh}{z_l}\right) \prod_{l=1}^{n}\left(1 - \frac{h}{z_l} + \frac{h^2}{z_l^2} - \ldots\right)^{n+2}$$

The given iterated residue formula is suitable to compute intersection numbers. Define

$$\chi(X, E_{k,m}T_X^*) = \sum_{i=0}^{n}(-1)^i \dim H^i(X, E_{k,m}T_X^*)$$

Well-known:

$$\chi(X, E_{k,m}) = \int_X [ch(E_{k,m}) \cdot Td(T_X)]_n$$

Theorem (Iterated residue formula for the Euler-characteristics)

$$\chi(X, \pi_* \mathcal{O}_{X_n}(m)) = \int_X \operatorname{Res}_{z=\infty} \frac{\prod_{i<j}(z_i - z_j)\, \mathcal{Q}_n(z_1 \ldots z_n) ch(\mathcal{O}_{X_n}(m)) Td(T_X)}{\prod_{1\le i+j\le l\le n}(z_i + z_j - z_l)(z_1 \ldots z_n)^n} \cdot$$

$$\cdot \prod_{l=1}^{n}\left(1 + \frac{dh}{z_l}\right) \prod_{l=1}^{n}\left(1 - \frac{h}{z_l} + \frac{h^2}{z_l^2} - \ldots\right)^{n+2}$$

where

$$ch(\mathcal{O}_{X_n}(1)) = e^{m(z_1+\ldots+z_n)},$$

The given iterated residue formula is suitable to compute intersection numbers. Define

$$\chi(X, E_{k,m}T_X^*) = \sum_{i=0}^{n}(-1)^i \dim H^i(X, E_{k,m}T_X^*)$$

Well-known:

$$\chi(X, E_{k,m}) = \int_X [ch(E_{k,m}) \cdot Td(T_X)]_n$$

Theorem (Iterated residue formula for the Euler-characteristics)

$$\chi(X, \pi_*\mathcal{O}_{X_n}(m)) =$$

$$\int_X \operatorname{Res}_{\mathbf{z}=\infty} \frac{\prod_{i<j}(z_i - z_j)\,\mathcal{Q}_n(z_1 \ldots z_n) ch(\mathcal{O}_{X_n}(m))\, Td(T_X)}{\prod_{1\le i+j\le l\le n}(z_i + z_j - z_l)(z_1 \ldots z_n)^n} \cdot$$

$$\cdot \prod_{l=1}^{n}\left(1 + \frac{dh}{z_l}\right) \prod_{l=1}^{n}\left(1 - \frac{h}{z_l} + \frac{h^2}{z_l^2} - \ldots\right)^{n+2}$$

where

$$ch(\mathcal{O}_{X_n}(1)) = e^{m(z_1+\ldots+z_n)}, \;\; Td(T_X) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \ldots$$

Appendix

Localization on the Demailly-tower gives an other iterated residue formula:

Theorem

$$\chi(X,\pi_*\mathcal{O}_{X_k}(\mathbf{a})) =$$

$$= (-1)^k \int_X \mathrm{Res}_{\mathbf{z}=\infty} \frac{\prod_{2\le t_1\le t_2\le k}(z_{t_1}+z_{t_1+1}+\ldots+z_{t_2})ch(\mathcal{O}_{X_k}(\mathbf{a}))\cdot Td(T_X)}{\prod_{1\le s_1<s_2\le k}(-z_{s_1}+z_{s_1+1}+\ldots+z_{s_2})\prod_{j=1}^k(z_1+\ldots+z_j)^n}$$

$$\prod_{j=1}^{k} \frac{1+\frac{dh}{z_1+\ldots+z_j}}{\left(1+\frac{h}{z_1+\ldots+z_j}\right)^{n+2}} d\mathbf{z}$$

Appendix

Localization on the Demailly-tower gives an other iterated residue formula:

Theorem

$$\chi(X, \pi_* \mathcal{O}_{X_k}(\mathbf{a})) =$$

$$= (-1)^k \int_X \operatorname{Res}_{\mathbf{z}=\infty} \frac{\prod_{2 \le t_1 \le t_2 \le k} (z_{t_1} + z_{t_1+1} + \ldots + z_{t_2}) ch(\mathcal{O}_{X_k}(\mathbf{a})) \cdot Td(T_X)}{\prod_{1 \le s_1 < s_2 \le k} (-z_{s_1} + z_{s_1+1} + \ldots + z_{s_2}) \prod_{j=1}^{k} (z_1 + \ldots + z_j)^n}$$

$$\prod_{j=1}^{k} \frac{1 + \frac{dh}{z_1 + \ldots + z_j}}{\left(1 + \frac{h}{z_1 + \ldots + z_j}\right)^{n+2}} d\mathbf{z}$$

where

$$ch(\mathcal{O}_{X_k}(\mathbf{a})) = e^{a_1 z_1 + \ldots + a_k z_k)},$$

Appendix

Localization on the Demailly-tower gives an other iterated residue formula:

Theorem

$$\chi(X, \pi_* \mathcal{O}_{X_k}(\mathbf{a})) =$$

$$= (-1)^k \int_X \operatorname{Res}_{\mathbf{z}=\infty} \frac{\prod_{2 \le t_1 \le t_2 \le k} (z_{t_1} + z_{t_1+1} + \ldots + z_{t_2}) ch(\mathcal{O}_{X_k}(\mathbf{a})) \cdot Td(T_X)}{\prod_{1 \le s_1 < s_2 \le k} (-z_{s_1} + z_{s_1+1} + \ldots + z_{s_2}) \prod_{j=1}^k (z_1 + \ldots + z_j)^n}$$

$$\prod_{j=1}^k \frac{1 + \frac{dh}{z_1 + \ldots + z_j}}{\left(1 + \frac{h}{z_1 + \ldots + z_j}\right)^{n+2}} d\mathbf{z}$$

where

$$ch(\mathcal{O}_{X_k}(\mathbf{a})) = e^{a_1 z_1 + \ldots + a_k z_k)}, \quad Td(T_X) = 1 + \frac{1}{2} c_1 + \frac{1}{12}(c_1^2 + c_2) + \ldots$$