Multidegrees of Singularities and Nonreductive Quotients

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Chapter 1

Introduction

This thesis gives a compact iterated residue formula for the multidegrees of A_n singularities. Multidegrees are generalisations of ordinary degree of algebraic geometry in the equivariant world; these are equivariant cohomology classes, and they coincide with equivariant Poincaré duals in topology.

More precisely, if we are given a complex representation V of a complex semisimple Lie group G and a G-invariant quasiprojective subvariety $\eta \subset V$, one can associate a multidegree $\mathrm{mdeg}[\eta, V] \in H^*_G(V) = H^*(BG)$ in the equivariant cohomology ring of V. This is a homogeneous polynomial in the variables associated to the generators of the Lie algebra of the maximal torus in G, whose degree equals to the codimension of (the maximal dimensional component of) η in V. As expected, one can derive from this the ordinary degree of η. However, the G-action is an extra structure on η, and it provides more information stored in the multidegree.

When $\eta \subset V$ is a quasi-projective variety, the characteristic properties of the equivariant dual can serve as axioms, and we will use the axiomatic viewpoint as the definition of multidegree. They are also applicable for subschemes of V, extending the definition to the non-reduced case.

Since the multidegree is a birational invariant, the definition naturally extends to constructible subsets of V, by defining the multidegree as the multidegree of the Zariski-closure. Moreover, only the maximal dimensional components of a quasi-projective variety contribute to the multidegree, allowing us to focus only on these components.

The first purely algebraic definition was introduced by Joseph in [27]. He introduced multidegree as the polynomial governing the asymptotic behavior of the character of the algebra of functions on the subvariety. Rossmann in [45] gives an extension of this definition to analytic subvarieties of V via an integral-limit representation.

An alternative term for these polynomials in the literature is equivariant Poincaré dual, or equivariant multiplicity. This goes back to the results of Vergne (cf. [48]), who gives an explicit form of the equivariant Thom class of V, and proves that the multidegree of η is equal to the integral of the Thom class on η. Moreover, for any equivariantly closed differential form μ with compact support, and $\Sigma \subset V$ subvariety, we have

$$\int_\Sigma \mu = \int_V \mathrm{eP}[\Sigma] \cdot \mu.$$

This formula serves as the motivation for the term equivariant Poincaré dual.

This thesis focuses on the rather special case when $V = \{(\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)\}$ is the vector space of polynomial map germs fixing the origo. Different parametrizations of $\mathbb{C}^n$ and $\mathbb{C}^k$ do not change the map, so we are concerned with $\mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$-invariant subsets of V under the left-right symmetry action $(A, B) \cdot f = AfB^{-1}$ for $(A, B) \in \mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$, $f \in V$.

Left-right symmetry orbits are called *singularities*. They are invariant under the action of the Lie-subgroup $\mathrm{GL}_k \times \mathrm{GL}_n \subset \mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$, and computing the multidegrees of singularities is a fundamental problem of global singularity theory. They are also called *Thom polynomials* since the pioneering work of Thom in the 1950's.

The main difficulty in computing Thom polynomials is the highly nontrivial symmetry group $\mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$. There are three known efficient methods for calculating these polynomials in the literature. The first, classical way, is the method of resolutions – this is also effective at computations of other invariants of singular varieties. The second is based on an idea of R. Rimányi and called the method of restriction equations. A standard reference of this is [16], and a nice summary of this method and its applications is [29].

This thesis translates the problem from singularity theory and topology into (almost) pure algebra. Using an algebraic model of Gaffney [21] and Porteous [40] for A_n singularities allows us to use methods in algebraic geometry and algebraic topology, especially localisation methods for studying group actions and their quotient spaces. The heart of this thesis is to handle the problem of quotienting by the diffeomorphism group. Since this is a non-reductive group, the standard methods of Mumford's Geometric Invariant Theory are not applicable. Instead of that, we give a new compactification method by embedding the quotient into a compact Schubert variety, and then using abelian localisation of Berline and Vergne on the closure. Finally, this can be turned into an iterated residue formula.

Overview

We begin with a quick summary of the notions of global singularity theory and the theory of Thom polynomials. For a more detailed review we refer the reader to [1, 28].

Consider a holomorphic map $f : N \to K$ between two complex manifolds, of dimensions $n \leq k$. We say that $p \in N$ is a *singular* point of f (or f has a singularity at p) if the rank of the differential $df_p : \mathrm{T}_pN \to \mathrm{T}_{f(p)}K$ less than n.

Often, the topology of the situation forces f to have singularities at some points of N. To introduce a finer classification of singular points, choose local coordinates near $p \in N$ and $f(p) \in K$, and consider the resulting map-germ $\check{f}_p : (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$, which may be thought of as a sequence of k power series in n variables without constant terms. The group of infinitesimal local coordinate changes $\mathrm{Diff}(\mathbb{C}^k) \times \mathrm{Diff}(\mathbb{C}^n)$ acts on the linear space $\mathcal{J}(n,k)$ of all such map-germs, and thus a reasonable notion of a *singularity* is a $\mathrm{Diff}(\mathbb{C}^k) \times \mathrm{Diff}(\mathbb{C}^n)$-orbit – or $\mathrm{Diff}(\mathbb{C}^k) \times \mathrm{Diff}(\mathbb{C}^n)$-invariant subset – in $\mathcal{J}(n,k)$. Given such orbit $O \subset \mathcal{J}(n,k)$, for any holomorphic map $f : N \to K$ we can define the set

$$Z_O[f] = \{p \in N;\ \check{f}_p \in O\},$$

which is an analytic subvariety of N, independent of any coordinate choices. Assuming N is compact and f is sufficiently generic this subvariety will have a Poincaré dual class $\alpha_O[f] \in H^*(N, \mathbb{Z})$; one of the fundamental problems of global singularity theory is the computation of this class. This is indeed useful: for example, if we can prove that $\alpha_O[f]$ does not vanish, then we can guarantee that the singularity O occurs at some point of the map f. The existence of a sufficient generic map can not be guaranteed, see Definition 2.2.7 and Remark 2.2.9.

A basic principle, introduced by René Thom, is that to every singularity O one can associate a bivariant characteristic class τ_O, which, when evaluated on the pair (TN, f^*TK) produces the Poincaré dual class $\alpha_f[O]$, assuming that f is sufficiently generic. Originally, Thom defined this class in the parallel theory of smooth maps between real manifolds [47, 25]. To my best knowledge, this result was not published by Thom himself, this is the reason for the terminology 'principle' we use, instead of 'theorem'. The first proof is due to Haefliger and Kosinski in [25], and their comment to the result is the following:
"Nous démontrerons le théorème suivant qui nous a été communiqué par R. Thom dans uns lettre."

They prove the principle for smooth maps between real, smooth manifolds. The corresponding statement for complex manifolds and holomorphic maps between them is proved in the PhD dissertation of Damon, see [10], but we give a new proof in §2.2.6.

Let us state this principle in more concrete terms. Denote by $\mathbb{C}[\boldsymbol{\lambda}, \boldsymbol{\theta}]^{\mathcal{S}_n \times \mathcal{S}_k}$ the space of those polynomials in the variables $(\lambda_1, \ldots, \lambda_n, \theta_1, \ldots, \theta_k)$ which are invariant under the permutations of the λs and the permutations of the θs. According to the structure theorem of symmetric polynomials $\mathbb{C}[\boldsymbol{\lambda}, \boldsymbol{\theta}]^{\mathcal{S}_n \times \mathcal{S}_k}$ itself is a polynomial ring in the elementary symmetric polynomials:

$$\mathbb{C}[\boldsymbol{\lambda}, \boldsymbol{\theta}]^{\mathcal{S}_n \times \mathcal{S}_k} = \mathbb{C}[c_1(\boldsymbol{\lambda}), \ldots, c_n(\boldsymbol{\lambda}), c_1(\boldsymbol{\theta}), \ldots, c_k(\boldsymbol{\theta})].$$

By the Chern-Weil map, any polynomial $b \in \mathbb{C}[\boldsymbol{\lambda}, \boldsymbol{\theta}]^{\mathcal{S}_n \times \mathcal{S}_k}$, and every pair of bundles (E, F) over N of ranks n and k, respectively, produces a characteristic class $b(E, F) \in H^*(N, \mathbb{C})$. Now Thom's principle reads:
For every $\mathrm{Diff}(\mathbb{C}^k) \times \mathrm{Diff}(\mathbb{C}^n)$*-orbit* O *of codimension* j *in* $\mathcal{J}(n,k)$*, there exists a*

homogeneous polynomial $\mathrm{Tp}_O \in \mathbb{C}[\boldsymbol{\lambda},\boldsymbol{\theta}]^{\mathcal{S}_n\times\mathcal{S}_k}$ *of degree* j *such that for an arbitrary, sufficiently generic map* $f : N \to K$*, the cycle* $Z_f[O] \subset N$ *is Poincaré dual to the characteristic class* $\mathrm{Tp}_O(TN, f^*TK)$*.*

The existence of a sufficiently generic map is not guaranteed, and in this case the geometric meaning of the Thom polynomial is not clear. For the definition of sufficiently generic, see Definition 2.2.7.

One of the consequences of this principle is that the class $\alpha_O[f]$ depends only on the homotopy class of f.

The statement may be formulated for more general $\mathrm{Diff}(\mathbb{C}^k)\times\mathrm{Diff}(\mathbb{C}^n)$-invariant subsets O as well; the corresponding polynomial Tp_O is called the *Thom polynomial* of O. We will see in Chapter 2 that Tp_O is the multidegree of O in $\mathcal{J}(n,k)$. The computation of these polynomials is a central problem of singularity theory.

The structure of the $\mathrm{Diff}(\mathbb{C}^k)\times\mathrm{Diff}(\mathbb{C}^n)$-action on $\mathcal{J}(n,k)$ is rather complicated; even the parametrization of the orbits is difficult. There is, however, a simple invariant on the space of orbits: to each map-germ $\check{f} : (\mathbb{C}^n,0) \to (\mathbb{C}^k,0)$, we can associate the finite-dimensional nilpotent algebra $A_{\check{f}}$ over $\mathbb{C}[[x_1,\dots,x_n]]$, defined as the quotient of the algebra of power series $\mathbb{C}[[x_1,\dots,,x_n]]$ by the pull-back subalgebra $\check{f}^*(\mathbb{C}[[y_1,\dots,,y_k]])$. This algebra $A_{\check{f}}$ is trivial if the map-germ $\check{f}$ is non-singular, and it does not change along a $\mathrm{Diff}(\mathbb{C}^k)\times\mathrm{Diff}(\mathbb{C}^n)$-orbit, in other words it can be defined without choosing coordinates.

According to Thom's principle, this last observation means that to each finite-dimensional nilpotent algebra A, and pair of integers (n,k) one can associate a doubly symmetric polynomial $\mathrm{Tp}_A^{n\to k} \in \mathbb{C}[\boldsymbol{\lambda},\boldsymbol{\theta}]^{\mathcal{S}_n\times\mathcal{S}_k}$; in the sense described above, this will serve as a universal Poincaré dual of points with nilpotent algebra A in the source spaces of holomorphic maps.

Again, it turns out in Chapter 2, that $\mathrm{Tp}_A^{n\to k}$ is the multidegree of the variety

$$\Sigma(A) = \left\{\check{f} \in \mathcal{J}(n,k); A_{\check{f}} \simeq A\right\}$$

in the complex vector space $\mathcal{J}(n,k)$.

The computation of this family of polynomials for general nilpontent algebras is a difficult problem. A few structural statements are known, however (cf. Chapter 2 for more details).

First, the polynomial $\mathrm{Tp}_A^{n\to k}$ lies in the subring of $\mathbb{C}[\boldsymbol{\lambda},\boldsymbol{\theta}]^{\mathcal{S}_n\times\mathcal{S}_k}$ generated by the relative Chern classes (cf. [10, 42]) defined by the generating series

$$1 + c_1q + c_2q^2 + \cdots = \frac{\prod_{j=1}^{k}(1+\theta_j q)}{\prod_{i=1}^{n}(1+\lambda_i q)}.$$

Next, the Thom polynomial, expressed in terms of these relative Chern classes, only depends on the codimension $j = k-n$. More precisely, there is a unique polynomial $\mathrm{TD}_A^c(c_1,c_2,\dots)$ such that

$$\mathrm{Tp}_A^{n\to k}(\boldsymbol{\lambda},\boldsymbol{\theta}) = \mathrm{TD}_A^{k-n}(c_1(\boldsymbol{\lambda},\boldsymbol{\theta}), c_2(\boldsymbol{\lambda},\boldsymbol{\theta}),\dots),,$$

as long as the relevant subset in $\mathcal{J}_d(k,n)$ has the right codimension.

Finally (cf. [17]), performing the substitution $c_i \mapsto c_{i-1}$ in the homogeneous part of TD_A^c of maximal degree produces TD_A^{c-1}.

In this thesis, we will concentrate on the so-called Morin singularities [35], which correspond to the situation when the algebra A is generated by a single element. The list of these algebras is simple: $A_d = t\mathbb{C}[t]/t^{d+1}$, $d = 1, 2, \ldots$.

The main result of this thesis is a compact iterated residue formula for the the Thom polynomial $\mathrm{Tp}_{A_d}^{n\to k}$ for arbitrary d, n and k. For simplicity of notation we will denote this polynomial by $\mathrm{Tp}_d^{n\to k}$, or sometimes simply by Tp_d, omitting the dependence on the parameters n and k.

This problem has a rich history. The case $d = 1$ is the classical formula of Porteous: $\mathrm{Tp}_1 = c_{k-n+1}$. The Thom polynomial in the $d = 2$ case was computed by Ronga in [42]. More recently, in [3], the authors proposed a formula for Tp_3; P. Pragacz has given a sketch of a proof for this conjecture [41]. Finally, using his method of restriction equations, Rimányi was able to treat the zero-codimension case [44] (cf. [21] for the case $d = 4$): he could compute $\mathrm{Tp}_d^{n\to n}$ for $d \le 8$.

Our approach combines the test-curve model of Porteous [40] and Gaffney [21] with localization techniques in equivariant cohomology [6, 45, 48].

The first observation is that the set $\Sigma(A_d)$ fibres equivariantly over a base space with respect to the action of a sub-torus of the diffeomorphism group. The base of this fibration is the quotient $\mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^d, \mathbb{C}^n)/H$ – which is a non-compact smooth algebraic variety – of the vector space of regular $n \times d$ matrices by a linear action of a non-reductive finite-dimensional subgroup H of the diffeomorphism group of $\mathbb{C}$. The main ingredient is a subtle compactification of this non-reductive quotient, which makes it possible to use a two-step localisation process:

1. First it is observed that $\mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^d, \mathbb{C}^n)/H$ fibres over the partial flag variety $\mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^d, \mathbb{C}^n)/B$, where $B \subset \mathrm{GL}_d$ is the Borel of upper triangular matrices, containing H. We transform the abelian localization on the flag manifold into an iterated residue formula, allowing attention to be restricted to the fibre over one distinguished fixed flag under the torus action.

2. We employ abelian localization on a subtle compactification of the fibre over the distinguished flag.

The result is a residue formula which is a sum of terms indexed by the fixed points in the compactified distinguished fibre. However something unexpected happens: all but one of these terms contribute zero to the result, so that all the relevant information is stored at one fixed point.

We obtain a formula which reduces the computation of $\mathrm{Tp}_d^{n\to k}$ to a certain problem of commutative algebra which only depends on d, namely the computation of the multidegree of a Borel-orbit in a vector space. Computation of multidegrees for toric varieties is a well-understood task (see [37]), but much less is known for Borel orbits. This problem is trivial for $d = 1, 2, 3$, hence we instantly recover essentially all known results. We also compute it for $d = 4, 5, 6$ explicitely in Chapter 5, where we also compute some part of these polynomials in general. However, the entire polynomial is not known for $d \geq 7$.

Organization of the thesis

The thesis is structured as follows: we describe the basic setup and notions of singularity theory in §2.2.1, essentially repeating the above construction using more formal notation. Next, in §2.2.2 we recall the notion of equivariant Poincaré duals, which is a convenient language in which one can describe Thom polynomials. We also present the localization formulas of Berline-Vergne [6] and Rossmann [45], which are crucial to our computations later. In Chapter 3 we develop a calculus localizing equivariant Poincaré duals by combining the localization principles with Vergne's integral formula for equivariant Poincaré duals. With these preparations, we proceed to describe the test curve model for Morin singularities in §4.4.1. The heart of our work is §4.4.2, where we reinterpret the model using a double fibration in a way which allows us to compactify our model space and apply the localization formulas. The following section, §4.4.3 is a rather straightforward application of the localization techniques of §2.2.2 to the double fibration constructed in §4.4.2. The resulting formula (4.3.24), in principle, reduces the computation of our Thom polynomials to a finite problem, but this formula is difficult to use for concrete calculations. Remarkably, however, the formula undergoes through several simplifications, which we explain in §4.4.4. At the end of §4.4.3, we summarize our constructions and results in a key diagram, which will hopefully orient the reader.

The simplifications bring us to our main result: Theorem 4.4.16 and formula (4.4.25). This formula is simple, but still contains an unknown quantity: a certain homogeneous polynomial $\widehat{Q}_d$ in d variables, which does not depend on n and k. This polynomial is the multidegree of the Borel-orbit $\widehat{\mathcal{O}}_d$ described in Theorem 4.4.16. The first few values of this polynomial are as follows:

$$\widehat{Q}_1 = \widehat{Q}_2 = \widehat{Q}_3 = 1,\ \widehat{Q}_4(z_1, z_2, z_3, z_4) = 2z_1 + z_2 - z_4.$$

The computation of $\widehat{Q}_d$ for general d is a finite but difficult problem. At the moment, we do not have an efficient algorithm for solving this problem. We discuss certain partial results in the final Chapter of this thesis; in §5.1 we give the computations for $d = 2, 3, 4, 5, 6$, and in §5.2 we prove for general d, that a certain part of $\widehat{Q}_d$ coincides with the multidegree of the toric part of $\widehat{\mathcal{O}}_d$.

Statement of Originality

Most of this thesis is based on common work with my supervisor András Szenes. We intend to present the results as a collaboration, and I have therefore based the structure of this thesis closely to the joint paper [4].

Chapter 5 is entirely my own work.

Chapter 2

Global Singularity Theory and Multidegrees

2.1 Basic notions of singularity theory

2.1.1 The setup

We start with a brief introduction to singularity theory. We suggest [34],[1],[47] as references for the subject.

Let $(e_1,\dots,e_n)$ be the basis of $\mathbb{C}^n$, and denote the corresponding coordinates by $(x_1,\dots,x_n)$. Introduce the notation $\mathcal{J}(n)=\{h\in\mathbb{C}[[x_1,\dots,x_n]];\ h(0)=0\}$ for the algebra of power series without a constant term, and let $\mathcal{J}_d(n)$ be the space of d-jets of holomorphic functions on $\mathbb{C}^n$ near the origin, i.e. the quotient of $\mathcal{J}(n)$ by the ideal of those power series whose lowest order term is of degree at least $d+1$. As a linear space, $\mathcal{J}_d(n)$ may be identified with polynomials on $\mathbb{C}^n$ of degree at most d without a constant term.

In this thesis, we will call an algebra *nilpotent* if it is finite-dimensional, and there exists a positive integer N such that the product of any N elements of the algebra vanishes. The algebra $\mathcal{J}_d(n)$, in particular, is nilpotent, since $\mathcal{J}_d(n)^{d+1}=0$.

Our basic object is $\mathcal{J}_d(n,k)$, the space of d-jets of holomorphic maps $(\mathbb{C}^n,0)\to(\mathbb{C}^k,0)$. This is a finite-dimensional complex vector space, which one can identify $\mathcal{J}_d(n)\otimes\mathbb{C}^k$; hence $\dim\mathcal{J}_d(n,k)=k\binom{n+d}{d}-k$. We will call the elements of $\mathcal{J}_d(k,n)$ *map-jets of order* d, or simply map-jets. In this thesis we will always assume $n\le k$.

One can compose map-jets via substitution and elimination of terms of degree greater than d; this leads to the composition maps

$$\mathcal{J}_d(n,k)\times\mathcal{J}_d(m,n)\to\mathcal{J}_d(m,k),\quad (\Psi_2,\Psi_1)\mapsto\Psi_2\circ\Psi_1. \tag{2.1.1}$$

When $d=1$, $\mathcal{J}_1(m,n)$ may be identified with n-by-m matrices, and (2.1.1) reduces to multiplication of matrices. More generally, by taking the linear parts of jets, we obtain a map

$$\mathrm{Lin}:\mathcal{J}_d(n,k)\to\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^k),$$

which is compatible with the compositions (2.1.1) and matrix multiplication.

Consider now the set

$$\mathrm{Diff}_d(n) = \{\Delta \in \mathcal{J}_d(n,n);\ \mathrm{Lin}(\Delta) \text{ invertible}\}.$$

The composition map (2.1.1) endows this set with the structure of an algebraic group, which has a faithful representation on $\mathcal{J}_d(n)$. Using the compositions (2.1.1) again, we obtain the so-called *left-right* action of the group $\mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$ on $\mathcal{J}_d(k,n)$:

$$[(\Delta_L, \Delta_R), \Psi] \mapsto \Delta_L \circ \Psi \circ \Delta_R^{-1}$$

Note that the action of $\mathrm{Diff}_d(n)$ is linear, while the action of $\mathrm{Diff}_d(k)$ is not. *Singularity theory*, in the sense that we are considering here, studies the left-right-invariant algebraic subsets of $\mathcal{J}_d(n,k)$.

A natural way to form such subsets is as follows. Observe that to each element $\Psi = (P_1, \ldots, P_k) \in \mathcal{J}_d(n,k)$, where $P_i \in \mathcal{J}_d(n)$ for $i = 1, \ldots, k$, we can associate the quotient algebra $A_\Psi = \mathcal{J}_d(n)/I\langle P_1, \ldots, P_k\rangle$: the algebra $\mathcal{J}_d(n)$ modulo the ideal generated by the elements of the sequence. Since $\mathcal{J}_d(n)^{d+1} = 0$, we also have $A_\Psi^{d+1}{=}0$. We will call A_Ψ the *nilpotent algebra*[1] of the map-jet Ψ. For $\Psi = 0$ this nilpotent algebra is $\mathcal{J}_d(n)$, while for a generic Ψ – in fact, as soon as $\mathrm{rank}[\mathrm{Lin}(\Psi)] = n$ – we have $A_\Psi = 0$.

Now let A be a nilpotent algebra, as defined above. Consider the subset

$$\Theta_A^{n\to k} = \{(P_1, \ldots, P_k) \in \mathcal{J}_d(n,k);\ \mathcal{J}_d(n)/I\langle P_1, \ldots, P_n\rangle \cong A\} \tag{2.1.2}$$

of the map-jets of order d. Again, the dependence on the parameters d, n and k will be usually omitted. Let us collect some simple properties of Θ_A.

Proposition 2.1.1 ([1]). *Let A be a nilpotent algebra. Assume that $A^{d+1} = 0$ and $n \geq \dim(A/A^2)$.*

- *For k sufficiently large, $\Theta_A^{n\to k}$ is a nonempty quasiprojective algebraic variety in $\mathcal{J}_d(n,k)$,*
- *The codimension of $\Theta_A^{n\to k}$ in $\mathcal{J}_d(n,k)$ equals $(k-n+1)\dim(A)$; in particular, it does not depend on d and depends only on the difference $k-n$.*
- *$\Theta_A^{n\to k}$ is $\mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$-invariant.*

A key observation is that although two map-jets with the same nilpotent algebra may be in different $\mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$-orbits, there is a group acting on $\mathcal{J}_d(n,k)$ whose orbits are exactly the sets $\Theta_A^{n\to k}$ for various nilpotent algebras A. This group is defined as the semidirect product

$$\mathcal{K}_d(n,k) = \mathrm{GL}_k(\mathbb{C} \oplus \mathcal{J}_d(n)) \rtimes \mathrm{Diff}_d(n), \tag{2.1.3}$$

[1] Instead of this algebra, it is customary to use the so-called *local algebra* of Ψ, which is simply the augmentation of A_Ψ by the constants.

using the natural action of $\mathrm{Diff}_d(n)$ on $\mathcal{J}_d(n)$; the algebra $\mathbb{C} \oplus \mathcal{J}_d(n)$ is the augmentation of $\mathcal{J}_d(n)$ by constants. The vector space $\mathcal{J}_d(n)$ is naturally a module over $\mathbb{C} \oplus \mathcal{J}_d(n)$, and hence $\mathcal{K}_d(n,k)$ acts on $\mathcal{J}_d(n,k)$ via

$$[(M, \Delta), \Psi] \mapsto (M \cdot \Psi) \circ \Delta^{-1}, \tag{2.1.4}$$

where "·" stands for matrix multiplication.

Proposition 2.1.2 ([33],[34],[1])**.** *Two map-jets in $\mathcal{J}_d(k,n)$ have the same nilpotent algebra if and only if they are in the same $\mathcal{K}_d$-orbit.*

Remark 2.1.3. Two jets in the same $\mathcal{K}_d$-orbit are called *contact equivalent*, or $\mathcal{K}$-equivalent (cf. [1]). The term V-equivalence is also used (e.g. [31]). The varieties Θ_A are called *contact singularity classes* or simply *contact singularities*.

As the group $\mathcal{K}_d$ is connected, we have the following

Corollary 2.1.4. *For a nilpotent algebra A, the set Θ_A is an irreducible quasiprojective variety in $\mathcal{J}_d(n,k)$.*

In the present thesis, we will study certain rough topological invariants of contact singularities; these invariants depend only on the closure of the sigularity locus in $\mathcal{J}_d(n,k)$. As it turns out, in an asymptotic sense, the closures of contact orbits are also closures of left-right orbits, hence, from our point of view, these two types of singlarity classes are closely related.

While we will not need this statement, we describe it in some details for reference. Roughly, we claim that for fixed A and r, and sufficiently large n, there is a dense left-right orbit in $\Theta_A[n, n+r]$.

Let r be a nonnegative integer. An *unfolding* of a map-jet $\Psi \in \mathcal{J}_d(n,k)$ is a map-jet $\widehat{\Psi} \in \mathcal{J}_d(k+r, n+r)$ of the form

$$(x_1, \ldots, x_n, y_1, \ldots, y_r) \mapsto (F(x_1, \ldots, x_n, y_1, \ldots, y_r), y_1, \ldots, y_r)$$

where $F \in \mathcal{J}_d(n+r, k)$ satisfies

$$F(x_1 \ldots, x_n, 0, \ldots, 0) = \Psi(x_1, \ldots, x_n).$$

The *trivial unfolding* is the map-jet

$$(x_1, \ldots, x_n, y_1, \ldots, y_r) \rightarrow (\Psi(x_1, \ldots, x_n), y_1, \ldots, y_r).$$

Definition 2.1.5 ([1],[34])**.** A map-jet $\Psi \in \mathcal{J}_d(n,k)$ is *stable* if all unfoldings of Ψ are left-right equivalent to the trivial unfolding.

Informally, a germ of a holomorphic map $f : N \rightarrow K$ of complex manifolds at a point $x \in N$ is *stable*, if for any small deformation $\tilde{f}$ of f, there is a point in the vicinity of x at which the germ of $\tilde{f}$ is left-right equivalent to the germ of f at x.

Now we can formulate the relationship between contact and left-right orbits precisely.

Proposition 2.1.6 ([1],[34]). *1. If $\widehat{\Psi}$ is an unfolding of Ψ, then $A_{\widehat{\Psi}} \cong A_\Psi$.*

2. Every map germ has a stable unfolding.

3. If a map germ is stable, then its left-right orbit is dense in its contact orbit.

2.1.2 Morin singularities

In this thesis, we will focus on the case of those nilpotent algebras A whch are generated by a single element. Such algebras form a one-parameter family:

$$A_d = t\mathbb{C}[t]/t^{d+1}, \quad d = 1, 2, \ldots$$

The corresponding singularity classes are called the *A_d-singularities* or *Morin-singularities* [1],[35]. We introduce the simplified notation

$$\Theta_d^{n\to k} \text{ instead of } \Theta_{A_d}^{n\to k} \tag{2.1.5}$$

for these varieties, and we will omit the parameters n and k when this causes no confusion.

Let us specialize the results nquoted in the previous paragraph to the case of the A_d algebras. We have

- $(A_d)^{d+1} = 0$, hence we can work in $\mathcal{J}_d(n,k)$.
- The variety $\Theta_d^{n\to k}$ is nonempty for any $n \le k$. For $n = k = 1$, we simply have $\Theta_d[1,1] = \{0\}$, the constant zero germ in $\mathcal{J}_d(1,1)$. This germ is not stable.
- There are stable map-jets in $\mathcal{J}_d(n,k)$ with nilpotent algebra A_d, whenever $n \ge d$. An example in $\mathcal{J}_N(d,d)$ for $N \ge d$ with minimal source dimension $n = d$ is

$$(x_1 \ldots, x_d) \mapsto (x_d^{d+1} + x_1 x_d^{d-1} + x_2 x_d^{d-2} + \ldots + x_{d-1}x_d, x_1, \ldots, x_{d-1}). \tag{2.1.6}$$

Finally, we recall that the A_d-singularities fit into a wider family of the so-called *Thom-Boardman* singularity classes. ([7],[1]). A Thom-Boardman class is specified by a nonincreasing sequence of positive integers $i_1 \ge \ldots \ge i_d$; the class corresponding to the special values $i_1 = \ldots = i_d = 1$, contains exactly those maps with nilpotent algebra isomorphic to A_d.

As the description of Θ_d as a Thom-Boardman class is rather different from (2.1.2), we provide it below for reference. Observe that

- eliminating the terms of degree d results in an algebra homomorphism $\pi_{d\to d-1} I : \mathcal{J}_d(n) \to \mathcal{J}_{d-1}(n)$, and
- partial differentiation $f \mapsto \partial f/\partial x_j$ is a well-defined map $\mathcal{J}_d(n) \to \mathcal{J}_{d-1}(n)$ for $j = 1, \ldots, n$.

Now, given a proper ideal I in the algebra $\mathcal{J}_d(n)$, denote by δI the ideal in $\mathcal{J}_{d-1}(n)$ generated by $\pi_{d\to d-1}I$ together with the truncated determinants of the n-by-n matrices of the form

$$\det\left(\frac{\partial Q_i}{\partial x_j}\right)_{i,j=1}^{n} \in \mathcal{J}_{d-1}(n),$$

with arbitrary $Q_1, \dots, Q_n \in I$.

Proposition 2.1.7. *Denoting by $I\langle P_1, \dots, P_k\rangle$ the ideal in $\mathcal{J}_d(n)$ generated by the elements $P_1, \dots, P_k$, we have*

$$\Theta_d^{n\to k} = \{(P_1, \dots, P_k) \in \mathcal{J}_d(n,k);\ \operatorname{codim}(\delta^{d-1}I\langle P_1, \dots, P_k\rangle \subset \mathcal{J}_1(n)) = 1\}. \quad (2.1.7)$$

2.2 Equivariant Poincaré duals and Thom polynomials

Let T be a complexified torus: $T \cong (\mathbb{C}^*)^r$. The *equivariant Poincaré dual* is an invariant $\Sigma \mapsto \mathrm{eP}[\Sigma]$ associated to algebraic or analytic T-invariant subvarieties of T-modules; this invariant takes values in homogeneous polynomials on the Lie algebra $\mathrm{Lie}(T)$ of T. The central objects of the present work, Thom polynomials, are special cases of equivariant Poincaré duals (cf. [44],[28]). We review the definitions and properties of equivariant Poincaré duals in some detail here in order to prepare ourselves for the localization formulas of the next section.

The equivariant Poincaré dual has appeared in the literature in several guises: as Joseph polynomial, equivariant multiplicity, multidegree, etc. One of the first definitions was given by Joseph [27], who introduced it as the polynomial governing the asymptotic behavior of the character of the algebra of functions on the subvariety. Rossmann in [45] defined this invariant for analytic subvarieties via an integral-limit representation, and then used it to write down a very general localization formula for equivariant integrals. This formula will play an important role in our computations.

We start with a list of properties in the algebraic context in §2.2.1, following the treatment of [37]; this will provide us with some useful computational tools. After studying an example in §2.2.2, we turn to the analytic picture. We first give an overview of Rossmann's localization formula, then we describe Vergne's integral representation, which places the equivariant Poincaré dual in the proper context of equivariant cohomology. Finally, in the last two paragraphs, we link the equivariant Poincaré dual to Thom polynomials. This allows us to formulate our problem precisely.

2.2.1 Basic properties

Let W be a complex N-dimensional vector space endowed with an action of the complexified torus $T = (\mathbb{C}^*)^r$. Denote the weight lattice of T by Λ; this is the lattice in $\mathrm{Lie}(T)^*$ generated by the standard weights $\lambda_1, \ldots, \lambda_r$. The action of T on W is diagonalizable, hence one can choose coordinates $y_1, \ldots, y_N$ on W in such a way that the action in the dual basis is diagonal; denote the respective weights by $\eta_1, \ldots, \eta_N$.

Note that we will *not* assume that the weights $\eta_1, \ldots, \eta_N$ all lie in an open half-space of $\mathrm{Lie}(T)^*$ (this is the so-called *convergent* case in [45, 37]), hence the Λ-graded pieces of the ring $S = \mathbb{C}[y_1, \ldots, y_N]$ of polynomial functions on W might be infinite dimensional.

Let Σ be a closed T-invariant algebraic subvariety of W. Denote the minimal codimension of the irreducible components of Σ by D. Then the equivariant Poincaré dual $\mathrm{eP}[\Sigma, W]_T$ is a homogeneous degree-D element of $\mathbb{C}[\lambda_1, \ldots, \lambda_r]$, the ring of polynomials on $\mathrm{Lie}(T)$.

Remark 2.2.1.
1. We will omit T and W from the notation $\mathrm{eP}[\Sigma, W]_T$ whenever this does not cause confusion.
2. The components of Σ of codimension higher than D will not contribute to $\mathrm{eP}[\Sigma, W]_T$.

Let $I(\Sigma) \subset S$ be the ideal of polynomials vanishing on Σ, and let $F(\Sigma) = S/I(\Sigma)$ be the ring of functions on Σ. According to Joseph, the polynomial $\mathrm{eP}[\Sigma]$ is responsible for the asymptotic behavior of the character

$$\chi^{F(\Sigma)} = \sum_{\eta \in \Lambda} \dim(F(\Sigma)^{[\eta]}) \cdot \exp(\eta),$$

where the superscript $[\eta]$ stands for the degree-η part of the ring. A family Σ_t of T-invariant subvarieties of W is defined to be *flat* if the character $\chi^{F(\Sigma_t)}$ does not depend on t. Then our first property is
1. Deformation invariance: the polynomial $\mathrm{eP}[\Sigma_t]$ remains unchanged in a flat T-invariant family of subvarieties of W.

The other properties are as follows.
2. Additivity: If $\{\Sigma_1, \ldots, \Sigma_m\}$ are the irreducible components of maximal dimension in Σ, then

$$\mathrm{eP}[\Sigma] = \sum_{i=1}^{m} \mathrm{eP}[\Sigma_i]. \tag{2.2.1}$$

3. Normalization: For T-invariant linear subspaces of W, the invariant is equal to the product of weights in the normal directions. More precisely, for any index set $\mathbf{i} \subset \{1, \ldots, N\}$, we have

$$\mathrm{eP}\left[\{w \in W;\ y_i(w) = 0,\, i \in \mathbf{i}\}, W\right] = \prod_{i \in \mathbf{i}} \eta_i. \tag{2.2.2}$$

The simplest example of (2.2.2) is the case of the point: $\Sigma = \{0\}$. We will often use the notation $\mathrm{Euler}^T(W)$ for $\mathrm{eP}[\{0\}, W]$, since, indeed, this is the equivariant Euler class of W thought of as a T-vector bundle over a point. We have thus

$$\mathrm{Euler}^T(W) = \prod_{i=1}^{N} \eta_i. \tag{2.2.3}$$

These 3 properties give us a blueprint for computing the equivariant Poincaré dual.

2.2.2 An example

A simple way to construct T-invariant subvarieties of W is to take the orbit $\Sigma = T{\cdot}p$ of a generic point $p \in W$. Because of deformation invariance, $\mathrm{eP}[\overline{T{\cdot}p}]$ then does not depend on the choice of p, hence this polynomial is an invariant of the T-action on W.

Consider the following example: let $W = \mathbb{C}^4$ endowed with a $T = (\mathbb{C}^*)^3$-action, whose weights η_1, η_2, η_3 and η_4 span $\mathrm{Lie}(T)^*$, and satisfy $\eta_1 + \eta_3 = \eta_2 + \eta_4$. In other words, the four weights, η_i, $i = 1, \ldots, 4$, form the vertices of a parallelogram in $\mathrm{Lie}(T)^*$ lying in a hyperplane which does not pass through the origin. Choose $p = (1, 1, 1, 1) \in W$; then the closure of the T-orbit of p is given by a single equation:

$$\overline{T{\cdot}p} = \{(y_1, y_2, y_3, y_4) \in \mathbb{C}^4;\ y_1 y_3 = y_2 y_4\}. \tag{2.2.4}$$

We will compute the equivariant Poincaré dual of this subvariety in a number of ways.

Method 1: degeneration. We use the axioms listed in §2.2.1. The equivariant deformation

$$\Sigma_t = \{(y_1, y_2, y_3, y_4) \in \mathbb{C}^4;\ y_1 y_3 = t y_2 y_4\}. \tag{2.2.5}$$

is flat, thus Σ_1 and Σ_0 must have the same equivariant Poincaré dual . We observe that $\Sigma_1 = \overline{T{\cdot}p}$, while Σ_0 is the union of two hyperplanes: $\{y_1 = 0\}$ and $\{y_3 = 0\}$. Then using the additivity and the normalization axioms, we arrive at the result that the equivariant Poincaré dual is $\mathrm{eP}[\Sigma_0] = \eta_1 + \eta_3 = \eta_2 + \eta_4$. Thus we have

$$\mathrm{eP}[\overline{T{\cdot}p}] = \eta_1 + \eta_3. \tag{2.2.6}$$

2.2.3 Multidegrees

Unfortunately, most varieties cannot be deformed to a union of linear subspaces of W. In a certain sense, this becomes possible, however, if we pass to schemes, i.e. to the defining ideals themselves. For completeness, below, we recall the extension of equivariant Poincaré duals to this more general setting.

We keep the notation of the previous paragraph; as we mentioned there, the equivariant Poincaré dual of a T-invariant subvariety $\Sigma \subset W$ is an invariant of the ideal $I(\Sigma)$. This ideal is reduced, i.e. has the property that $f^n \in I(\Sigma) \Rightarrow f \in I(\Sigma)$. Below, we sketch how the correspondence $\Sigma \mapsto \mathrm{eP}[\Sigma]$ may be extended to a more general one: $I \mapsto \mathrm{mdeg}[I, S]$, where I is an arbitrary T-invariant ideal in $S = \mathbb{C}[y_1, \ldots, y_N]$. For every closed variety $\Sigma \subset W$ then, we will have

$$\mathrm{eP}[\Sigma, W] = \mathrm{mdeg}[I(\Sigma), \mathbb{C}[y_1, \ldots, y_N]].$$

Consider the 3 properties listed in the previous paragraph. The property of deformation invariance naturally extends to flat families of arbitrary ideals; additivity is generalized as follows:
Additivity for ideals: Let $I \subset S$ be an ideal, and denote $\Sigma(I)$ the variety of common zeros of the polynomials in I:

$$\Sigma(I) = \{p \in W;\ f(p) = 0\ \forall f \in I\}.$$

Let $\Sigma_1(I), \Sigma_2(I), \ldots, \Sigma_m(I)$ be the list of maximal-dimensional irreducible components of $\Sigma(I)$. Then for each of these components $\Sigma_i(I)$, one can define a positive integer $\mathrm{mult}(\Sigma_i(I))$, the multiplicity of $\Sigma_i(I)$, as follows. Each $\Sigma_i(I)$ corresponds to a prime ideal $\mathfrak{p}_i \subset S$, and $\mathrm{mult}(\Sigma_i(I))$ is the length of the $S_{\mathfrak{p}_i}$-module $(S/I)_{\mathfrak{p}_i}$, where $S_{\mathfrak{p}_i}$ (resp. $(S/I)_{\mathfrak{p}_i}$) is the localization of S (resp. S/I) at $\mathfrak{p}_i$, see section II.3.3 in [13]. Then we have

$$\mathrm{mdeg}[I, S] = \sum_{i=1}^{m} \mathrm{mult}(\Sigma_i(I)) \cdot \mathrm{mdeg}[\mathfrak{p}_i, S]. \tag{2.2.7}$$

In particular, the normalization axiom generalizes as follows.
Normalization for ideals: Let $\mathbf{i} \subset \{1, \ldots, N\}$ be a subset. Then

$$\mathrm{mdeg}\left[\langle y_i = 0,\ i \in \mathbf{i}\rangle, S\right] = \prod_{i \in \mathbf{i}} \eta_i, \tag{2.2.8}$$

where $\langle\cdot\rangle$ stands for the ideal generated by the listed polynomials.
Consider the following three examples:

1. Set $N = 4$, and consider the ideal $I = \langle y_1^2, y_2^3, y_3\rangle$ in $S = \mathbb{C}[y_1, y_2, y_3, y_4]$. This is the line $\{y_1 = y_2 = y_3 = 0\}$ with multiplicity 6, so its multidegree is

$$\mathrm{mdeg}[I, S] = 6\eta_1\eta_2\eta_3.$$

2. The ideal $I = \langle y_1^2 y_2^3 y_3\rangle$ in $S = \mathbb{C}[y_1, y_2, y_3]$ correponds to the union of the hyperplanes $y_1 = 0, y_2 = 0, y_3 = 0$ with multiplicities $2, 3, 1$, respectively. By the normalization and additivity properties

$$\mathrm{mdeg}[I, S] = 2\eta_1 + 3\eta_2 + \eta_3$$

3. The ideal $I = \langle y_1y_2, y_2y_3, y_1y_3\rangle = \langle y_1, y_2\rangle \cap \langle y_2, y_3\rangle \cap \langle y_1, y_3\rangle$ in $S = \mathbb{C}[y_1, y_2, y_3]$ has three components with multiplicity 1, corresponding to the given decomposition, so

$$\mathrm{mdeg}[I, S] = \eta_1\eta_2 + \eta_2\eta_3 + \eta_1\eta_3$$

Following [37] §8.5, now we sketch an algorithm for computing $\mathrm{mdeg}[I, S]$, and hence $\mathrm{eP}[\Sigma, W]$.

An ideal $M \subset S$ generated by a set of monomials in $y_1, \ldots, y_N$ is called a *monomial ideal*. The examples above are those of monomial ideals, and for such ideals the multidegree may be computed from the axioms as follows. If the codimension of $\Sigma(M)$ in W is s, then the associated primes corresponding to the maximal dimensional components of $\Sigma(M)$ have the form

$$\mathfrak{p}_\Delta = \langle y_i : i \in \Delta\rangle$$

where $\Delta \in \binom{N}{s}$ is an s-tuple of indices. By the normalization and additivity axiom it is sufficient to describe the length of the $S_{\mathfrak{p}_\Delta}$-module $(S/M)_{\mathfrak{p}_\Delta}$. Without loss of generality assume that $\Delta = \{1, \ldots, s\}$. The elements of this module have the form

$$\sum_{\mathbf{a}\in\mathcal{A}, \mathbf{b}\in\mathbb{Z}_{\geq 0}^{N-s}} \alpha_{\mathbf{a},\mathbf{b}} \frac{y_1^{a_1}\cdots y_s^{a_s}}{y_{s+1}^{b_{s+1}}\cdots y_N^{b_N}},$$

where

$$\mathcal{A} = \{\mathbf{a} \in \mathbb{N}^s : y_1^{a_1}\cdots y_s^{a_s} \notin M\}$$

Therefore, the $S_{\mathfrak{p}_\Delta}$-submodules are parametrized by subsets $\mathcal{B} \subset \mathcal{A}$ with the property that for $\mathbf{b} \in \mathcal{B}, \mathbf{x} \in \mathbb{Z}_{\geq 0}^s$, and $\mathbf{b} + \mathbf{x} \in \mathcal{A}$ we have $\mathbf{b} + \mathbf{x} \in \mathcal{B}$. A chain of submodules corresponds to a chain of subsets of $\mathcal{A}$, each containing the next. The longest chain has length $|\mathcal{A}|$ so it must be a composition chain, since the quotient of the consecutive modules have dimension 1 as a complex vector space, proving that the multiplicity is $|\mathcal{A}|$.

Finally, we observe that every ideal I has a flat deformation to a monomial ideal. Indeed, fixing a monomial order, one may consider the *initial* ideal, $\mathrm{in}(I)$ of I, linearly spanned by the largest with respect to the order monomials of the elements of I, see [14] §15.2 for details. This ideal is clearly monomial, and I has a flat deformation to $\mathrm{in}(I)$ ([14], Theorem 15.17.) So by the deformation invariance property $\mathrm{mdeg}[I, S] = \mathrm{mdeg}[\mathrm{in}(I), S]$. See also §5.2.2 for the definition of initial ideal and monomial order.

As an example, consider the deformation (2.2.5) in our example above: this is a special case of this construction: $I(\Sigma) = I(\Sigma_1)$ deforms to $I(\Sigma_0) = \mathrm{in}(I(\Sigma_1))$ with respect to the lexicographic monomial order corresponding to the order $y_1 > y_2 > y_3 > y_4$ on the variables, see [14] §15.2.

2.2.4 Some technical statements

In the previous paragraph we sketched the construction and properties of the equivariant Poincaré dual. Here we will discuss a few simple consequences of these properties.

We retain the notation of the previous paragraph; thus we have a T-module W endowed with coordinates $y_1, \dots, y_N$, which are of weight $\eta_1, \dots, \eta_N$, respectively. The following technical lemma will be crucial in our computations.

Lemma 2.2.2. *Let $I \subset \mathbb{C}[y_1, \dots, y_N]$ be a T-invariant ideal, and assume that for some j, $1 \le j \le N$, there is an element $R \in I$ which expesses the variable y_j as a polynomial of the remaining variables:*

$$R: \; y_j = f(y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_N). \tag{2.2.9}$$

Then denoting by I_j the ideal in $\mathbb{C}[y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_N]$ obtained from I by performing the substitution (2.2.9)*, we have*

$$\mathrm{mdeg}[I, \mathbb{C}[y_1, \dots, y_N]] = \eta_j \cdot \mathrm{mdeg}[I_j, \mathbb{C}[y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_N]] \tag{2.2.10}$$

Proof. One can easily define a monomial order on S such that the initial term of R is y_j: give weights -2 to y_j, and the weight -1 to y_i, $i \neq j$. (see [14], sect. 15.2). So y_j is a generator of $\mathrm{in}(I(\Sigma))$, therefore the prime monomial ideals corresponding to the maximal dimensional components of the underlying variety of $\mathrm{in}(I(\Sigma))$ contain y_j as a generator. The lemma now follows from the additivity and normalization properties. □

Remark 2.2.3. The geometric version of Lemma 2.2.2, corresponding to the case when I is reduced, reads as follows. Let $\Sigma \subset W$ be a closed T-invariant subvariety, and assume that the conditions of Lemma 2.2.2 hold for $I(\Sigma)$. Let $\pi_j : W \to W_j$ denote the projection onto the hyperplane $W_j = \{y_j = 0\}$. Then $\pi_j(\Sigma)$ is a closed subvariety in W_j and

$$\mathrm{eP}[\Sigma, W] = \eta_j \cdot \mathrm{eP}[\pi_j(\Sigma), W_j]$$

Note that in this case the polynomial $\mathrm{eP}[\Sigma, W]$ is divisible by η_j.

Remark 2.2.4. The normalization axiom may be reformulated as follows: given a surjective equivariant linear map $\gamma : W \to E$ from W to another T-module E, we have

$$\mathrm{eP}[\gamma^{-1}(0)] = \mathrm{Euler}^T(E). \tag{2.2.11}$$

2.2.5 Integration and equivariant multiplicities

A fundamental observation of Rossmann in [45] is, that the notion of equivariant Poincaré dual can be extended to the following situations: first, to the case of analytic T-invariant varieties defined in a neighborhood of the origin in T-representations;

secondly, the setup can be further generalized to the nonlinear situation in the following sense.

Let Z be a complex manifold with a holomorphic T-action, and let $M \subset Z$ be a T-invariant analytic subvariety with an isolated fixed point $p \in M^T$. Then one can find local analytic coordinates near p, in which the action is linear and diagonal. Using these coordinates, one can identify a neighborhood of the origin in T_pZ with a neighborhood of p in Z. We denote by $\hat{T}_pM$ the part of T_pZ which corresponds to M under this identification; informally, we will call $\hat{T}_pM$ the T-invariant *tangent cone* of M at p. This tangent cone is not quite canonical: it depends on the choice of coordinates, however, the equivariant Poincaré dual of $\Sigma = \hat{T}_pM$ in $W = \mathrm{T}_pZ$ does not. Rossmann named this equivariant Poincaré dual the *equivariant multiplicity of M in Z at p*:

$$\mathrm{emult}_p[M, Z] \stackrel{\mathrm{def}}{=} \mathrm{eP}[\hat{T}_pM, \mathrm{T}_pZ]. \tag{2.2.12}$$

Remark 2.2.5. In the algebraic framework one might need to pass to the *tangent scheme* of M at p (cf. [18]). This is canonically defined, but we will not use this notion.

An important application of the equivariant multiplicity is Rossmann's localization formula [45]. The reader will find the necessary background material about equivariant differential forms and equivariant integration in [23, 5]. For technical reasons, we need to pass to the compact versions of our reductive groups. We will use the notation $G_\circ$ for the compact form of the complex reductive group G; for example $T_\circ$ will be a product of copies of the circle group $U(1)$. The introduction of these group into our framework means an implicit choice of a Hermitian metric.

Let $\mu : \mathrm{Lie}(T_\circ) \to \Omega^\bullet(Z)$ be a holomorphic equivariant map with values in smooth differential forms on Z. Then *Rossmann's localization formula* states that

$$\int_M \mu = \sum_{p \in M^T} \frac{\mathrm{emult}_p[M, Z]}{\mathrm{Euler}^T(\mathrm{T}_pZ)} \cdot \mu^{[0]}(p), \tag{2.2.13}$$

where $\mu^{[0]}(p)$ is the differential-form-degree-zero component of μ evaluated at p. Recall that $\mathrm{Euler}^T(\mathrm{T}_pZ)$ stands for the product of the weights of the T-action on T_pZ.

This formula generalizes the equivariant integration formula of Berline and Vergne [6], which applies when M is smooth. In this case the tangent cone of M at p is a well-defined linear subspace $\mathrm{T}_pM \subset \mathrm{T}_pZ$, and $\mathrm{emult}_p[M]$ is the equivariant Poincaré dual of this subspace. Then the fraction in (2.2.13) simplifies: the ambient space Z is eliminated from the picture, and one arrives at (cf. [6])

$$\int_M \mu = \sum_{p \in M^T} \frac{\mu^{[0]}(p)}{\mathrm{Euler}^T(\mathrm{T}_pM)}. \tag{2.2.14}$$

Rossmann proves (2.2.13) by writing down a local integral-limit formula for the equivariant multiplicity, and then applying an adaptation of Stokes theorem, following the method of Bott [8].

As showed by Vergne [48], such a local integration formula for equivariant Poincaré duals may be given in the framework of equivariant cohomology. To describe this formula, we return to our setup of a T-invariant subvariety Σ in a vector space W. The starting point is the Thom isomorphism in equivariant cohomology:

$$H^*_{T_\circ,\mathrm{cpt}}(W) = H^*_{T_\circ}(W) \cdot \mathrm{Thom}_{T_\circ}(W), \tag{2.2.15}$$

which presents compactly supported equivariant cohomology as a module over usual equivariant cohomology. The class $\mathrm{Thom}_T(W) \in H^{\dim W}_{T_\circ,\mathrm{cpt}}(W)$ may be represented by an explicit equivariant differential form with compact support (cf. [32, 12]). Then *Verge's integration formula* (cf. [48]) reads as follows:

$$\mathrm{eP}[\Sigma] = \int_\Sigma \mathrm{Thom}_{T_\circ}(W). \tag{2.2.16}$$

Compared to Rossmann's formula (2.2.13), this result turns things upside down, and describes $\mathrm{eP}[\Sigma]$ as an integral in equivariant cohomology. As we explain in the next section, this allows us to *localize* the equivariant Poincaré dual.

We complete this review by noting that a consequence of (2.2.16) is the following formula. For an equivariantly closed differential form μ with compact support, we have

$$\int_\Sigma \mu = \int_W \mathrm{eP}[\Sigma] \cdot \mu.$$

This formula serves as the motivation for the term *equivariant Poincaré dual.*

2.2.6 Thom polynomials and equivariant Poincaré duals

Let us apply our newfound invariant to the setup of global singularity theory described in §2.1. Recall that, for integers d and $n \le k$, we have an irreducible variety $\Theta_d \subset \mathcal{J}_d(n,k)$ which is invariant under the natural action of the group $\mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$.

Now observe that the quotient map $\mathrm{Lin} : \mathrm{Diff}_d(n) \to \mathrm{Diff}_1(n) = \mathrm{GL}_n$ has a canonical section, consisting of linear substitutions. In other words we have a canonical group embedding

$$\mathrm{GL}_n \hookrightarrow \mathrm{Diff}_d(n),$$

and we can restrict the action of the diffeomorphism groups $\mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$ on $\mathcal{J}_d(n,k)$ to the group $\mathrm{GL}_k \times \mathrm{GL}_n$. Denoting the subgroup of diagonal matrices of GL_k and GL_n by T_k and T_n, respectively, we can consider

$$\mathrm{eP}[\Theta_d, \mathcal{J}_d(n,k)]_{T_k \times T_k}$$

which is a polynomial of degree $d(k-n+1)$ in the basic weights of T_k and T_n, which we will denote by $\theta_1, \ldots, \theta_k$ and $\lambda_1, \ldots, \lambda_n$, respectively.

Starting with the next section we will focus on the computation of this polynomial. In the remainder of this section, however, we would like to argue that this polynomial is, in fact, the Thom polynomial of the A_d-singularity, in the sense formulated in the introduction. This is considered to be known by many experts, and indeed may be deduced from the homotopy equivalence of $\mathrm{Diff}_d(n)$ and GL_n (cf [25]), but we have not found a precise reference for this statement. Thus, for completeness sake, we have decided to sketch the following rather low-brow argument.

We start with a simple observation. In case the torus action extends to the action of the general linear group, the symmetric group $\mathcal{S}_n$, thought of as the Weyl group, naturally acts on the weights of T by permuting the λs. Thus we can conclude the following.

Lemma 2.2.6. *Let $T = (\mathbb{C}^*)^n$ be the subgroup of diagonal matrices of the complex group* GL_n*, and denote by $\lambda_1, \ldots, \lambda_n$ its basic weights. If Σ is a* GL_n*-invariant subvariety of the* GL_n*-module W, then the equivariant Poincaré dual* $\mathrm{eP}[\Sigma, W]_T$ *is a* symmetric *polynomial in $\lambda_1, \ldots, \lambda_n$.*

Our next goal is to describe the topological meaning of the equivariant Poincaré dual as a certain universality property with respect to the ordinary Poincaré dual.

Again, fix the notation $G = \mathrm{GL}_n$ and $G_\circ = U_n$. Let F be a principal $G_\circ$-bundle over a compact oriented manifold M. Then, using the Chern-Weil map, any symmetric polynomial $P \in \mathbb{C}[\lambda_1, \ldots, \lambda_n]^{\mathcal{S}_n}$ defines a characteristic class $P(F) \in H^*(M, \mathbb{C})$. Now let Σ be G-invariant subvariety of the G-module W, and, denote by W_F the associated vector bundle $F \times_{G_\circ} W$ over M, and by Σ_F the subset of W_F corresponding to Σ.

$$\begin{array}{ccccc} F \times_{G_\circ} W & = & W_F & \hookleftarrow & \Sigma_F \quad = \quad F \times_{G_\circ} \Sigma \\ & & s \uparrow \downarrow & \swarrow & \\ & & M & & \end{array} \tag{2.2.17}$$

Then by Poincaré duality on the manifold W_F, there is a cohomology class $\alpha_\Sigma \in H^{2\mathrm{codim}(\Sigma)}(W_F)$ such that

$$\int_{W_F} \alpha_\Sigma \cdot \beta = \int_{\Sigma_F} \beta$$

for any compactly supported cohomology class on W_F. Then the universality property maybe written as follows:

$$\alpha_\Sigma = \mathrm{eP}[\Sigma, W](F) \text{ in } H^*(W_F),$$

i.e. the Chern-Weil image of the equivariant Poincaré dual is the ordinary Poincaré dual of the induced variety.

We will prove this statement in a geometric form which will be more convenient for our purposes. As most of what follows is well-known to the experts, our arguments will be rather sketchy. In this setup $\mathrm{eP}[\Sigma, W](F)$ will appear as the Poincaré dual of $s^{-1}(\Sigma_F)$ in M for an appropriate section $s : M \to \Sigma_F$. To make this more precise, we make the following

Definition 2.2.7. Consider the diagram (2.2.17), and assume for simplicity that Σ is equidimensional. We say that a *smooth* section $s : M \to W_F$ is *transversal* to Σ_F at some point $p \in M$ if $s(p)$ is a smooth point of Σ_F and the intersection $ds(\mathrm{T}_pM) \cap \mathrm{T}_{s(p)}\Sigma_F$ of vector spaces in $\mathrm{T}_{s(p)}W_F$ has the smallest possible dimension. We say that $s : M \to W_F$ is *generically transversal* to Σ_F if the we have

$$\overline{\{p \in M; s \text{ is transversal to } \Sigma_F \text{ at } p\}} = s^{-1}(\Sigma_F).$$

With this technical notion out of the way, we are ready to formulate our main statement.

Proposition 2.2.8. *For a smooth section $s : M \to W_F$ generically transversal to Σ_F, the cycle $s^{-1}(\Sigma_F) \subset M$ is Poincaré dual to the characteristic class* $\mathrm{eP}[\Sigma](F)$ *of F corresponding to the symmetric polynomial* $\mathrm{eP}[\Sigma]$.

Remark 2.2.9. We can not guarantee the existence of a generically transversal holomorphic section, even the existence of a holomorphic section. The Proposition says, that *if* the generically transversal section exists, then we have a nice geometric meaning of these polynomials.

Proof. If we consider (2.2.16) as the definition of the equivariant Poincaré dual, then this statement is almost tautological. Indeed, recall Cartan's correspondence which associates to equivariantly closed differential form μ on a G-manifold X an ordinary closed differential form $C(\mu)$ on the manifold $X_F = F \times_G X$. This construction is very simple if one takes the Weyl algebra model for equivariant cohomology and the only input necessary is a connection on F [32]. In particular, when $X = \mathrm{pt}$, then C reduces to the usual Chern-Weil correspondence. As this map C trivially commutes with integration and restriction, considering forms with compact support, we obtain the following commutative diagram:

$$\begin{array}{ccc} H^*_{G,\mathrm{cpt}}(W) & \xrightarrow{C} & H^*_{\mathrm{cpt}}(W) \\ \Big\downarrow {\textstyle\int_\Sigma} & & \Big\downarrow \pi^\Sigma_* \\ H^*_G(\mathrm{pt}) & \xrightarrow{C} & H^*(M) \end{array} \qquad (2.2.18)$$

The symbol $\int_\Sigma$ here stands for integrating on $\Sigma \subset W$, while π^Σ_* is the push-forward along the fibers of the bundle $\Sigma_F \to M$.

Now starting with $\mathrm{Thom}_{G_\circ}(W) \in H^*_{G,\mathrm{cpt}}(W)$ defined by (2.2.15) in the upper left corner of the diagram, we arrive exactly at our statement. Indeed, according to (2.2.16), we have

$$C\left(\int_\Sigma \mathrm{Thom}_{G_\circ}(W)\right) = C(\mathrm{eP}[\Sigma]) = \mathrm{eP}[\Sigma](F).$$

On the other hand, the Cartan correspondence takes $\mathrm{Thom}_{G_\circ}(W)$ to the Thom class of the bundle $W_F \to M$, which is also the Poincaré dual of M thought of as the zero section in W_F. Now, using the properties of the Poincaré dual (cf. [9]), it is a simple exercise to check that the push-forward is Poincaré dual to $s^{-1}(\Sigma_F) \subset M$ for a section $s : M \to W_F$, generically transversal to Σ_F. □

Now let us look at the situation of singularity loci of holomorphic maps described in the introduction; this appears to be similar to the setup we have just considered.

Indeed, for a complex manifold N of dimension n and an integer d, the space of choices of local coordinates up to order d at each point forms a principal $\mathrm{Diff}_d(n)$-bundle $\mathrm{Diff}_d(N)$. The $\mathrm{Diff}_d(n)$-module $\mathcal{J}_d(n)$ then induces the associated bundle of algebras $\mathcal{J}_d(N)$ of the d-jets of holomorphic functions on N. Given a holomorphic map $f : N \to K$, for each $p \in N$, the pull-back of the functions from K to N induces and algebra homomorphism $\mathcal{J}_d(K)_{f(p)} \to \mathcal{J}_d(K)_{f(p)}$. These algebra homomorphisms assemble into a section of the bundle $\mathrm{Hom}_{\mathrm{alg}}(f^*\mathcal{J}_d(K), \mathcal{J}_d(N))$ over N.

Lemma 2.2.10. *The bundle* $\mathrm{Hom}_{\mathrm{alg}}(f^*\mathcal{J}_d(K), \mathcal{J}_d(N))$ *is associated to the principal* $\mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$*-bundle* $f^*\mathrm{Diff}_d(K) \times \mathrm{Diff}_d(N)$ *via the representation* $\mathcal{J}(n,k)$ *of* $\mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$.

The reason for this is that the only relation in the algebra $\mathcal{J}_d(k)$ are those stating that the product of any $d+1$ elements vanishes. Since this also holds in $\mathcal{J}_d(n)$, then any element of $\mathcal{J}(n,k)$, which may be thought of as a map of k generators of $\mathcal{J}_d(k)$ into $\mathcal{J}_d(n)$, extends to and algebra homomorphism $\mathcal{J}_d(k) \to \mathcal{J}_d(n)$.

Hence we will introduce the notation

$$\mathcal{J}_d(N_f K) \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathrm{alg}}(f^*\mathcal{J}_d(K), \mathcal{J}_d(N)).$$

Again, note that even though the space $\mathcal{J}_d(n,k)$ has a linear structure the action of the group $\mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$ on it is not linear, and hence this bundle is not a vector bundle.

In any case, now can repeat the construction of the previous paragraph. Let $\Sigma \subset \mathcal{J}_d(n,k)$ be a closed $\mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$-invariant subvariety, and consider a holomorphic map $f : N \to K$ as above. Then Σ induces the associated subbundle $\Sigma_f \subset \mathcal{J}_d(N_f K)$, and f itself gives rise to a section $s_f : N \to \mathcal{J}_d(N_f K)$. It is reasonable then to ask for the obstruction for s_f to avoid Σ_f, just as in Proposition 2.2.8. There is a small problem though, because the group $\mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$ is not reductive, and we do not want to deal with its cohomology directly.

Instead, observe that the quotient map $\mathrm{Lin} : \mathrm{Diff}_d(n) \to \mathrm{GL}_n$ has a canonical section, consisting of linear substitutions. In other words we have a canonical group embedding

$$\mathrm{GL}_n \hookrightarrow \mathrm{Diff}_d(n),$$

and we can restrict the action of the diffeomorphism groups $\mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$ on $\mathcal{J}_d(n,k)$ to the group $\mathrm{GL}_k \times \mathrm{GL}_n$.

Now we can formulate our main technical statement which is a precise version of Thom's principle which we formulated in the introduction.

Proposition 2.2.11. *Let $\Sigma \subset \mathcal{J}_d(n,k)$ and $f : N \to K$ be as above. Assume that the section s_f is generically transversal to the subbundle $\Sigma_f \subset \mathcal{J}_d(N_f K)$. Next, denote by $Q_\Sigma(\lambda_1, \dots, \lambda_n, \theta_1, \dots, \theta_k)$ the equivariant Poincaré dual $\mathrm{eP}[\Sigma, \mathcal{J}_d(n,k)]$ with respect to the group $\mathrm{GL}_k \times \mathrm{GL}_n \subset \mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$. Then the coholomogy class $Q_\Sigma(TN, f^*TK) \in H^*(N)$ is Poincaré dual to the subvariety $s_f^{-1}(\Sigma_f)$.*

Proof. Observe that this statement becomes a special case of Proposition 2.2.8 as soon as the structure group of the bundle $\mathcal{J}_d(N_f K)$ reduces to $\mathrm{GL}_k \times \mathrm{GL}_n$. This, in turn, follows if we show that the structure group of the vector bundle $\mathcal{J}_d(N)$ reduces to GL_n for any complex manifold N.

To understand what this means, note that the quotient map $\mathcal{J}_d(n) \to \mathcal{J}_1(n) \cong \mathbb{C}^n$ induces a natural holomorphic bundle map $\pi : \mathcal{J}_d(N) \to \mathcal{J}_1(N) \cong \mathrm{T}^*N$. Now a section

$$\gamma : \mathrm{T}^*N \to \mathcal{J}_d(N) \tag{2.2.19}$$

of this map π naturally extends to an algebra isomorphism

$$\tilde{\gamma} : \oplus_{j=1}^d \mathrm{Sym}^j \mathrm{T}^*N \longrightarrow \mathcal{J}_d(N),$$

and this isomorphism clearly constitutes the reduction of the structure group we were looking for. Of course, a surjective map π of holomorphic vector bundles does not always have a holomorphic section γ, as in (2.2.19), but there always exists a C^∞-section. Indeed, if we fix an Hermitian structure on $\mathcal{J}_d(N)$, the inverse of the map π restricted to the orthogonal complement $(\ker \pi)^\perp$ provides us with such a section.

In short, there is a C^∞ reduction of the structure group from $\mathrm{Diff}_d(n)$ to GL_n, and once we are in the framework of ordinary vector bundles we can apply Proposition 2.2.8. This completes the proof. □

2.2.7 Thom polynomials of contact singularities

Now we restrict our attention to contact singularities which we discussed in § 2.1. Thus fix the integers $n \leq k$, and let A be a nilpotent algebra satisfying $A^{d+1} = 0$ for some $d \geq 1$.

The definition of the Thom polynomial, as in the Overview, is the following:

Definition 2.2.12.

$$\mathrm{Tp}_A^{n\to k}(\boldsymbol{\lambda},\boldsymbol{\theta}) = \mathrm{eP}[\Theta_A, \mathcal{J}_d(n,k)]$$

One of the natural questions to ask is how the Thom polynomials for fixed A and different pairs (n,k) are related. Denote the ring of bisymmetric polynomials in the λs and θs by $\mathbb{C}[\boldsymbol{\lambda},\boldsymbol{\theta}]^{\mathcal{S}_n\times\mathcal{S}_k}$. We collect the known facts [1, 10, 17] in Proposition 2.2.13 below. For simplicity, we will formulate the statements for the algebra $A_d = t\mathbb{C}[t]/t^{d+1}$ we study, although essentially the same properties are satisfied by the Thom polynomials of any other contact singularity (see [17] for details).

Recall from §2.2.1, that for $1 \le d$ and $1 \le n \le k$, $\Theta_d = \Theta_d^{n\to k}$ is a nonempty subvariety of $\mathcal{J}_d(n,k)$ of codimension $d(k-n+1)$. Consider the infinite sequence of homogeneous polynomials $c_i \in \mathbb{C}[\boldsymbol{\lambda},\boldsymbol{\theta}]^{\mathcal{S}_n\times\mathcal{S}_k}$, $\deg c_i = i$, defined by the generating series

$$\mathrm{RC}(q) = 1 + c_1 q + c_2 q^2 + \dots = \frac{\prod_{m=1}^{k}(1+\theta_m q)}{\prod_{l=1}^{n}(1+\lambda_l q)}; \qquad (2.2.20)$$

we will call c_i the ith *relative Chern class.*

Proposition 2.2.13 ([17])**.** *Let $1 \le d$ and $1 \le n \le k$. Then for each nonnegative integer j, there is a polynomial $\mathrm{TD}_d^j(b_0, b_1, b_2, \dots)$ in the indeterminates $b_0, b_1, b_2, \dots$ with the following properties*

1. TD_d^j *is homogeneous of degree d, and*

2. *if we set $\deg(b_i) = i$, then TD_d^j is homogeneous of degree $d(k-n+1)$;*

3. $$\mathrm{Tp}_d^{n\to k}(\boldsymbol{\lambda},\boldsymbol{\theta}) = \mathrm{TD}_d^{k-n}(1, c_1(\boldsymbol{\lambda},\boldsymbol{\theta}), c_2(\boldsymbol{\lambda},\boldsymbol{\theta}), \dots), \qquad (2.2.21)$$

   where the polynomials $c_i(\boldsymbol{\lambda},\boldsymbol{\theta})$, $i = 1, \dots,$ are defined by (2.2.20);

4. *the polynomial TD_d^{j-1} may be obtained from TD_d^j via the following substitution:*

   $$\mathrm{TD}_d^{j-1}(b_0, b_1, b_2, \dots) = \mathrm{TD}_d^j(0, b_0, b_1, b_2, \dots),$$

The notation TD stands for Thom-Damon series. The first two statements are straightforward from the definition of the Thom polynomial. The third and fourth statements are proved in [17], but our formula (4.4.25) provides a new proof.

For fixed j and sufficiently large n and k, the polynomials $c_i(\boldsymbol{\lambda},\boldsymbol{\theta})$, $i = 1, , \dots, j$ are algebraically independent. This means that for fixed codimension j and large enough n, the Thom polynomial $\mathrm{Tp}_d^{n\to n+j}(\boldsymbol{\lambda},\boldsymbol{\theta})$ determines TD_d^j. However, as we will see in the example below, for small values of n we can write $\mathrm{Tp}_d^{n\to n+j}(\boldsymbol{\lambda},\boldsymbol{\theta})$ in several different way as a polynomial of $c_0, c_1, \dots$. In short: $\mathrm{Tp}_d^{n\to k}(\boldsymbol{\lambda},\boldsymbol{\theta})$ and TD_d^j are well defined but $\mathrm{Tp}_d^{n\to n+j}(c_0, c_1 \dots)$ is not.

Example 2.2.14. For $d = 4, n = 1, k = 1$,

$$\mathrm{RC}(q) = \frac{1+\theta q}{1+\lambda q} = 1 + (\theta - \lambda)q - \lambda(\theta-\lambda)q^2 + \dots,$$

so

$$c_0(\theta,\lambda) = 1, c_1(\theta,\lambda) = \theta - \lambda, c_2(\theta,\lambda) = -\lambda(\theta-\lambda).$$
$$c_3(\theta,\lambda) = \lambda^2(\theta-\lambda), c_4(\theta,\lambda) = -\lambda^3(\theta-\lambda).$$

We have (see [44], or Section 5.1.5 below)

$$\mathrm{TD}_4^0 = c_1^4 + 6c_1^2c_2 + 2c_2^2 + 9c_1c_3 + 6c_4c_0$$

But

$$c_1(\theta,\lambda)c_3(\theta,\lambda) = c_2(\theta,\lambda)^2,$$

so

$$\mathrm{Tp}_4^{1\to 1}(\theta,\lambda) = c_1^4 + 6c_1^2c_2 + \alpha c_2^2 + (11-\alpha)c_1c_3 + 6c_4c_0$$

holds for any $\alpha \in \mathbb{R}$. For $n > 1$, however,

$$\mathrm{Tp}_4^{n\to n}(\theta,\lambda) = \mathrm{TD}_4^0.$$

Next, following [17], observe that property (4) allows us to define a universal object, the Thom series $\mathrm{Ts}(a_i,\, i \in \mathbb{Z})$, which is an infinite formal series in infinitely many variables with the following properties:

- it is homogeneous of degree d;
- setting $\deg(a_i) = i$ for $i \in \mathbb{Z}$, the series $\mathrm{Ts}_d(a_i,\, i \in \mathbb{Z})$ is homogeneous of degree 0;
- the Thom-Damon series maybe expressed via the following substitution:

$$\mathrm{TD}_d^j(b_0, b_1, b_2, \dots) = \mathrm{Ts}_d\left(\begin{cases} a_i = b_{i+k-n+1}, \text{ if } i \geq -(k-n+1), \\ a_i = 0 \,\text{otherwise.}\end{cases}\right)$$

The goal of this thesis is to compute the Thom polynomials for the algebras A_d. As we mentioned in the introduction, results in this direction go back 4 decades, to the works of Thom, Porteous, Gaffney, etc (see the Introduction for references). We were particularly interested in the meaning of the coefficients of the Thom series. The known results were Porteous's formula $\mathrm{Ts}_1 = a_0$, and Ronga's formula: $\mathrm{Ts}_2 = a_0^2 + \sum_{i=0}^{\infty} 2^{i-1}a_ia_{-i}$.

We obtained a rather satisfactory answer, which manifestly has the structure described above; the final result is contained in Theorem 4.4.16. Computationally, this answer can be explicitly implemented for $d \leq 6$ (cf. §8).

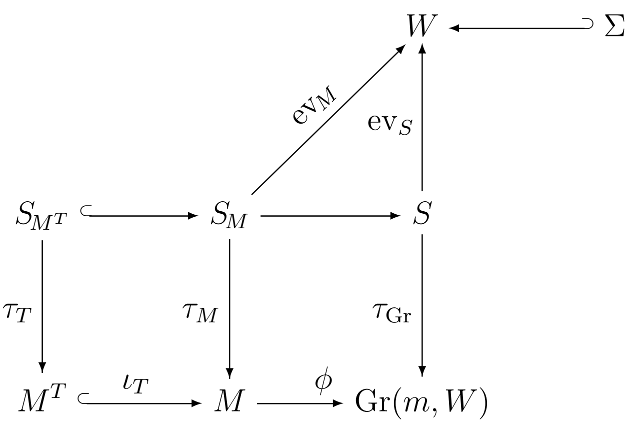
Chapter 3

Localizing Poincaré duals

In this chapter we develop the idea introduced at the end of §2.2.5: the localization of equivariant Poincaré duals based on Vergne's integration formula. Roughly, we show that if the T-invariant subvariety $\Sigma \subset W$ is equivariantly fibered over a parameter space M, then the equivariant Poincaré dual $\mathrm{eP}[\Sigma, W]$ may be read of from local data near fixed points of the T action on M. The final form of the statement is Proposition 3.3.2. We will start, however, with the more regular case of a smooth parameter space.

3.1 Localization in the smooth case

Let Σ be a T-invariant closed subvariety of the T-module W. Consider the following diagram:



(3.1.1)

Here

- $\mathrm{Gr}(m, W)$ is the Grassmannian of m-planes in W, S is the tautological bundle over $\mathrm{Gr}(m, W)$, and $\tau_{\mathrm{Gr}} : S \to \mathrm{Gr}(m, W)$ is the tautological projection; we denoted by ev_S the tautological evaluation map. Observe that the map $\mathrm{ev}_S : S \to W$ is proper.

- M is a smooth compact complex manifold, endowed with a T-action; as usual, the notation M^T stands for the set $\{y \in M;\ Ty = y\}$ of fixed points of the T-action; suppose that M^T is a finite set of points. The embedding $M^T \hookrightarrow M$ is denoted by ι_T.

- Let $\phi : M \to \mathrm{Gr}(m, W)$ be a T-equivariant map, and introduce the pull-back bundles $S_M = \phi^* S$ and $S_{M^T} = \iota_T^* S_M$. We denoted by ev_M the induced evaluation map $S_M \to W$.

- For clarity, we indexed our spaces and maps, but these indices will be omitted whenever this does not cause confusion. For example if $p \in M$, then we will denote by S_p the fiber of the bundle S_M over the point p.

Literally, to say that Σ is fibered over M would mean that the map $\mathrm{ev}_M : S_M \to W$ establishes a diffeomorphism of S_M with Σ. Since this essentially never happens, we weaken this condition as follows.

Recall (see e.g. [9]) that to a smooth proper map $f : X \to Y$ between connected oriented manifolds of equal dimensions one can associate an integer $\deg(f)$ called the *degree*. This constant may be defined via the equality

$$\int_X f^* \mu = \deg(f) \int_Y \mu,$$

for any compactly supported form μ on Y.

An alternative definition of $\deg(f)$ is signed sum of the preimages of a regular value; the sign associated to a preimage depends on whether the map is orientation-preserving or reversing at the point. Since a holomorphic map is orientation-preserving everywhere, we have the following simple statement.

Lemma 3.1.1. *Let f be a holomorphic map between complex manifolds. Then f is of degree 1 if and only if there is dense open $U \subset X$ such that f restricted to U is a biholomorphism onto a dense open subset of Y.*

The definition of a degree-1 map may be extended to the following situation. Let $f : X \to Y$ be a smooth, proper map of complex manifolds, $U \subset X$ and $V \subset Y$ not necessarily smooth closed analytic subvarieties.

Definition 3.1.2. We say that f establishes a degree-1 map between U and V if there are Zariski open subsets $U^o \subset U$ and $V^o \subset V$, not containing singular points, such that $f|_{U^o} : U^o \to V^o$ is biholomorphic. Here Zariski open means that the complement is a closed analytic subvariety.

A key fact is that integration of differential forms with compact support may be extended to not necessarily smooth analytic subvarieties of complex manifolds.

Indeed, let X be a complex manifold, $U \subset X$ be a closed analytic subvariety, $U^s \subset U$ the set of smooth points. One can show (see [24], page 32) that in a local chart, with respect to the euclidean metric, the submanifold of smooth points $U^s \subset U$ has finite volume in bounded regions. Since a differential form μ with compact support in this chart will have bounded coefficients with respect to this same metric, we can conclude that the a priori improper integral $\int_{U^s} \mu$ is absolutely convergent. This allows us to define

$$\int_U \mu \stackrel{\text{def}}{=} \int_{U^s} \mu. \tag{3.1.2}$$

Proposition 3.1.3. *If the map f in Definition 3.1.2 establishes a degree 1 map between U and V, then*

$$\int_U f^*\mu = \int_V \mu \quad \textit{for every compactly supported smooth form } \mu \textit{ on } Y.$$

Proof. By definition, there are $U^o \subset U$, $V^o \subset V$ Zariski open subsets such that f establishes an orientation-preserving diffeomorphism between these.

First, since these sets differ from the sets of corresponding sets of smooth points by positive codimensional subsets, we have $\int_{U^s} \mu = \int_{U^o} \mu$ and $\int_{V^s} \mu = \int_{V^o} \mu$, where all the integrals are absolutely convergent.

On the other hand,

$$\int_{V^o} \mu = \int_{U^o} f^*\mu,$$

since, just as in the compactly supported case, for a form with absolutely convergent integral, integration commutes with pull-back by an oriented diffeomorphism. In view of the definition (3.1.2) this completes the proof. □

Another convenient way to describe our notion is

Proposition 3.1.4. *Let $f : X \to Y$ be a proper map of complex manifolds, $U \subset X$ possibly singular closed analytic subvariety. Suppose that there is $U^o \subset U$ Zariski open subset, not containing singular points, such that $f|_{U^o}$ is injective. Then f establishes a degree-1 map between U and $f(U)$.*

Proof. Since f is proper, $f(U)$ is a closed analytic subvariety of Y, (see [24], page 34). Injectivity implies that $\dim(U^o) = \dim(V^0)$, and hence there is a possibly smaller Zariski open $U' \subset U^o$ such that $f(U')$ is in the smooth part of $f(U)$. Since an injective holomorphic map between manifolds is biholomorphic, can conclude that f restricted to U' is a biholomorphism, and this completes the proof. □

Finally, for later reference, we note that the defintion of integrating over analytic subvarieties that we described above satisfies Stokes' theorem (see [24], page 34), and also allows us to define the push-forward along singular fibers.

We will use this latter statement in the following form.

Proposition 3.1.5. *Let M be a complex manifold, V be a complex vector bundle over M, and let $S \hookrightarrow V$ be a locally trivial subbundle with fibers which are possibly singular analytic subvarieties of the correspoinding linear fibers of V. Denote by $\pi : S \to M$ the projection. Then for any smooth compactly supported form μ on V, the push-forward of the restriction: $\pi_*\mu$ is a smooth form on M, moreover,*

$$\int_S \mu = \int_M \pi_*\mu.$$

Proof. In any local trivialization, the push-forward will take the form a smoothly varying parametric, uniformly absolutely convergent integral along an analytic subvariety in a linear space, where the parameters are the local coordinates on M. This immediately inplies the statement. □

Now we are ready to formulate our first localization formula.

Proposition 3.1.6. *Assume that in diagram* (3.1.1) *the fixed point set M^T is finite, and ev_M establishes a degree-1 map from S_M to Σ. Then we have*

$$\mathrm{eP}[\Sigma, W] = \sum_{p\in M^T} \frac{\mathrm{eP}[\mathrm{ev}_M(S_p), W]}{\mathrm{Euler}^T(\mathrm{T}_p M)}. \tag{3.1.3}$$

Remark 3.1.7.

1. The most natural situation is when M is a smooth submanifold of $\mathrm{Gr}(m, W)$. The more general setup we are considering in Proposition 3.1.6 works, however, even when the image $\phi(M)$ is singular.
2. Since the space $\mathrm{ev}_M(S_p)$ is a linear T-invariant subspace of W for $p \in M^T$, the polynomial $\mathrm{eP}[\mathrm{ev}_M(S_p)]$ is determined by the normalization axiom: it simply equals the product of those weights of W which are not weights of $\mathrm{ev}_M(S_p)$ (with multiplicities taken into account).
3. The equivariant Euler class in the denominator is also a product of weights (cf. (2.2.3)), hence each term in the sum is a rational function. After the summation, however, the denominators cancel, and one ends up with a polynomial result.

Proof. Vergne's integral formula, (2.2.16) combined with our assumption that $\mathrm{ev}_M : S_M \to \Sigma$ is degree-1, implies that

$$\mathrm{eP}[\Sigma] = \int_{S_M} \mathrm{ev}_M^* \mathrm{Thom}(W).$$

Integrating first along the fibers, we obtain that

$$\mathrm{eP}[\Sigma] = \int_M \tau_* \mathrm{ev}_M^* \mathrm{Thom}(W),$$

where the integrand $\tau_* \mathrm{ev}_M^* \mathrm{Thom}(W)$ is a smooth equivariant form on M. Now we apply the Berline-Vergne equivariant integration formula (2.2.14) to this form, and obtain that

$$\mathrm{eP}[\Sigma] = \sum_{p \in M^T} \frac{\left(\tau_* \mathrm{ev}_M^* \mathrm{Thom}(W)\right)^{[0]}(p)}{\mathrm{Euler}^T(\mathrm{T}_p M)}, \tag{3.1.4}$$

where, as usual, we denote by $\mu^{[0]}$ the differential-form-degree-zero part of the equivariant form μ. Since ev_M is a linear injective map on each fiber, the numerator of (3.1.4) is simply the integral $\int_{\mathrm{ev}_M(S_p)} \mathrm{Thom}(W)$. Now, using Vergne's formula (2.2.16) once again, we arrive at (3.1.3).

□

Using remark 2.2.4, formula (3.1.3) may be rewritten as follows. Let E be an equivariant vector bundle over M, and let $\gamma_p : W \to E_p$ for $p \in M$ be an equivariant family of surjective linear maps. Assume, that this establishes a degree-1 map between the subbundle

$$\{(p, w) \in M \times W;\ \gamma_p(w) = 0\}$$

and Σ. Then according to Remark 2.2.4, we have $\mathrm{eP}[\mathrm{ev}_M(S_p), W] = \mathrm{Euler}^T(E_p)$, which leads to the following variant of (3.3.1):

$$\mathrm{eP}[\Sigma] = \sum_{p \in M^T} \frac{\mathrm{Euler}^T(E_p)}{\mathrm{Euler}^T(\mathrm{T}_p M)}. \tag{3.1.5}$$

As a quick application, we will give yet another way of computing the equivariant Poincaré dual for the basic example introduced in §2.2.2.

Method 2: localization on the projectivized cone. Consider the smooth, T-invariant projective variety $\mathbb{P}\Sigma \subset \mathbb{P}^3$ cut out by the homogeneous equation $x_1 x_3 = x_2 x_4$. In the notation of (3.1.1), we have $M = \mathbb{P}\Sigma$, $m = 1$ and $W = \mathbb{C}^4$. Then the fixed point set $\mathbb{P}\Sigma^T$ consists of the four fixed points on $\mathbb{P}^3$ corresponding to the four coordinate axes.

Pick one of these fixed points, say, $p = (1 : 0 : 0 : 0)$, which corresponds to the coordinate line $S_p = \{x_2 = x_3 = x_4 = 0\}$. Using the normalization axiom, we have then $\mathrm{eP}[S_p] = \eta_2 \eta_3 \eta_4$.

Turning to the denominator in (3.1.3), it is not hard to see that

$$\mathrm{Euler}^T(\mathrm{T}_p \mathbb{P}\Sigma) = (\eta_2 - \eta_1)(\eta_4 - \eta_1).$$

Indeed, this is the standard yoga of toric geometry: consider the parallelogram formed by the weights η_1, η_2, η_3 and η_4; the fixed points of the torus action correspond to the vertices of this parallelogram, and the weights at a particular fixed point are the edge-vectors emanating from the associated vertex.

The contributions at the other fixed points may be computed likewise, and the result is the following complicated formula for the equivariant Poincaré dual:

$$\mathrm{eP}[\Sigma] = \frac{\eta_2\eta_3\eta_4}{(\eta_2-\eta_1)(\eta_4-\eta_1)} + \frac{\eta_1\eta_3\eta_4}{(\eta_1-\eta_2)(\eta_3-\eta_2)} + \frac{\eta_1\eta_2\eta_4}{(\eta_2-\eta_3)(\eta_4-\eta_3)} + \frac{\eta_1\eta_2\eta_3}{(\eta_1-\eta_4)(\eta_3-\eta_4)}. \tag{3.1.6}$$

This rational function is *not* a polynomial, however, assuming $\eta_1+\eta_3=\eta_2+\eta_4$ holds, it can be easily shown to reduce to the simple form (2.2.6).

We note that this procedure may be applied, inductively, to more general toric varieties, and, again, the data may be read off the corresponding polytope. However, if the polytope is not simple, then the prescription is more involved.

3.2 An interlude: the case of $d = 1$

In this paragraph, we consider the case $d=1$ of the A_d-singularities introduced in §2.1.2, and recover the classical result of Porteous.

We have $\mathcal{J}_1(n,k) = \mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^k)$, and $\Theta_1 \subset \mathcal{J}_1(n,k)$ consists of those linear maps $\mathbb{C}^n \to \mathbb{C}^k$ whose kernel is 1-dimensional. These maps may be identified with k-by-n matrices, and the weight of the action on the entry e_{ji} is equal to $\theta_j - \lambda_i$. Then the closure $\overline{\Theta}_1$ consist of those k-by-n matrices which have a nontrivial kernel:

$$\overline{\Theta}_1 = \{A \in \mathrm{Hom}(\mathbb{C}^k,\mathbb{C}^n);\ \exists v\in\mathbb{C}^n,\ v\neq 0:\ Av=0\}. \tag{3.2.1}$$

This description immediately suggests us an equivariant birational fibration of $\overline{\Theta}_1$ over $\mathbb{P}^{n-1}$, fitting the conditions of Proposition 3.1.6: the fiber over a point $[v]\in\mathbb{P}^{n-1}$ is the linear subspace $\{A;\ Av=0\}\subset\Theta_1$; where $[v]$ stands for the point in $\mathbb{P}^{n-1}$ corresponding to the nonzero vector $v\in\mathbb{C}^n$.

Again, we simply need to collect our fixed-point data, and then apply (3.1.3). There are n fixed points, $p_1,\dots,p_n$ in $\mathbb{P}^{n-1}$, corresponding to the coordinate axes. The weights of $\mathrm{T}_{p_i}\mathbb{P}^{n-1}$ are $\{\lambda_s-\lambda_i;\ s\neq i\}$. The fiber at p_i is the set of matrices A with all entries in the ith column vanishing. Again, using the normalization axiom, this shows that the equivariant Poincaré dual of the fiber at p_i is $\prod_{j=1}^k(\theta_j-\lambda_i)$, so our localization formula looks as follows:

$$\mathrm{eP}[\Theta_1] = \sum_{i=1}^n \frac{\prod_{j=1}^k(\theta_j-\lambda_i)}{\prod_{s\neq i}(\lambda_s-\lambda_i)} \tag{3.2.2}$$

This is a finite sum for fixed n, but as n increases, the number of terms also increases. There is a way, however, to further "localize" this expression, and obtain a formula, which only depends on the local behavior of a certain function at a single point.

Indeed, consider the following rational differential form on $\mathbb{P}^1$:

$$-\frac{\prod_{j=1}^k(\theta_j-z)}{\prod_{i=1}^n(\lambda_i-z)}\,dz.$$

Observe that the residues of this form at finite poles: $\{z = \lambda_i;\ i = 1, \ldots, n\}$ exactly recover the terms of the sum (3.2.2). Then, applying the Residue theorem, we obtain

$$\mathrm{eP}[\Theta_1] = \operatorname*{Res}_{z=\infty} \frac{\prod_{j=1}^{k}(\theta_j - z)}{\prod_{i=1}^{n}(\lambda_i - z)}\, dz.$$

Finally, after the change of variables $z \to -1/q$, we end up with

$$\mathrm{eP}[\Theta_1] = \operatorname*{Res}_{q=0} \frac{\prod_{j=1}^{k}(1 + q\theta_j)}{\prod_{i=1}^{n}(1 + q\lambda_i)} \frac{dq}{q^{k-n+2}},$$

which, according to (2.2.20), is exactly the relative Chern class c_{k-n+1}. Thus we recovered the well-known Giambelli-Thom -Porteous formula ([39]; [24] Chapter I.5).

As a final remark, note that our basic example introduced in §2.2.2 is a special case of Θ_1, corresponding to the values $n = k = 2$. Hence this computation provides us with a 3RD METHOD of arriving at (2.2.6). This computation uses localization, similarly to the 2ND METHOD, but the two constructions are different.

$$\mathrm{eP}[\Sigma] = \frac{\eta_1\eta_2}{\eta_3 - \eta_2} + \frac{\eta_3\eta_4}{\eta_2 - \eta_3} \tag{3.2.3}$$

Using $\eta_1 + \eta_3 = \eta_2 + \eta_4$, we arrive to the formula (2.2.6).

3.3 Variations of the localization formula

We will need to amend and generalize Proposition 3.1.6 in order to be able deal with Θ_d for $d > 1$. We generalize in two directions: first, we drop the assumption on the linearity of the fibers, secondly, we will allow M to be singular.

3.3.1 Nonlinear fibers

Observe that, during the proof of Proposition 3.1.6, we never used the assumption that the fibers are linear spaces. In fact, using Proposition 3.1.5, the same formula and the same argument holds if the fibers of S are possibly singular analytic subvarieties.

Proposition 3.3.1. *Let Σ be a closed subvariety of the complex vector space W. Assume that M is a smooth compact complex manifold, V is a complex vector bundle over M, and let $S \hookrightarrow V$ be a locally trivial subbundle with fibers which are possibly singular analytic subvarieties of the correspoinding linear fibers of V. Suppose that we have a proper map: $\mathrm{ev}_V : V \to W$, which establishes a degree-1 map from S to Σ. Then*

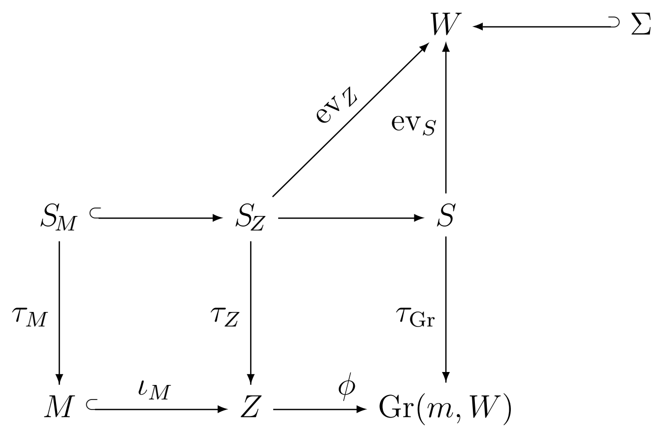
$$\mathrm{eP}[\Sigma, W] = \sum_{p \in M^T} \frac{\mathrm{eP}[\mathrm{ev}_V(S_p), W]}{\mathrm{Euler}^T(\mathrm{T}_p M)}. \tag{3.3.1}$$

We will use this variant of the localization in 4.3.1, for the localization on a flag variety.

3.3.2 Fibrations over a singular base

Finally, we remove the assumption that M is smooth. For brevity, below, without explicitly stating this, we will assume that every space and map is in the T-equivariant category. We will apply the following proposition for the localization in $\mathcal{O}$ in 4.3.3.

Proposition 3.3.2. *1. Let Σ be a closed subvariety of the complex vector space W. Assume that Z is a compact, smooth complex manifold, and $M \subset Z$ is a possibly singular, closed subvariety with a finite set of fixed points M^T. Consider the following analog of diagram 3.1.1:*



(3.3.2)

Assume that ev_Z *establishes a degree-1 map between* $\tau_Z^{-1}(M)$ *and* Σ.

Then

$$\mathrm{eP}[\Sigma] = \sum_{p \in M^T} \frac{\mathrm{eP}[\mathrm{ev}_Z(S_p)]\,\mathrm{emult}_p[M, Z]}{\mathrm{Euler}^T(\mathrm{T}_p Z)}. \tag{3.3.3}$$

2. *Assume that there is a T-equivariant vector bundle E over M, and an equivariant family of surjective linear maps $\gamma_p : W \to E_p$ for $p \in M$, such that the set*

$$\{(p, w) \in M \times W;\ \gamma_p(w) = 0\}$$

is a subbundle of the trivial bundle $M \times W$, and it maps to Σ in a birational fashion. Then

$$\mathrm{eP}[\Sigma] = \sum_{p \in M^T} \frac{\mathrm{Euler}^T(E_p)\,\mathrm{emult}_p[M, Z]}{\mathrm{Euler}^T(\mathrm{T}_p Z)}.$$

Proof. The second part is the combination of the first part and (3.1.5). The proof of the first part is analogous to that of Proposition 3.1.6; when passing to (3.1.4), however, one needs to use Rossmann's integration formula (2.2.13).

Using Vergne's integral formula, and the fact that ev is birational:

$$\mathrm{eP}[\Sigma] = \int_{S_{\tau_Z^{-1}(M)}} \mathrm{ev}_Z^* \mathrm{Thom}(W).$$

Integrating first along the fibres:

$$\mathrm{eP}[\Sigma] = \int_M \tau_* \mathrm{ev}_Z^* \mathrm{Thom}(W),$$

Using Rossmann's formula (2.2.13) we get

$$\mathrm{eP}[\Sigma] = \sum_{p \in M^T} \frac{\left(\tau_* \mathrm{ev}_S^* \mathrm{Thom}(W)\right)^{[0]}(p) \mathrm{emult}_p[M, Z]}{\mathrm{Euler}^T(T_p M)},$$

Since ev_Z is a linear injective morphism on each fiber, the numerator is simply the integral $\int_{\mathrm{ev}_Z(S_p)} \mathrm{Thom}(W)$. Now, using Vergne's formula (2.2.16) once again, we arrive at (3.3.3). □

Chapter 4

Multidegrees of A_n singularities

4.1 The test curve model

In §2.1, we described the variety Θ_d in two different ways: as an example of a contact singularity class defined in (2.1.2), and as the Boardman class corresponding to the sequence $(1, 1, \ldots, 1)$ (cf. Prop. 2.1.7). In this section, we recall another, birationally equivalent description of Θ_d – the so-called "test curve model" – which goes back to the works of Porteous, Ronga, and Gaffney [40, 43, 21]. Roughly, the idea of the construction is to generalize (3.2.1) to $d > 1$ by requiring that the map-jet $\Psi \in \mathcal{J}_d(n,k)$ carry a d-jet of a curve in $\mathbb{C}^n$ to zero. As we have not found a complete proof of the appropriate statement (Theorem 4.1.1) in the literature, we give one below.

Recall the notation $\mathrm{Lin} : \mathcal{J}_d(n,k) \to \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^k)$ for the linear part of map-jets. A d-jet of a curve in $\mathbb{C}^n$ is simply an element of $\mathcal{J}_d(1,n)$. We will call such a curve γ *regular* if $\mathrm{Lin}(\gamma) \neq 0$; introduce the notation $\mathcal{J}_d^{\mathrm{reg}}(1,n)$ for the set of these curves:

$$\mathcal{J}_d^{\mathrm{reg}}(1,n) \stackrel{\mathrm{def}}{=} \{\gamma \in \mathcal{J}_d(1,n);\ \mathrm{Lin}(\gamma) \neq 0\}\,. \tag{4.1.1}$$

Now define the set

$$\Theta'_d = \{\Psi \in \mathcal{J}_d(n,k);\ \exists \gamma \in \mathcal{J}_d^{\mathrm{reg}}(1,n) \text{ such that } \Psi \circ \gamma = 0\}\,. \tag{4.1.2}$$

In words: Θ'_d is the set of those d-jets of maps, which take at least one regular curve to zero. By definition, Θ'_d is the image of the closed subvariety of $\mathcal{J}_d(n,k) \times \mathcal{J}_d^{\mathrm{reg}}(1,n)$ defined by the algebraic equations $\Psi \circ \gamma = 0$, under the projection to the first factor. By a theorem of Chevalley (see [26], Ex. 3.19, page 94), Θ'_d is constructible. We will not use the set Θ'_d itself in this thesis, rather its Zariski closure. This closure is identified via the following

Theorem 4.1.1. *The Zariski closures of Θ_d and Θ'_d in $\mathcal{J}_d(n,k)$ coincide.*

Proof. Since K_d is an algebraic group acting on $\mathcal{J}_d(n,k)$, to prove the theorem, it is sufficient to show that

- Θ'_d is $\mathcal{K}_d$-invariant,
- $\Theta'_d \cap \Theta_d$ is nonempty,
- $\operatorname{codim}(\overline{\Theta'}_d) = \operatorname{codim}(\Theta_d)$ in $\mathcal{J}_d(n,k)$, and that
- the subvariety $\overline{\Theta'}_d \subset \mathcal{J}_d(n,k)$ is irreducible.

To prove that these 4 statements are sufficient, we observe that according to Proposition 2.1.2 and Corollary 2.1.4, Θ_d is a single $\mathcal{K}_d$-orbit. This fact, with the first two properties above induces that $\Theta_d \subset \Theta'_d$, so $\overline{\Theta}_d \subset \overline{\Theta'}_d$. Since $\overline{\Theta'}$ is irreducible of the same dimension as $\overline{\Theta}$, $\overline{\Theta}_d = \overline{\Theta'}_d$ must hold.

To show the $\mathcal{K}_d$-invariance of Θ', observe that if $\gamma \in \mathcal{J}_d(1,n)$ is regular and $\Delta \in \operatorname{Diff}_d(n)$, then $\Delta \circ \gamma$ is also regular. Indeed, in this case

$$\operatorname{Lin}(\Delta \circ \gamma) = \operatorname{Lin}(\Delta) \cdot \operatorname{Lin}(\gamma) \neq 0.$$

Now, if $\Psi \in \mathcal{J}_d(n,k)$ such that $\Psi \circ \gamma = 0$ for some regular γ, and $(M, \Delta) \in \mathcal{K}_d$, then recalling the action (2.1.4), we have

$$[(M,\Delta)\cdot\Psi]\circ(\Delta\circ\gamma) = (M\cdot\Psi)\circ\Delta^{-1}\circ(\Delta\circ\gamma) = (M\cdot\Psi)\circ\gamma = (M\circ\gamma)\cdot(\Psi\circ\gamma) = 0.$$

This shows that $\Delta \circ \gamma$ is an appropriate test curve for the transformed map-jet $(M,\Delta)\cdot\Psi$.

To find an element in the intersection of Θ_d and Θ'_d, consider the map-jet

$$\Psi_0(x_1,\dots,x_n) = (0, x_2, \dots, x_n, 0, \dots, 0).$$

It obviously belongs to Θ_d; on the other hand, for the test curve $\gamma(t) = (t, 0, \dots, 0)$, we have $\operatorname{Lin}(\gamma) \neq 0$ and $\Psi_0 \circ \gamma = 0$ in $\mathcal{J}_d(n,k)$, hence $\Psi_0 \in \Theta'_d$.

Regarding the codimensions, we have $\operatorname{codim}(\Theta_d) = d(k-n+1)$ according to Proposition 2.1.1. The proof of the irreducibility of Θ'_d and the computation of its codimension (cf. Corollary 4.1.5) will follow from the more detailed study of its structure, to which we devote the rest of this section.

□

Our first project is to write down the equation $\Psi \circ \gamma = 0$ in coordinates. This is a rather mechanical exercise, and we will spend some time setting up the notation.

A curve $\gamma \in \mathcal{J}_d(1,n)$ is parametrized by d vectors $v_1, \dots, v_d$ in $\mathbb{C}^n$:

$$\gamma(t) = tv_1 + t^2 v_2 + \dots + t^d v_d, \tag{4.1.3}$$

In this explicit form, the condition of regularity, $\operatorname{Lin}(\gamma) \neq 0$, simply means that $v_1 \neq 0$.

Next, we switch to a new parametrization of our space $\mathcal{J}_d(k,n)$. Separating the similar homogeneous components of the k polynomials, $P_1, \dots, P_k$, and thinking of a homogeneous degree-l polynomial as an element of $\mathrm{Hom}(\mathrm{Sym}^l\mathbb{C}^n, \mathbb{C})$, we may represent $\Psi \in \mathcal{J}_d(k,n)$ as a linear map

$$\Psi = (\Psi^1, \dots, \Psi^d) : \oplus_{l=1}^d \mathrm{Sym}^l\mathbb{C}^n \to \mathbb{C}^k. \tag{4.1.4}$$

The standard basis of the vector space $\oplus_{l=1}^d \mathrm{Sym}^l\mathbb{C}^n$ may be parametrized by non-decreasing sequences of positive integers, or, alternatively – and this is the language we will prefer – by *partitions*. Namely, to the partition $[i_1,, \dots, i_l]$ of the integer $i_1 + \dots + i_l$ with $1 \le i_m \le n$, we associate the basis element $e_{i_1} \cdots e_{i_l} \in \mathrm{Sym}^l\mathbb{C}^n$.

In what follows, certain integer characteristics of partitions will be used.

Notation 4.1.2. For a partition $\tau = [i_1, \dots, i_l]$ of the integer $i_1 + \ldots + i_l$, introduce

- the *length*: $|\tau| = l$,
- the *sum*: $\mathrm{sum}(\tau) = i_1 + \ldots + i_l$,
- the *maximum*: $\max(\tau) = \max(i_1, \dots, i_l)$,
- and the *number of permutations*: $\mathrm{perm}(\tau)$, which is the number of different sequences consisting of the numbers $i_1, \dots, i_l$; e.g. $\mathrm{perm}([1,1,1,3]) = 4$.

Denoting the set of all nonempty partitions by Π, we can parametrize the basis elements of $\oplus_{l=1}^d \mathrm{Sym}^l\mathbb{C}^n$ by the finite set

$$\{\tau \in \Pi;\ |\tau| \le d,\ \max(\tau) \le n\}.$$

We will also use the notation $\Pi[m]$ for the set of all partitions of the positive integer m:

$$\Pi[m] = \{\tau \in \Pi;\ \mathrm{sum}(\tau) = m\}. \tag{4.1.5}$$

Next, for a map-jet $\Psi \in \mathcal{J}_d(n,k)$, a sequence $\mathbf{v} = (v_1, v_2, \dots)$ of vectors in $\mathbb{C}^n$, and a partition $\tau = [i_1, \dots, i_l]$ satisfying $l \le d$, $\max(\tau) \le n$, introduce the shorthand

$$\mathbf{v}_\tau = \prod_{j=1}^{l} v_{i_j} \in \mathrm{Sym}^l\mathbb{C}^n \quad \text{and} \quad \Psi(\mathbf{v}_\tau) = \Psi^l(v_{i_1}, \dots, v_{i_l}) \in \mathbb{C}^k. \tag{4.1.6}$$

Armed with this new notation, we can write down the equation $\Psi \circ \gamma = 0$ more explicitly, as follows.

Lemma 4.1.3. *Let $\gamma \in \mathcal{J}_d(1,n)$ be given in the form* (4.1.3)*. Then, using the notation* (4.1.6)*, the equation $\Psi \circ \gamma = 0$ is equivalent to the following system of d linear equations with values in $\mathbb{C}^k$ on the components Ψ^l of $\Psi \in \mathcal{J}_d(n,k)$, $l = 1, \dots, d$:*

$$\sum_{\tau \in \Pi[m]} \mathrm{perm}(\tau)\, \Psi(\mathbf{v}_\tau) = 0, \quad m = 1, 2, \dots, d. \tag{4.1.7}$$

Let us see what the system of equations (4.1.7) looks like for small d. To make the formulas easier to follow, we will use the lth capital letter of the alphabet for the symmetric multi-linear map Ψ^l introduced in (4.1.4): we will write A for the linear part Ψ^1 of Ψ, B for its second order part, etc. With this convention (see also (4.1.3)), the system of equations for $d = 4$ reads as follows:

$$
\begin{gathered}
A(v_1) = 0, \qquad (4.1.8)\\
A(v_2) + B(v_1, v_1) = 0,\\
A(v_3) + 2B(v_1, v_2) + C(v_1, v_1, v_1) = 0,\\
A(v_4) + 2B(v_1, v_3) + B(v_2, v_2) + 3C(v_1, v_1, v_2) + D(v_1, v_1, v_1, v_1) = 0.
\end{gathered}
$$

For a curve $\gamma \in \mathcal{J}_d^{\mathrm{reg}}(1, n)$, introduce the notation $\varepsilon(\gamma)$ for the system of equations (4.1.7), and

$$\mathrm{Sol}_{\varepsilon(\gamma)} \text{ for the space of solutions of this system.} \qquad (4.1.9)$$

Then, according to (4.1.2),

$$\Theta'_d = \bigcup \left\{ \mathrm{Sol}_{\varepsilon(\gamma)};\ \gamma \in \mathcal{J}_d^{\mathrm{reg}}(1, n) \right\}. \qquad (4.1.10)$$

In the following Proposition, we collect some simple facts about the system (4.1.7).

Proposition 4.1.4. *1. Let $0 \neq v \in \mathbb{C}^n$, and assume that $\gamma \in \mathcal{J}_d^{\mathrm{reg}}(1, n)$ is such that $\mathrm{Lin}(\gamma)$ is parallel to v. Pick a hyperplane H in $\mathbb{C}^n$ which is complementary to v. Then there is a unique $\delta \in \mathrm{Diff}_d(1)$ such that*

$$\gamma \circ \delta = tv + t^2 v_2 + \cdots + t^d v_d \quad \text{with } v_2, v_3, \ldots, v_d \in H. \qquad (4.1.11)$$

2. *For $\gamma \in \mathcal{J}_d^{\mathrm{reg}}(1, n)$, the set of solutions $\mathrm{Sol}_{\varepsilon(\gamma)} \subset \mathcal{J}_d(n, k)$ is a linear subspace of codimension dk.*

3. *Set*

$$\mathcal{J}_d(n, k)^0 = \{\Psi \in \mathcal{J}_d(n, k) |\, \dim \ker(\mathrm{Lin}(\Psi)) = 1\}.$$

For any $\gamma \in \mathcal{J}_d^{\mathrm{reg}}(1, n)$, $\mathrm{Sol}_{\varepsilon(\gamma)} \cap \mathcal{J}_d(n, k)^0$ is a dense subset of $\mathrm{Sol}_{\varepsilon(\gamma)}$.

4. *If $\Psi \in \mathcal{J}_d(n, k)^0$, then Ψ may belong to at most one of the spaces $\mathrm{Sol}_{\varepsilon(\gamma)}$. More precisely,*

$$\text{if } \gamma, \gamma' \in \mathcal{J}_d^{\mathrm{reg}}(1, n),\ \dim(\ker \mathrm{Lin}(\Psi)) = 1, \text{ and } \Psi \circ \gamma = \Psi \circ \gamma' = 0,$$

then there exists $\delta \in \mathrm{Diff}_d(1)$ such that $\gamma' = \gamma \circ \delta$.

5. *Given $\gamma, \gamma' \in \mathcal{J}_d^{\mathrm{reg}}(1, n)$, we have $\mathrm{Sol}_{\varepsilon(\gamma)} = \mathrm{Sol}_{\varepsilon(\gamma')}$ if and only if there is a $\delta \in \mathrm{Diff}_d(1)$ such that $\gamma' = \gamma \circ \delta$.*

Proof. For (1), write explicitly $\gamma(s) = sw_1 + \cdots + s^d w_d$ and $\delta = \lambda_1 t + \cdots + \lambda_d t^d$. After making the substitution $s \mapsto \delta$, we obtain a curve $\gamma \circ \delta = tv + t^2 v_2 + \cdots + t^d v_d$, where $v_l = \lambda_l w_1$+terms with λs which have lower indices than l; this clearly implies the statement.

The second statement follows from the presence of the term $\Psi^l(v_1, \ldots v_1)$ in the lth equation of (4.1.7), which is clearly linearly independent of the rest of the terms in the first l equations.

For the third statement, let $\gamma = (v_1, \ldots, v_d)$, $v_1 \neq 0$ be as in (4.1.3). Then $\Psi = (A, B, \ldots) \in \mathrm{Sol}_{\varepsilon(\gamma)}$ iff (4.1.8) equations hold. For arbitrary $A = \mathrm{Lin}(\Psi)$, with $v_1 \in \ker(A)$, the solution set is a $(d-1)k$-codimensional subspace of $\mathrm{Hom}(\oplus_{l=2}^d \mathrm{Sym}^l \mathbb{C}^n, \mathbb{C}^k)$, i.e the set of $(B, C, \ldots)$s. The statement now follows from the fact, that

$$\left\{A \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^k) \,|\, \dim\ker A = 1\right\}$$

is dense in

$$\left\{A \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^k) \,|\, \dim\ker A \geq 1\right\}$$

.

To prove statement (4), we assume that γ and γ' are normalized according to (4.1.11) with respect to some $v \in \ker(\mathrm{Lin}(\Psi))$; then we show that $\gamma = \gamma'$ using induction. Assume, for example, that the two curves coincide up to the third order, i.e. $v_1 = v_1'$, $v_2 = v_2'$, $v_3 = v_3'$. Then we see from (4.1.8) that $A(v_4) = A(v_4')$. We have $A = \mathrm{Lin}(\Psi)$ and $\ker(A) = \mathbb{C}v_1$, hence $v_4, v_4' \in H$ and $A(v_4) = A(v_4')$ imply $v_4 = v_4'$. This completes the inductive step.

The last statement is an immediate consequence of statement (2),(3) and (4). $\square$

We can summarize the results of this section in the following diagram:

$$\begin{array}{ccccccc}
\Theta_d & \hookrightarrow & \Theta_d' & \hookrightarrow & \mathcal{J}_d(n,k) & \xleftarrow{\mathrm{ev}_S} & S \\
 & & & & & \swarrow_{\tau_{\mathrm{Gr}}} & \\
 & & \mathcal{J}_d^{\mathrm{reg}}(1,n) & \xrightarrow{\tilde{\phi}} & \mathrm{Gr}(\mathrm{codim} = dk, \mathcal{J}_d(n,k)) & &
\end{array} \qquad (4.1.12)$$

Explanations:

- Each space in the diagram carries an action of the group $GL(k) \times GL(n)$, and the maps are equivariant with respect to this action.
- As usual, we denote by S the tautological bundle over the Grassmannian and by ev_S the tautological evaluation map (cf. diagram 3.1.1). The rank of the bundle S equals to $\dim(\mathcal{J}_d(n,k)) - dk$.
- $\tilde{\phi} : \mathcal{J}_d^{\mathrm{reg}}(1,n) \to \mathrm{Gr}(\mathrm{codim} = dk, \mathcal{J}_d(n,k))$; $\gamma \mapsto \mathrm{Sol}_{\varepsilon(\gamma)}$ was introduced after (4.1.8).

Now we have

Proposition 4.1.5. *1. The map $\tilde{\phi}$ is $\mathrm{Diff}_d(1)$-invariant, and the induced map ϕ_{Gr} on the orbits is injective.*

2. The map ev_S restricted to $[\tau_{\mathrm{Gr}}]^{-1}\overline{\mathrm{im}(\tilde{\phi})}$ is of degree 1 onto $\overline{\Theta'_d}$.

3. $\overline{\Theta'_d}$ is an irreducible subvariety of $\mathcal{J}_d(n,k)$.

4. $\mathrm{codim}(\overline{\Theta'}_d) = d(k-n+1)$.

Proof. The first statement immediately follows from Proposition 4.1.4 (2) and (5), while the second is a consequence of Proposition 4.1.4 (3),(4) and Proposition 3.1.4: ev_S is injective on

$$\mathcal{J}_d(n,k)^0 \cap [\tau_{\mathrm{Gr}}]^{-1}\mathrm{im}(\tilde{\phi}),$$

a dense subset of the total space of this vector bundle.

To prove the third statement, we rewrite (4.1.10) in terms of diagram (4.1.12) as follows:

$$\Theta'_d = \mathrm{ev}_S\left([\tau_{\mathrm{Gr}}]^{-1}\mathrm{im}(\tilde{\phi})\right) \tag{4.1.13}$$

Moreover, ev_S is proper, so

$$\overline{\Theta'_d} = \mathrm{ev}_S\left([\tau_{\mathrm{Gr}}]^{-1}\overline{\mathrm{im}(\tilde{\phi})}\right) \tag{4.1.14}$$

Moreover, the Zariski closure of the image under any morphism of an irreducible variety is irreducible. Apply this to the morphism $\tilde{\phi} : \mathcal{J}_d^{\mathrm{reg}}(1,n) \to \mathrm{Gr}(\mathrm{codim} = dk, \mathcal{J}_d(n,k))$, and we get that $\overline{\mathrm{im}(\tilde{\phi})}$ is an irreducible closed subvariety of $\mathrm{Gr}(\mathrm{codim} = dk, \mathcal{J}_d(n,k))$.

Hence, by (4.1.14), ev_S is a map onto $\overline{\Theta'_d}$ from a vector bundle whose base $\overline{\mathrm{im}(\tilde{\phi})}$ is irreducible; this implies the third statement, because the closure of the image under any morphism of an irreducible variety is irreducible.

Finally, note that the fibers of this vector bundle are codimension-dk vector spaces in $\mathcal{J}_d(n,k)$ (Proposition 4.1.4, (2)), while the base $\overline{\mathrm{im}(\tilde{\phi})}$ has dimension $d(n-1)$ by the first statement: the dimension of $\mathcal{J}_d^{\mathrm{reg}}(1,n)$ is dn, and the dimension of $\mathrm{Diff}_d(1)$ is d. Hence the codimension of Θ'_d equals $dk - d(n-1) = d(k-n+1)$. □

Remark 4.1.6. • We can use the first part of Proposition 4.1.4 for defining a complex manifold structure on the $\mathrm{Diff}_d(1)$-orbits of $\mathcal{J}_d^{\mathrm{reg}}(1,n)$. We construct a bundle Q_d on $\mathbb{P}^{n-1}$, and a $\mathrm{Diff}_d(1)$-invariant holomorphic map $\rho : \mathcal{J}_d^{\mathrm{reg}}(1,n) \to Q_d$.

Let

$$\mathbb{C}_i^{n-1} = \left\{[x_1 : \ldots : x_n] \in \mathbb{P}^{n-1} : x_i \neq 0\right\},\ i = 1, \ldots n$$

be the standard coordinate charts on $\mathbb{P}^{n-1}$, and

$$H_i = \{(x_1 : \ldots : x_n) \in \mathbb{C}^n : x_i = 0\},\ i = 1, \ldots, n$$

be the $n-1$-dimensional coordinate subspaces of $\mathbb{C}^n$.

Take

$$U_i = \mathbb{C}_i^{n-1} \times H_i^{\times(d-1)} \quad i = 1, \dots, n$$

and define

$$\mathcal{J}_d^{\text{reg}}(1,n)_i = \{\gamma = (v_1, \dots v_d) \in \mathcal{J}_d^{\text{reg}}(1,n) : \text{ the } i\text{th coordinate of } v_1 \neq 0\}$$

We define a map

$$\rho_i : \mathcal{J}_d^{\text{reg}}(1,n)_i \to U_i$$

using the first part of Proposition 4.1.4: for $\gamma = (v_1, \dots, v_d) \in \mathcal{J}_d^{\text{reg}}(1,n)_i$ there is a unique $h^i(\gamma) \in \text{Diff}_d(1)$ such that $\gamma \cdot h^i(\gamma) \in U_i$. If we think of γ as a $n \times d$ matrix with v_i in the ith column, (see (4.2.2)), then $\text{Diff}_d(1)$ is a $d \times d$ matrix group (see Lemma 4.2.10), and U_i consists of matrices with 1 in the $(i,1)$ entry, and 0 in the (i,j) entries for $j = 2, \dots, d$.

The entries of $h^i(\gamma)$ are polynomials in the coordinates of $v_2, \dots, v_d$, and rational expressions of the coordinates of v_1.

If we glue the patches $U_i, \dots, U_n$ with the holomorphic transition functions

$$\phi_{i,j} : U_i \to U_j$$

$$\gamma \mapsto \gamma \cdot h^j(\gamma),$$

then we get a complex bundle over $\mathbb{P}^{n-1}$, which we denote by Q_d, and the maps ρ_i glue together to give a map

$$\rho : \mathcal{J}_d^{\text{reg}}(1,n) \to Q_d$$

which is invariant on the $\text{Diff}_d(1)$ orbits, and the induced map on the orbits is injective. Note that the domain of $\phi_{i,j}$ is formed by those $n \times d$ matrices of U_i wich have nonzero $(j,1)$ entry.

Since the entries of $h^j(\gamma)$ are polynomials in the coordinates of $v_2, \dots, v_d$, we get that Q_d is a complex bundle over $\mathbb{P}^{n-1}$, but not a complex vector bundle.

- As a result, we can think of the quotient $\mathcal{J}_d^{\text{reg}}(1,n)/\text{Diff}_d(1)$ as a complex smooth manifold. However, we will not use this complex manifold structure in the construction of a proper compactification of $\mathcal{J}_d^{\text{reg}}(1,n)/\text{Diff}_d(1)$.

The fact that $\tilde{\phi}$ defines an embedding of this quotient into $\text{Gr}(\text{codim} = dk, \mathcal{J}_d(n,k))$ suggests a reasonable compactification of $\mathcal{J}_d^{\text{reg}}(1,n)/\text{Diff}_d(1)$: the closure of $\text{im}(\tilde{\phi})$ in $\text{Gr}(\text{codim} = dk, \mathcal{J}_d(n,k))$. The choice of the compactification is very important from the point of view of the efficiency of the resulting formulas, and we will be very careful in constructing one. This is the subject of the next section.

4.1.1 Nonreductive quotients

The standard construction of quotient spaces in algebraic geometry uses Mumford's Geometric Invariant Theory (GIT) [36]. Given a *reductive* algebraic group G acting linearly on a quasi-projective variety X, GIT identifies two important G-invariant subsets of X: the stable locus, X^s, a Zariski open subset whose topological quotient space X^s/G is an algebraic variety, and the semistable locus X^{ss}, a Zariski open subset which contains the stable locus and maps to the 'GIT quotient' of X, denoted $X//G$, such that $X//G$ is a 'categorical' quotient of X^{ss}.

Mumford's GIT construction of quotients $X//G$ depends in a crucial way, however, on the group G being reductive (that is, G is the complexification of a maximal compact subgroup). On the other hand, there are many natural geometric problems where the group G is not reductive, so that GIT does not apply, and these include the diffeomorphism groups like $\mathrm{Diff}_d(1)$.

On the other hand, Frances Kirwan and Brent Doran have been developing a generalisation of GIT for non-reductive group actions, including various notions of stable and semistable points and compactified quotients [11]. They transfer the problem to conventional GIT by choosing a reductive group G which contains as a subgroup the non-reductive group H acting on X, and consider the associated reductive G-actions on projective compactifications $\overline{G \times_H X}$ of $G \times_H X$. The GIT quotients $\overline{G \times_H X}//G$ provide compactifications of X^s/H, and (in principle, at least) existing localisation methods can be applied to the action of G on $\overline{G \times_H X}$ to study the topology of these compactified quotients of the non-reductive action. For more details, see [11]

At the moment, the methods of Kirwan and Doran do not handle the difficulties related to localizations on nonreductive quotients. We hope, that our work represents a step in the direction of creating an effective theory of localization on nonreductive quotients.

4.2 The compactification

As we observed at the end of the previous section, the morphism $\tilde{\phi}$ in diagram (4.1.12) may be used to compactify $\mathcal{J}_d^{\mathrm{reg}}(1,n)/\mathrm{Diff}_d(1)$, and, in principle, allows us to apply the localization techniques of §3. The resulting formulas turn out to be intractable, however, and the purpose of this section is to replace the Grassmanian by a "smaller" space, which provides us with a better compactification and, hopefully, with more efficient formulas.

The constructions of this section form the backbone of the thesis; we will employ two ideas. The first is straightforward: we note that the system of equations (4.1.7) has a special form respecting a certain filtration, and thus not every dk-codimensional linear subspace of the Grassmanian may appear as the solution space of a system of our equations. Restricting out attention to these special systems gives us a smaller space to consider (cf. §4.2.1).

The second idea, detailed in §4.2.2, is a bit more involved. The main features of this construction are removing a certain part of the space of regular curves, thus breaking the $\mathrm{Diff}_d(1)$-symmetry, and then fibering the remainder over the space of full flags of d-dimensional subspaces of $\mathbb{C}^n$. This leads to a double fibration, whose study we are able to reduce to that of a single fiber.

4.2.1 Embedding into the space of equations

We start by rewriting the linear system $\Psi \circ \gamma = 0$ associated to $\gamma \in \mathcal{J}_d(1,n)$ in a dual form (cf. Lemma 4.1.3). The system is based on the standard composition map (2.1.1):

$$\mathcal{J}_d(n,k) \times \mathcal{J}_d(1,n) \longrightarrow \mathcal{J}_d(1,k),$$

which, in view of $\mathcal{J}_d(n,k) = \mathcal{J}_d(n,1) \otimes \mathbb{C}^k$, is derived from the map

$$\mathcal{J}_d(n,1) \times \mathcal{J}_d(1,n) \longrightarrow \mathcal{J}_d(1,1)$$

via tensoring with $\mathbb{C}^k$. Observing that composition is linear in its first argument, and passing to linear duals, we may rewrite this correspondence in the form

$$\psi : \mathcal{J}_d(1,n) \longrightarrow \mathrm{Hom}(\mathcal{J}_d(1,1)^*, \mathcal{J}_d(n,1)^*). \tag{4.2.1}$$

To present this map explicitly, we recall (cf. (4.1.3)) that a d-jet of a curve $\gamma \in \mathcal{J}_d(1,n)$ is given by a sequence of d vectors in $\mathbb{C}^n$, and thus, as a vector space, we can

$$\text{identify } \mathcal{J}_d(1,n) \text{ with } \mathrm{Hom}(\mathbb{C}^d, \mathbb{C}^n). \tag{4.2.2}$$

Also, according to (4.1.4), the dual of $\mathcal{J}_d(n,1)$ is the vector space $\mathrm{Sym}_d^\bullet \mathbb{C}^n = \oplus_{l=1}^d \mathrm{Sym}^l \mathbb{C}^n$, hence a system of d linear equations on $\mathcal{J}_d(n,1)$ may be thought of as a linear map $\varepsilon \in \mathrm{Hom}(\mathbb{C}^d, \mathrm{Sym}_d^\bullet \mathbb{C}^n)$; the solution set of this system is the linear subspace orthogonal to the image of ε: $\mathrm{im}(\varepsilon)^\perp \subset \mathcal{J}_d(n,1)$ (cf. Definition 4.2.4 below).

Using these identifications, we can recast the map ψ in (4.2.1) as

$$\psi : \mathrm{Hom}(\mathbb{C}_L^d, \mathbb{C}^n) \longrightarrow \mathrm{Hom}(\mathbb{C}_R^d, \mathrm{Sym}_d^\bullet \mathbb{C}^n), \tag{4.2.3}$$

which may be written out explicitly as follows (cf. (4.1.8)):

$$\psi : (v_1, \ldots, v_d) \longmapsto \left(v_1, v_2 + v_1^2, v_3 + 2v_1v_2 + v_1^3, \ldots, \sum_{\mathrm{sum}(\tau)=m} \mathrm{perm}(\tau)\, \mathbf{v}_\tau, \ldots \right).$$

Note that in (4.2.3) – anticipating what is to come - we marked the two copies of $\mathbb{C}^d$ with different indices: L for left and R for right (cf. **Convention** after Lemma 4.2.1 below).

The constructions of this section will be based on the observation that the spaces of map germs $\mathcal{J}_d(n,1)$ and $\mathcal{J}_d(1,1)$ – and hence their duals – have natural filtrations, and these filtrations are preserved by the map ψ.

The filtration on the dual of $\mathcal{J}_d(n,1)$ (cf. (4.1.4)) is

$$\mathrm{Sym}_d^{\bullet}\mathbb{C}^n = \oplus_{l=1}^{d}\mathrm{Sym}^l\mathbb{C}^n \supset \oplus_{l=1}^{d-1}\mathrm{Sym}^l\mathbb{C}^n \supset \cdots \supset \mathbb{C}^n \oplus \mathrm{Sym}^2\mathbb{C}^n \supset \mathbb{C}^n; \qquad (4.2.4)$$

setting $n=1$, this reduces to $\mathbb{C}^d$ with the standard filtration:

$$\mathbb{C}^d \supset \oplus_{l=1}^{d-1}\mathbb{C}e_l \supset \cdots \supset \mathbb{C}e_1 \oplus \mathbb{C}e_2 \supset \mathbb{C}e_1. \qquad (4.2.5)$$

Now introduce the notation $\mathrm{Hom}^{\triangle}(\cdot,\cdot)$ for the linear space of morphisms of filtered vector spaces. Then we have

$$\mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Sym}_d^{\bullet}\mathbb{C}^n) = \{\varepsilon \in \mathrm{Hom}(\mathbb{C}^d_R, \mathrm{Sym}_d^{\bullet}\mathbb{C}^n);\ \varepsilon(e_l) \in \oplus_{m=1}^{l}\mathrm{Sym}^m\mathbb{C}^n, l=1,\dots,d\}. \qquad (4.2.6)$$

We will also need two open subsets of $\mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Sym}_d^{\bullet}\mathbb{C}^n)$: the set of *nondegenerate systems*

$$\mathcal{F}_d(n) = \{\varepsilon \in \mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Sym}_d^{\bullet}\mathbb{C}^n);\ \ker(\varepsilon)=0\}, \qquad (4.2.7)$$

and the set of *regular nondegenerate systems*

$$\mathcal{F}_d^{\mathrm{reg}}(n) = \{\varepsilon \in \mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Sym}_d^{\bullet}\mathbb{C}^n);\ \varepsilon(e_l) \notin \oplus_{m=1}^{l-1}\mathrm{Sym}^m\mathbb{C}^n, l=1,\dots,d\}. \qquad (4.2.8)$$

The following property of the map ψ is manifest (cf. Proposition 4.1.4(2)):

Lemma 4.2.1. *The correspondence ψ given in* (4.2.3) *takes values in* $\mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Sym}_d^{\bullet}\mathbb{C}^n)$.

Convention: The group of linear automorphisms of $\mathbb{C}^d$ will be denoted, as usual by GL_d, its subgroup of diagonal matrices by T_d, and its subgroup of upper-triangular matrices by B_d. In what follows, the two (left and right) copies of $\mathbb{C}^d$ appearing in (4.2.3) will play a rather different but important role. To avoid any confusion we will use the following notation for the corresponding groups:

$$T_L \subset B_L \subset \mathrm{GL}_L \quad \text{and} \quad T_R \subset B_R \subset \mathrm{GL}_R.$$

The space $\mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Sym}_d^{\bullet}\mathbb{C}^n)$ carries a left action of GL_n, and also a right action of the Borel subgroup B_R of GL_R preserving the filtration (4.2.5). Indeed, we have

$$B_R = \{b \in \mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathbb{C}^d_R);\ b \text{ invertible}\}. \qquad (4.2.9)$$

Lemma 4.2.2. *The subspaces $\mathcal{F}_d(n)$ and $\mathcal{F}_d^{\mathrm{reg}}(n)$ of* $\mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Sym}_d^{\bullet}\mathbb{C}^n)$ *are invariant under both GL_n and B_R. The quotient $\widetilde{\mathcal{F}}_d(n) = \mathcal{F}_d(n)/B_R$ is a compact, smooth manifold endowed with a GL_n-action, $\widetilde{\mathcal{F}}_d^{\mathrm{reg}}(n) = \mathcal{F}_d^{\mathrm{reg}}(n)/B_R \subset \widetilde{\mathcal{F}}_d(n)$ is a GL_n-invariant open subset.*

Proof. To check the invariance with respect to the group actions is straightforward. The statement about the smoothness and compactness of $\widetilde{\mathcal{F}}_d(n)$ may be verified in a number of ways (see, e.g. Remark 4.2.3 below). It also follows from results of Lakshmibai and Sandhya in [30] (see also [22], Theorem 1.1) on special Schubert varieties, of which $\widetilde{\mathcal{F}}_d(n)$ is an example. For reference, we recall the necessary definitions below.

By definition,

$$\widetilde{\mathcal{F}}_d(n) = \mathcal{F}_d(n)/B_R \subset \mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^d, \mathrm{Sym}_d^{\bullet}\mathbb{C}^n)/B_R = \mathrm{Flag}_d(\mathrm{Sym}_d^{\bullet}\mathbb{C}^n), \qquad (4.2.10)$$

where $\mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^d, \mathrm{Sym}_d^{\bullet}\mathbb{C}^n)$ denotes the maps of rank d, and $\mathrm{Flag}_d(\mathrm{Sym}_d^{\bullet}\mathbb{C}^n)$ is the compact partial flag manifold of full flags of d-dimensional subspaces of $\mathrm{Sym}_d^{\bullet}\mathbb{C}^n$:

$$\mathrm{Flag}(\mathrm{Sym}_d^{\bullet}\mathbb{C}^n) = \{0 = F_0 \subset F_1 \subset \cdots \subset F_d \subset \mathrm{Sym}_d^{\bullet}\mathbb{C}^n,\ \dim F_l = l\}\,. \qquad (4.2.11)$$

For simplicity, we use temporarily the notation $\dim_i = \dim(\oplus_{l=1}^i \mathrm{Sym}^l\mathbb{C}^n)$ for $i = 1, \ldots, d$. Let

$$\mathbb{C} \subset \mathbb{C}^2 \subset \ldots \subset \mathbb{C}^{\dim_d} = \oplus_{l=1}^d \mathrm{Sym}^l\mathbb{C}^n \qquad (4.2.12)$$

be a full flag of subspaces, refining (4.2.4), i.e $\mathbb{C}^{\dim_i} = \oplus_{l=1}^i \mathrm{Sym}^l\mathbb{C}^n$ for $i = 1, \ldots d$.

If $1 \le w_1 \le \ldots \le w_d \le \dim_d$ is a sequence of integers, then the set

$$\mathrm{Sch}_{w_1,\ldots,w_d} = \left\{(F_1 \subset F_2 \subset \ldots \subset F_d)\,;\ \dim(F_i \cap \mathbb{C}^l) \ge \sharp\,\{k \le i; w_k \le l\}\right\} \qquad (4.2.13)$$

is called a *Schubert variety* of the partial flag manifold $\mathrm{Flag}(\mathrm{Sym}_d^{\bullet}\mathbb{C}^n)$. In this language, $\widetilde{\mathcal{F}}_d(n)$ is the Schubert variety corresponding to the sequence $w_i = \dim_i$, $i = 1, \ldots d$. □

Remark 4.2.3. $\widetilde{\mathcal{F}}_d(n)$ may also be described as the total space of a tower of d fibrations as follows. The base of the tower is $\mathbb{P}(\mathbb{C}^n)$, and a fiber of the first fibration over a line $l_1 \in \mathbb{P}(\mathbb{C}^n)$ is $\mathbb{P}((\mathbb{C}^n \oplus \mathrm{Sym}^2\mathbb{C}^n)/l_1)$. Next, the fiber of the second fibration over a point $(l_1, l_2) \in (\mathbb{P}(\mathbb{C}^n), P(\mathbb{C}^n \oplus \mathrm{Sym}^2\mathbb{C}^n/l_1))$ is $\mathbb{P}(\mathbb{C}^n \oplus \mathrm{Sym}^2\mathbb{C}^n \oplus \mathrm{Sym}^3\mathbb{C}^n)/(l_1 + l_2)$, etc.

Before proceeding, we introduce some notation associated with the quotient in Lemma 4.2.2.

Definition 4.2.4. For $\varepsilon \in \mathrm{Hom}^{\triangle}(\mathbb{C}_R^d, \mathrm{Sym}_d^{\bullet}\mathbb{C}^n)$, thought of as a system of equations, introduce the notation

- Sol_ε for the solution set $\mathrm{im}(\varepsilon)^\perp \otimes \mathbb{C}^k \subset \mathcal{J}_d(n,k)$, and
- $\tilde{\varepsilon}$ for the point in $\widetilde{\mathcal{F}}_d(n)$ corresponding to ε.
- Clearly, $\mathrm{Sol}_\varepsilon = \mathrm{Sol}_{\varepsilon b}$ for $\varepsilon \in \mathrm{Hom}^{\triangle}(\mathbb{C}_R^d, \mathrm{Sym}_d^{\bullet}\mathbb{C}^n)$ and $b \in B_R$, hence to each element $\tilde{\varepsilon} \in \widetilde{\mathcal{F}}_d(n)$ we can associate a solution space $\mathrm{Sol}_{\tilde{\varepsilon}}$, and

- these spaces form a vector bundle $\tau_{\widetilde{\mathcal{F}}} : \mathrm{Sol}_{\widetilde{\mathcal{F}}} \longrightarrow \widetilde{\mathcal{F}}_d(n)$.
- This bundle induces a map (cf. (4.1.12))

$$\alpha : \widetilde{\mathcal{F}}_d(n) \to \mathrm{Gr}(\mathrm{codim} = dk, \mathcal{J}_d(n,k)). \tag{4.2.14}$$

We can identify $\mathrm{Sol}_{\widetilde{\mathcal{F}}}$ as a bundle associated to the quotienting described in Lemma 4.2.2, so we can rewrite the model in (4.1.13) as follows.

Lemma 4.2.5. *Consider the bundle V over $\widetilde{\mathcal{F}}_d(n)$ associated to the standard representation of B_R: $V = \mathcal{F}_d(n) \times_{B_R} \mathbb{C}^d_R$. Then the canonical pairing*

$$\mathcal{F}_d(n) \times \mathcal{J}_d(n,1) \to \mathrm{Hom}(\mathbb{C}^d_R, \mathbb{C}) \tag{4.2.15}$$

induces a linear bundle map from the trivial bundle with fiber $\mathcal{J}_d(n,1)$ over $\widetilde{\mathcal{F}}_d(n)$ to V^*:

$$s : \widetilde{\mathcal{F}}_d(n) \to \mathrm{Hom}(\mathcal{J}_d(n,1), V^*)$$

such that for $\tilde{\varepsilon} \in \widetilde{\mathcal{F}}_d(n)$, we have $\ker(s(\tilde{\varepsilon})) \otimes \mathbb{C}^k = \mathrm{Sol}_{\tilde{\varepsilon}} \subset \mathcal{J}_d(n,k)$.

The upshot of this identification is the following exact sequence of vector bundles over $\widetilde{\mathcal{F}}_d(n)$:

$$0 \longrightarrow \mathrm{Sol}_{\widetilde{\mathcal{F}}} \xrightarrow{\mathrm{ev}} \mathcal{J}_d(n,k) \xrightarrow{s} V^* \otimes \mathbb{C}^k \longrightarrow 0 \tag{4.2.16}$$

After these preparations we return to our main task: the construction of a refinement of the map $\tilde{\phi}$ in diagram (4.1.12). Lemmas 4.2.1 and 4.2.2 imply that $\mathrm{im}\tilde{\phi} \subset \mathrm{im}(\alpha)$, and hence, were α injective, we could argue that the map ψ induces a $\mathrm{Diff}_d(1)$-invariant algebraic map of $\mathcal{J}^{\mathrm{reg}}_d(1,n)$ in to $\widetilde{\mathcal{F}}_d(n)$, which induces an injective map is on the orbits. This seems reasonable since $\tilde{\phi}$ clearly factors through the map α. There is a subtlety here, however: the map (4.2.14) is not injective, thus we have to be more careful. Indeed, for example, let $d = 3$, and take the points

$$\varepsilon_1 = (v_1, v_2, v_1^2) \text{ and } \varepsilon_2 = (v_1, v_1^2, v_2)$$

in $\widetilde{\mathcal{F}}_3(n)$. Then $\mathrm{Sol}_{\varepsilon_1} = \mathrm{Sol}_{\varepsilon_2}$, hence $\alpha(\tilde{\varepsilon}_1) = \alpha(\tilde{\varepsilon}_2)$, but $\tilde{\varepsilon}_1 \neq \tilde{\varepsilon}_2$.

As the next Lemma shows, however, ε_1 and ε_2 in this example are not in the image $\psi(\mathcal{J}^{\mathrm{reg}}_d(1,n))$.

Lemma 4.2.6. *We have*

- *$\psi(\mathcal{J}^{\mathrm{reg}}_d(1,n)) \subset \mathcal{F}^{\mathrm{reg}}_d(n)$, and*
- *the map α (defined in (4.2.14)) restricted to $\widetilde{\mathcal{F}}^{\mathrm{reg}}_d(n)$ is an injective algebraic map.*

Proof. If $(v_1,\dots,v_d)\in\mathcal{J}_d^{\rm reg}(1,n)$ then $v_1\neq 0$. Since

$$\psi((v_1,\dots,v_d))(e_d)=v_1^d+(d-1)v_1^{d-2}v_2+\dots$$

and $v_1^d\neq 0$, the first part is proved. Let $\pi_d:\widetilde{\mathcal{F}}_d(n)\to \mathrm{Gr}(d,\mathrm{Sym}_d^\bullet\mathbb{C}^n)$ denote the projection, which maps a flag (4.2.11) to the d-dimensional subspace $F_d\subset \mathrm{Sym}_d^\bullet\mathbb{C}^n$. Observe, that for $\varepsilon\in\mathcal{F}_d(n)$

$$\mathrm{im}(\varepsilon)=\pi_d(\tilde{\varepsilon}), \tag{4.2.17}$$

hence the second part follows if π_d restricted to $\widetilde{\mathcal{F}}_d^{\rm reg}(n)$ is injective. (See also Definition 4.2.4.) This comes from the following explicit form of its inverse on $\pi_d(\widetilde{\mathcal{F}}_d^{\rm reg}(n))$:

$$(\pi_d|_{\widetilde{\mathcal{F}}_d^{\rm reg}(n)})^{-1}(U)=$$
$$((U\cap\mathbb{C}^n)\subset(U\cap(C^n\oplus\mathrm{Sym}^2\mathbb{C}^n))\subset(U\cap(\oplus_{l=1}^3\mathrm{Sym}^l\mathbb{C}^n))\subset\ldots\subset U)\in\widetilde{\mathcal{F}}_d^{\rm reg}(n). \tag{4.2.18}$$

□

Remark 4.2.7. If $U\in\mathrm{Gr}(d,\mathrm{Sym}_d^\bullet\mathbb{C}^n)$ in (4.2.18) is not in the image of $\widetilde{\mathcal{F}}_d^{\rm reg}(n)$, then some of the subspaces on the RHS of (4.2.18) will coincide, so they do not determine a flag.

From the definition of $\tilde{\phi},\psi,\alpha$ and Lemma 4.2.6 follows that ψ can be written in the form

$$\psi=\alpha^{-1}\circ\tilde{\phi}, \tag{4.2.19}$$

where the domain of definition of α^{-1} is understood to be $\mathrm{im}(\alpha|_{\widetilde{\mathcal{F}}_d^{\rm reg}(n)})$, and by Lemma 4.2.6 we have

Corollary 4.2.8. *The map ψ in (4.2.3) is a $\mathrm{Diff}_d(1)$-invariant algebraic morphism*

$$\psi:\mathcal{J}_d^{\rm reg}(1,n)\to\widetilde{\mathcal{F}}_d(n),$$

which induces an injective map $\phi_{\widetilde{\mathcal{F}}}$ on the $\mathrm{Diff}_d(1)$-orbits. Moreover, $\psi^*(\mathrm{Sol}_{\widetilde{\mathcal{F}}})=\tilde{\phi}^*(S)$, hence by (4.1.13),(4.1.14):

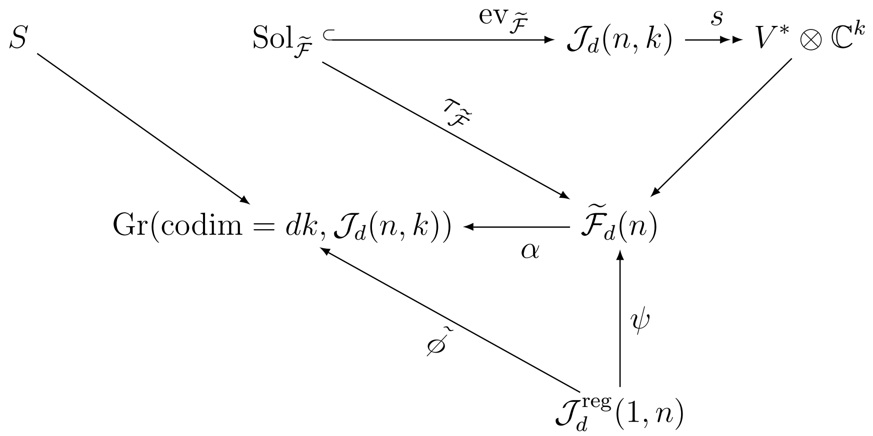
$$\Theta'_d=\mathrm{ev}_{\widetilde{\mathcal{F}}}\left(\tau_{\widetilde{\mathcal{F}}}^{-1}[\mathrm{im}(\psi)]\right) \tag{4.2.20}$$

and

$$\overline{\Theta}_d=\overline{\Theta'}_d=\mathrm{ev}_{\widetilde{\mathcal{F}}}\left(\tau_{\widetilde{\mathcal{F}}}^{-1}[\overline{\mathrm{im}(\psi)}]\right) \tag{4.2.21}$$

Finally, $\mathrm{ev}_{\widetilde{\mathcal{F}}}$ is a degree-1 map onto $\overline{\Theta'}$ in (4.2.21).

Combining diagram (4.1.12) and sequence (4.2.16), we arrive at the following picture:



(4.2.22)

Remark 4.2.9. Corollary 4.2.8 gives our way of thinking of the quotient $\mathcal{J}_d^{\mathrm{reg}}(1,n)/\mathrm{Diff}_d(1)$, namely, as $\mathrm{im}(\psi) \subset \widetilde{\mathcal{F}}_d(n)$. The closure $\overline{\mathrm{im}(\psi)}$ is a closed subvariety of $\widetilde{\mathcal{F}}_d(n)$, and we will call a subset of $\mathcal{J}_d^{\mathrm{reg}}(1,n)/\mathrm{Diff}_d(1)$ Zariski open, if it is Zariski open in $\overline{\mathrm{im}(\psi)}$.

4.2.2 Fibration over the flag variety

In the previous paragraph we took advantage of the special "filtered" form of the system (4.1.7), and replaced the Grassmannian from (4.1.12) with the space of linear systems $\widetilde{\mathcal{F}}_d(n)$. In this second part of the section, we further refine this construction.

We start with a closer look at the "natural" identification (4.2.2). In fact, the two objects are rather different: $\mathcal{J}_d(1,n)$ is a module over $\mathrm{Diff}_d(n) \times \mathrm{Diff}_d(1)$ while $\mathrm{Hom}(\mathbb{C}^d, \mathbb{C}^n)$ is a module over $\mathrm{GL}_n \times \mathrm{GL}_d$; in addition, note that we have the following somewhat odd inclusions:

$$\mathrm{Diff}_d(1) \subset \mathrm{GL}_d, \quad \mathrm{GL}_n \subset \mathrm{Diff}_d(n). \tag{4.2.23}$$

By a straightforward computation, the first of the two inclusions may be made more precise as follows.

Lemma 4.2.10. *Under the identification* (4.2.2)*, a substitution*

$$\alpha_1 t + \alpha_2 t^2 + \ldots + \alpha_d t^d \in \mathrm{Diff}_d(1)$$

corresponds to the upper-triangular matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_d \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \ldots & 2\alpha_1\alpha_{d-1} + \ldots \\ 0 & 0 & \alpha_1^3 & \ldots & 3\alpha_1^2\alpha_{d-2} + \ldots \\ 0 & 0 & 0 & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \alpha_1^d \end{pmatrix};$$

the coefficient in the ith row and jth column is

$$\sum_{\{\tau\in\Pi[j];\,|\tau|=i\}} \mathrm{perm}(\tau)\,\alpha_\tau,$$

where the notation $\alpha_\tau = \prod_{i\in\tau}\alpha_i$ was used. This correspondence establishes an isomorphism of $\mathrm{Diff}_d(1)$ *with a d-dimensional subgroup H_d of the Borel subgroup $B_d \subset \mathrm{GL}_d$.*

Remark 4.2.11. In accordance with the convention introduced after Lemma 4.2.1, we will use the notation H_L when working with the copy of the group H_d in the "left" Borel subgroup B_L.

Lemma 4.2.10 implies that there is map

$$\mathcal{J}_d(1,n) \longrightarrow \mathrm{Hom}(\mathbb{C}^d_L, \mathbb{C}^n),$$

which is equivariant with respect to the $\mathrm{Diff}_d(1)$ action on the left and H_d action on the right. Consequaently, it defines a map from the $\mathrm{Diff}_d(1)$-orbits to the B_L-orbits on the right. To make this map geometric, consider the subspace of injective linear maps:

$$\mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^d_L, \mathbb{C}^n) = \{\gamma \in \mathrm{Hom}(\mathbb{C}^d_L, \mathbb{C}^n);\ \ker(\gamma) = 0\} \tag{4.2.24}$$

The following statements are standard:

Lemma 4.2.12.
- *Under the identification* (4.2.2)*, the space* $\mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^d_L, \mathbb{C}^n)$ *is a dense, open subset of* $\mathcal{J}^{\mathrm{reg}}_d(1,n)$*.*
- *The action of B_L on* $\mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^d_L, \mathbb{C}^n)$ *is free, and the quotient* $\mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^d_L, \mathbb{C}^n)/B_L$ *is the compact, smooth variety of full flags of d-dimensional subspaces of $\mathbb{C}^n$:*

$$\mathrm{Flag}_d(\mathbb{C}^n) = \{0 = F_0 \subset F_1 \subset \cdots \subset F_d \subset \mathbb{C}^n,\ \dim F_l = l\}\,.$$

- *The residual action of* GL_n *on* $\mathrm{Flag}_d(\mathbb{C}^n)$ *is transitive.*

Since fibrations over $\mathrm{Flag}_d(\mathbb{C}^n)$ will play a major role in what follows, we introduce some notation related to the quotient described in Lemma 4.2.12.

Definition 4.2.13.
- Denote by γ_{ref} the *reference* sequence

$$\gamma_{\mathrm{ref}} = (e_1, \ldots, e_d) \in \mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^d_L, \mathbb{C}^n),$$

  where e_i is the ith basis vector of $\mathbb{C}^n$, and and we use the identification (4.2.2). Let $\mathbf{f}_{\mathrm{ref}}$ denote the corresponding flag in $\mathrm{Flag}_d(\mathbb{C}^n)$.
- For a space X endowed with a left B_L-action, denote by $\mathrm{Ind}(X)$ the induced space $\mathrm{Ind}(X) = \mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^d_L, \mathbb{C}^n) \times_{B_L} X$.

Note that, in particular, we have $\mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}_L^d, \mathbb{C}^n) = \mathrm{Ind}(B_L)$, and

$$\mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}_L^d, \mathbb{C}^n)/H_L = \mathrm{Ind}(B_L/H_L). \tag{4.2.25}$$

Observe that this equality means that we managed to fiber a Zariski-open part of $\mathcal{J}_d^{\mathrm{reg}}(1,n)/\mathrm{Diff}_d(1)$ over $\mathrm{Flag}_d(\mathbb{C}^n)$, see Remark 4.2.9. This suggests investigating the systems of equations (4.1.7) in a single fiber of this fibration; we will take a closer look at the fiber $\gamma_{\mathrm{ref}} B_L$ lying over the point $\mathbf{f}_{\mathrm{ref}} \in \mathrm{Flag}_d(\mathbb{C}^n)$.

To inspect these systems, we will write them down in the standard basis of $\mathrm{Sym}_d^\bullet \mathbb{C}^n$; using the notation introduced in § 4.1, this consists of the elements

$$e_\tau = e_{i_1} \cdot \ldots \cdot e_{i_m}, \quad \text{where } \tau = [i_1, \ldots, i_m],\ m = |\tau| \leq d, \text{ and } \max(\tau) \leq n.$$

We will denote the corresponding components of $\Psi \in \mathcal{J}_d(n,k)$ by

$$\Psi_\tau = \Psi^m(e_{i_1}, \ldots, e_{i_m}).$$

We start with the *reference system* $\varepsilon_{\mathrm{ref}} = \psi(\gamma_{\mathrm{ref}})$:

$$\varepsilon_{\mathrm{ref}} = \left\{ \sum_{\mathrm{sum}(\tau)=l} \mathrm{perm}(\tau)\, \Psi_\tau = 0, \quad l = 1, 2, \ldots, d \right\}. \tag{4.2.26}$$

With the convention of using the mth capital letter of the alphabet for Ψ^m, the first four equations of $\varepsilon_{\mathrm{ref}}$ look as follows:

$$\begin{gathered} A_1 = 0 \\ A_2 + B_{11} = 0 \\ A_3 + 2B_{12} + C_{111} = 0 \\ A_4 + 2B_{13} + B_{22} + 3C_{112} + D_{1111} = 0 \end{gathered} \tag{4.2.27}$$

Now consider a general element of $\gamma_{\mathrm{ref}} B_L$, a test curve over the reference flag:

$$\gamma_{\mathrm{ref}} \cdot \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \cdot \\ 0 & \beta_{22} & \beta_{23} & \cdot \\ 0 & 0 & \beta_{33} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} = (\beta_{11} e_1, \beta_{22} e_2 + \beta_{12} e_1, \beta_{33} e_3 + \beta_{23} e_2 + \beta_{13} e_1 \ldots).$$

The first 3 equations of the corresponding system (4.1.7) are

$$\begin{gathered} \beta_{11} A_1 = 0 \\ \beta_{22} A_2 + \beta_{12} A_1 + (\beta_{11})^2 B_{11} = 0 \\ \beta_{33} A_3 + \beta_{23} A_2 + \beta_{13} A_1 + 2\beta_{11}\beta_{22} B_{12} + 2\beta_{11}\beta_{12} B_{11} + (\beta_{11})^3 C_{111} = 0; \end{gathered} \tag{4.2.28}$$

these are thus of the form

$$\begin{gathered} u_1^1 A_1 = 0 \\ u_2^2 A_2 + u_1^2 A_1 + u_{11}^2 B_{11} = 0 \\ u_3^3 A_3 + u_2^3 A_2 + u_1^3 A_1 + 2u_{12}^3 B_{12} + u_{11}^3 B_{11} + u_{111}^3 C_{111} = 0, \end{gathered} \tag{4.2.29}$$

with some complex coefficients of the form u_τ^m, where m is the ordinal number of the equation, while τ marks the component of Ψ. We observe that in the lth equations of these systems, only the components Ψ_τ satisfying $\mathrm{sum}(\tau) \le l$ appear.

Lemma 4.2.14. *The system of equations* (4.1.7) *corresponding to a test curve* $\gamma \in \gamma_{\mathrm{ref}} B_L$ *is of the form*

$$\sum_{\mathrm{sum}(\tau)\le l} \mathrm{perm}(\tau)\, u_\tau^l\, \Psi_\tau = 0, \quad l = 1, 2, \ldots, d, \tag{4.2.30}$$

where u_τ^l, $\mathrm{sum}(\tau) \le l \le d$, *are some complex coefficients.*

We can formalize this simple point as follows: introduce a new filtered vector space $\mathrm{Ym}^\bullet\mathbb{C}_L^d$:

$$\mathrm{Ym}^\bullet\mathbb{C}_L^d = \bigoplus_{\mathrm{sum}(\tau)\le d} \mathbb{C}e_\tau \supset \bigoplus_{\mathrm{sum}(\tau)\le d-1} \mathbb{C}e_\tau \supset \cdots \supset \mathbb{C}e_2 \oplus \mathbb{C}e_1^2 \oplus \mathbb{C}e_1 \supset \mathbb{C}e_1; \tag{4.2.31}$$

the notation is motivated by the fact that $\mathrm{Ym}^\bullet\mathbb{C}_L^d$ is a truncation of $\mathrm{Sym}_d^\bullet\mathbb{C}^n$. Recall the notation $\mathrm{Hom}^\triangle(\cdot,\cdot)$ for filtration preserving linear maps, and observe that the space $\mathrm{Hom}^\triangle(\mathbb{C}_R^d, \mathrm{Ym}^\bullet\mathbb{C}_L^d)$ is a left-right representation of the group $B_L \times B_R$. Consider the diagram

$$\begin{array}{ccc} & & \mathrm{Hom}^\triangle(\mathbb{C}_R^d, \mathrm{Sym}_d^\bullet\mathbb{C}^n) \\ & \nearrow^{\psi} & \uparrow \kappa \\ \mathrm{Hom}(\mathbb{C}_L^d, \mathbb{C}^n) & \longrightarrow & \mathrm{Hom}(\mathbb{C}_L^d, \mathbb{C}^n) \times_{B_L} \mathrm{Hom}^\triangle(\mathbb{C}_R^d, \mathrm{Ym}^\bullet\mathbb{C}_L^d) \end{array} \tag{4.2.32}$$

where the horizontal arrow is the correspondence $\gamma \mapsto (\gamma, \varepsilon_{\mathrm{ref}})$, while κ is obtained by composing the linear map $\mathbb{C}_R^d \to \mathrm{Ym}^\bullet\mathbb{C}_L^d$ with the substitution $\mathbb{C}_L^d \to \mathbb{C}^n$. One can easily see that the diagram (4.2.32) is commutative, and hence it establishes a factorization of the map ψ defined in (4.2.1). A key point here is that we represent the set of systems (4.2.28) as an orbit of the B_L-action on $\mathrm{Hom}^\triangle(\mathbb{C}_R^d, \mathrm{Ym}^\bullet\mathbb{C}_L^d)$.

Now we introduce an analog of $\mathcal{F}_d(n)$ as follows:

$$\mathcal{E} = \{\varepsilon \in \mathrm{Hom}^\triangle(\mathbb{C}_R^d, \mathrm{Ym}^\bullet\mathbb{C}_L^d);\ \ker(\varepsilon) = 0\}, \tag{4.2.33}$$

Proposition 4.2.15.

1. *The open subset* $\mathcal{E} \subset \mathrm{Hom}^\triangle(\mathbb{C}_R^d, \mathrm{Ym}^\bullet\mathbb{C}_L^d)$ *is invariant under the left-right action of* $B_L \times B_R$.

2. *The quotient* $\widetilde{\mathcal{E}} = \mathcal{E}/B_R$ *is a smooth, compact variety endowed with a left action of* B_L.

3. *The map* κ *in diagram* (4.2.32) *is* B_R*-equivariant, and induces a map* $\tilde{\kappa} : \mathrm{Ind}(\widetilde{\mathcal{E}}) \to \widetilde{\mathcal{F}}_d(n)$.

4. The horizontal map in the diagram induces an algebraic map

$$\overline{\phi_{\widetilde{\mathcal{E}}}} : \mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^d_L, \mathbb{C}^n) \to \mathrm{Ind}(\widetilde{\mathcal{E}}),$$

which is H_L-invariant, and the induced map $\phi_{\widetilde{\mathcal{E}}}$ on the orbits is injevtive. Moreover, the restriction of the map (cf. diagram (4.2.22)) $\phi_{\widetilde{\mathcal{F}}} : \mathcal{J}^{\mathrm{reg}}_d(1,n)/\mathrm{Diff}_d(1) \hookrightarrow \widetilde{\mathcal{F}}_d(n)$ to $\mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^d_L, \mathbb{C}^n)/H_L$ factorizes as $\tilde{\kappa} \circ \phi_{\widetilde{\mathcal{E}}}$.

Proof. The first and the third statements are obvious, while the second may be proved the same way as Lemma 4.2.2.

For the last statement, consider the subset $\gamma_{\mathrm{ref}} \cdot B_L \subset \mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^d_L, \mathbb{C}^n) \subset \mathcal{J}^{\mathrm{reg}}_d(1,n)$. The injective map $\phi_{\widetilde{\mathcal{F}}} : \mathcal{J}^{\mathrm{reg}}_d(1,n)/\mathrm{Diff}_d(1) \longrightarrow \widetilde{\mathcal{F}}_d(n)$ of Corollary 4.2.8 then restricts to the map

$$(\gamma_{\mathrm{ref}} \cdot B_L)/H_L \hookrightarrow \widetilde{\mathcal{F}}_d(n), \tag{4.2.34}$$

which is induced from ψ of the diagram (4.2.32). According to Lemma 4.2.14, $\psi(\gamma_{\mathrm{ref}} B_L) \subset \mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Ym}^{\bullet}\mathbb{C}^d_L)$, hence the image of (4.2.34) is in $\widetilde{\mathcal{E}} \subset \widetilde{\mathcal{F}}_d(n)$, and so (4.2.34) has the form

$$\phi : (\gamma_{\mathrm{ref}} B_L)/H_L \hookrightarrow \widetilde{\mathcal{E}}.$$

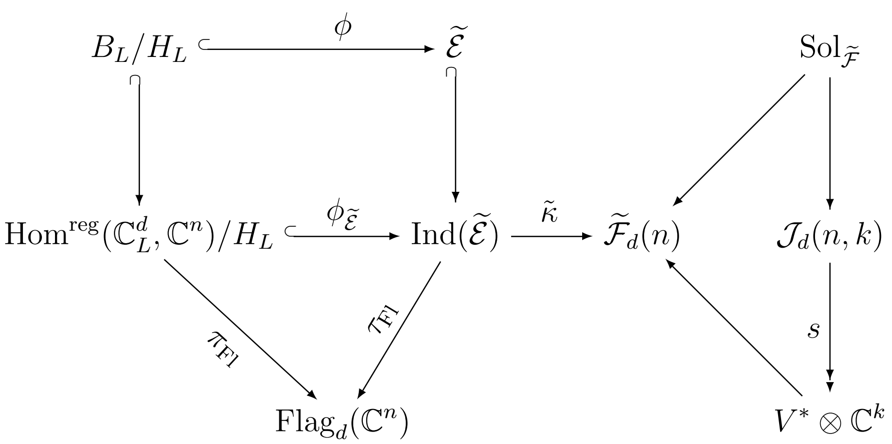
Now inducing over $\mathrm{Flag}_d(\mathbb{C}^n)$, we obtain the injective morphism

$$\phi_{\widetilde{\mathcal{E}}} : \mathrm{Ind}(B_L/H_L) \hookrightarrow \mathrm{Ind}(\widetilde{\mathcal{E}}).$$

The second half of the last statement follows from the construction of $\phi_{\widetilde{\mathcal{E}}}$. □

Corollary 4.2.16. *Let $\tilde{\varepsilon}_{\mathrm{ref}} \in \widetilde{\mathcal{E}}$ be the reference point $\mathrm{pr}_{\mathcal{E}}(\varepsilon_{\mathrm{ref}})$, where $\mathrm{pr}_{\mathcal{E}} : \mathcal{E} \to \widetilde{\mathcal{E}}$ is the projection. The stabilizer of the B_L-action on $\widetilde{\mathcal{E}}$ of the point $\tilde{\varepsilon}_{\mathrm{ref}}$ is the subgroup $H_L \subset B_L$.*

Combining the results of Proposition 4.2.15 with diagram (4.2.22), we arrive at the following picture:



(4.2.35)

Now we are ready to formulate our model in its final form.

- Consider the fibered product $V = \mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^d_L, \mathbb{C}^n) \times_{B_L} \mathcal{E} \times_{B_R} \mathbb{C}^d_R$, resulting in the double fibration

$$\mathrm{Flag}_d(\mathbb{C}^n) \longleftarrow \mathrm{Ind}(\widetilde{\mathcal{E}}) \overset{\tau_V}{\longleftarrow} V$$

  where $\mathcal{E}$ is defined in (4.2.33), and $\widetilde{\mathcal{E}} = \mathcal{E}/B_R$.

- Since $\kappa(\mathrm{Ind}(\mathcal{E})) \subset \mathcal{F}_d(n)$, (4.2.15) induces a canonical pairing

$$\mathrm{Ind}(\mathcal{E}) \times \mathcal{J}_d(n,1) \to \mathrm{Hom}(\mathbb{C}^d_R, \mathbb{C}) \tag{4.2.36}$$

  through κ. The associated linear bundle map from the trivial bundle with fiber $\mathcal{J}_d(n,1)$ to V^*, tensored with $\mathbb{C}^k$ fits into the short exact sequence of bundles over $\mathrm{Ind}(\widetilde{\mathcal{E}})$, as an analog of (4.2.16):

$$0 \longrightarrow \mathrm{Sol}_{\widetilde{\mathcal{E}}} \xrightarrow{\mathrm{ev}_{\widetilde{\mathcal{E}}}} \mathcal{J}_d(n,k) \xrightarrow{s} V^* \otimes \mathbb{C}^k \longrightarrow 0$$

We have the following analog of (4.2.21).

Proposition 4.2.17. *Let $\tilde{\varepsilon}_{\mathrm{ref}} \in \widetilde{\mathcal{E}}$ be the point corresponding to the system* (4.2.26) *(cf. Corollary 4.2.16). Then the orbit $B_L\tilde{\varepsilon}_{\mathrm{ref}}$ is an irreducible B_L-invariant subvariety in $\widetilde{\mathcal{E}}$ of dimension $\binom{d}{2}$, and*

$$\mathrm{ev}_{\widetilde{\mathcal{E}}}\left(\tau_{\widetilde{\mathcal{E}}}^{-1}\left(\mathrm{Ind}(\overline{B_L\tilde{\varepsilon}_{\mathrm{ref}}})\right)\right) = \overline{\Theta'}_d = \overline{\Theta}_d.$$

Moreover, $\mathrm{ev}_{\widetilde{\mathcal{E}}}$ *is a degree-1 map onto its image.*

Proof. The first half of the statement follows from Corollary 4.2.16 once we note that the image of the map ϕ is exactly $B_L\tilde{\varepsilon}_{\mathrm{ref}}$. For the second half consider the following facts:

- Comparing the construction of the bundle V we have given above with Lemma 4.2.5, we can conclude that $\tilde{\kappa}^*\mathrm{Sol}_{\widetilde{\mathcal{F}}} = \mathrm{Sol}_{\widetilde{\mathcal{E}}}$.

- The evaluation map $\mathrm{ev}_{\widetilde{\mathcal{E}}}$ is proper.

- According to Proposition 4.2.15 (4), we have $\phi_{\widetilde{\mathcal{F}}} = \tilde{\kappa} \circ \phi_{\widetilde{\mathcal{E}}}$ on the Zariski open part $\mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^d_L, \mathbb{C}^n)/H_L$ in $\mathcal{J}_d^{\mathrm{reg}}(1,n)/\mathrm{Diff}_d(1)$.

- The closure of Θ_d coincides with that of Θ'_d.

Now the statement follows from our previous "model" construction, (4.2.20).

□

4.3 Application of the localization formulas

Recall that our aim is the computation of the equivariant Poincaré dual $\mathrm{eP}[\overline{\Theta}_d]$, where the subvariety $\Theta_d \subset \mathcal{J}_d(n,k)$ represents the A_d-singularity (cf. § 2.1). The symmetry group of the problem is the product of matrix groups $\mathrm{GL}_n \times \mathrm{GL}_k$; the respective subgroups of diagonal matrices are T_n with weights $(\lambda_1, \ldots, \lambda_n)$ and T_k with weights $(\theta_1, \ldots, \theta_k)$, hence $\mathrm{eP}[\Theta_d]$ is a bisymmetric polynomial in these two sets of variables.

In this section, we apply the localization techniques of §3 to the computation of $\mathrm{eP}[\Theta_d]$, using the model described in §4.2.2. As our model is a double fibration, the application of the localization formula is a 2-step process.

Before we proceed we set the following **convention**: when describing the action of B_L on the B_R-quotient $\widetilde{\mathcal{E}}$, we will revert to the notation B_d, since here there is only one copy of the Borel group is acting.

4.3.1 Localization in $\mathrm{Flag}_d(\mathbb{C}^n)$

The model of Proposition 4.2.17 is an equivariant fibration over the smooth homogeneous space $\mathrm{Flag}_d(\mathbb{C}^n)$, hence, in this case, we can use Proposition 3.1.6 with the extension (cf. § 3.3.1) that the fibers of S are not necessarily linear and smooth. The result of our calculation is Proposition 4.3.3 below.

The data needed for formula (3.3.1)) is

- the fixed point set of the T_n-action on $\mathrm{Flag}_d(\mathbb{C}^n)$,
- the weights of this action on the tangents spaces $\mathrm{T}_p\mathrm{Flag}_d(\mathbb{C}^n)$ at these fixed points,
- the equivariant Poincaré duals of the fibers at these fixed points.

The following general statement will be helpful in organizing our fixed point data. Its proof is straightforward and will be omitted.

Lemma 4.3.1. *Assume that the torus action in Proposition 3.1.6 is obtained by a restriction of a GL_n-action to its subgroup of diagonal matrices T_n. Then the Weyl group of permutation matrices $\mathcal{S}_n$ acts on M^{T_n}, and we have*

$$\mathrm{eP}[S_{\sigma\cdot p}, W] = \sigma \cdot \mathrm{eP}[S_p, W] \text{ and } \mathrm{Euler}^{T_n}(\mathrm{T}_{\sigma\cdot p}M) = \sigma \cdot \mathrm{Euler}^{T_n}(\mathrm{T}_p M).$$

for all $\sigma \in \mathcal{S}_n$ and $p \in M_{T_n}$.

Our situation is fortunate in the sense that the action of $\mathcal{S}_n$ on the fixed point set is transitive. Indeed, the fixed point set $\mathrm{Flag}_d(\mathbb{C}^n)^{T_n}$ is the set of partial flags obtained from sequences of d elements of the basis $(e_1, \ldots, e_n)$ of $\mathbb{C}^n$; in particular, $|\mathrm{Flag}_d(\mathbb{C}^n)^{T_n}| = n(n-1)\ldots(n-d+1)$.

Recall the notation $\mathbf{f}_{\text{ref}}$ for the reference flag associated to the sequence $(e_1, \ldots, e_d)$. The stabilizer subgroup of $\mathbf{f}_{\text{ref}}$ in $\mathcal{S}_n$ is the subgroup $\mathcal{S}_{n-d}$ permuting the numbers starting with $d+1$, and the map $\sigma \mapsto \sigma \cdot \mathbf{f}_{\text{ref}}$ induces a bijection between $\mathrm{Flag}_d(\mathbb{C}^n)^{T_n}$ and the quotient $\mathcal{S}_n/\mathcal{S}_{n-d}$.

According to Lemma 4.3.1, it is sufficient for us to compute the equivariant Poincaré dual of the fiber and the weights of the tangent space at the reference flag $\mathbf{f}_{\text{ref}}$. The weights of $\mathrm{T}_{\mathbf{f}_{\text{ref}}}\mathrm{Flag}_d(\mathbb{C}^n)$ are well-known:

$$\{\lambda_i - \lambda_m;\ 1 \le m \le d,\ m < i \le n\};$$

the weights at the other fixed points are obtained by applying the corresponding permutation this set.

The numerators of the summands of (3.3.1) in our case are much harder to compute, although, thanks to Lemma 4.3.1, it is sufficient to know the numerator for the fixed point $\mathbf{f}_{\text{ref}}$. The situation over $\mathbf{f}_{\text{ref}}$ is reflected in the following diagram:

$$\begin{array}{ccccc} \mathrm{Sol}_{\tilde{\mathcal{E}}} & \xrightarrow{\mathrm{ev}_{\tilde{\mathcal{E}}}} & \mathcal{J}_d(n,k) & \xrightarrow{s} & V^* \otimes \mathbb{C}^k \\ & \searrow^{\tau_{\tilde{\mathcal{E}}}} & & \swarrow & \\ \mathcal{O} = \overline{B_d \tilde{\varepsilon}_{\text{ref}}} & \hookrightarrow & \widetilde{\mathcal{E}} & & \end{array} \tag{4.3.1}$$

The fiber of our model (4.2.20) over the fixed point $\mathbf{f}_{\text{ref}}$ is the set $\tau_{\tilde{\mathcal{E}}}^{-1}(\mathcal{O})$, where we introduced the notation $\mathcal{O}$ for the closure of the B_d-orbit of $\tilde{\varepsilon}_{\text{ref}}$. Using this notation we can write the numerator of the term corresponding to $\mathbf{f}_{\text{ref}}$ in the sum (3.3.1) as follows:

$$\mathrm{eP}\left[\mathrm{ev}_{\tilde{\mathcal{E}}}\left(\tau_{\tilde{\mathcal{E}}}^{-1}(\mathcal{O})\right), \mathcal{J}_d(n,k)\right]; \tag{4.3.2}$$

this is a polynomial in two sets of variables: $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$. Since $\mathcal{O}$ is invariant under B_d only, this polynomial is not necessarily symmetric in the λs. The following statement is straightforward.

Lemma 4.3.2. *The equivariant Poincaré dual* (4.3.2) *does not depend on the last $n-d$ basic λ-weights: $\lambda_{d+1}, \ldots, \lambda_n$.*

Proof. Indeed, recall that $\mathrm{ev}_{\tilde{\mathcal{E}}}\tau_{\tilde{\mathcal{E}}}^{-1}(B_d\tilde{\varepsilon}_{\text{ref}})$ consists of all possible solutions of the systems of equations of the form $B_L\varepsilon_{\text{ref}}$. We wrote down these systems explicitly in (4.2.28), and saw in § 4.2.2 that all these systems are in $\mathcal{E}$. The systems of equations in $\mathcal{E}$, however, impose conditions only on those components of Ψ which do not have indices higher than d, and this implies the statement of the Lemma. □

As a consequence of Lemma 4.3.2, the equivariant Poincaré dual (4.3.2) may be considered as being taken with respect to the group $T_d \times T_k$, which has weights $\mathbf{z} = (z_1, \ldots, z_d)$ and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$.

Putting together Lemmas 4.3.1 and 4.3.2 and the description of the fixed point set $\mathrm{Flag}_d(\mathbb{C}^n)^{T_d}$ given above, we arrive at the following form of (3.3.1) applied to our situation:

Proposition 4.3.3. *We have*

$$\mathrm{eP}[\Theta_d] = \sum_{\sigma\in\mathcal{S}_n/\mathcal{S}_{n-d}} \frac{Q_{\mathrm{Fl}}(\lambda_{\sigma\cdot 1},\dots,\lambda_{\sigma\cdot d},\boldsymbol{\theta})}{\prod_{1\le m\le d}\prod_{i=m+1}^{n}(\lambda_{\sigma\cdot i}-\lambda_{\sigma\cdot m})}, \tag{4.3.3}$$

where

$$Q_{\mathrm{Fl}}(\mathbf{z},\boldsymbol{\theta}) = \mathrm{eP}\left[\mathrm{ev}_{\widetilde{\mathcal{E}}}\left(\tau_{\widetilde{\mathcal{E}}}^{-1}(\mathcal{O})\right), \mathcal{J}_d(n,k)\right]_{T_d\times T_k}. \tag{4.3.4}$$

4.3.2 Residue formula for the cohomology pairings of $\mathrm{Flag}_d(\mathbb{C}^n)$

Usually formulas such as (4.3.3) are difficult to use: they have the form of a finite sum of rational functions, and only after adding up the terms of this sum and performing some cancellations is that we obtain a polynomial. This obscures the result, Moreover, the number of terms of the sum grows very quickly with n and d.

In this paragraph, we derive an efficient residue formula for the right hand side of (4.3.3). While the geometric meaning of this formula is not entirely clear, our summation procedure yields an effective, "truly" localized formula; by this we mean that for its evaluation one only needs to know the behavior of a certain function at a single point, rather than at a large, albeit finite number of points.

To describe this formula, we will need the notion of an *iterated residue* (cf. e.g. [46]) at infinity. Let $\omega_1,\dots,\omega_N$ be affine linear forms on $\mathbb{C}^d$; denoting the coordinates by $z_1,\dots,z_d$, this means that we can write $\omega_i = a_i^0 + a_i^1 z_1 + \ldots + a_i^d z_d$. We will use the shorthand $h(\mathbf{z})$ for a function $h(z_1,\dots,z_d)$, and $d\mathbf{z}$ for the holomorphic d-form $dz_1\wedge\cdots\wedge dz_d$. For a Laurent polynomial $h(\mathbf{z})$ and define

$$\operatorname*{Res}_{z_1=\infty}\dots\operatorname*{Res}_{z_d=\infty}\frac{h(\mathbf{z})\;d\mathbf{z}}{\prod_{i=1}^N\omega_i} \stackrel{\text{def}}{=} \int_{|z_1|=R_1}\cdots\int_{|z_d|=R_d}\frac{h(\mathbf{z})\;d\mathbf{z}}{\prod_{i=1}^N\omega_i}, \tag{4.3.5}$$

where $1\ll R_1\ll\ldots\ll R_d$. At first sight this is the same integral as the one defining the residue at the origin, but here we give the opposite orientation of the cycle $|z_i|=R_i$, e.g. $\int_{|z|=R} dz/z = -2\Pi i$. Below we will use the following simplified notation:

$$\operatorname*{Res}_{\mathbf{z}=\infty} \stackrel{\text{def}}{=} \operatorname*{Res}_{z_1=\infty}\operatorname*{Res}_{z_2=\infty}\dots\operatorname*{Res}_{z_d=\infty}.$$

In practice, the iterated residue 4.3.5 may be computed using the following **algorithm**: for each i, use the expansion

$$\frac{1}{\omega_i} = \sum_{j=0}^{\infty}(-1)^j\frac{(a_i^0 + a_i^1 z_1 + \ldots + a_i^{q(i)-1}z_{q(i)-1})^j}{(a_i^{q(i)}z_{q(i)})^{j+1}}, \tag{4.3.6}$$

where $q(i)$ is the largest value of m for which $a_i^m \neq 0$, then multiply the product of these expressions with $(-1)^d h(z_1, \ldots, z_d)$, and then take the coefficient of $z_1^{-1} \ldots z_d^{-1}$ in the resulting Laurent series.

We have the following *iterated residue theorem.*

Proposition 4.3.4. *For a polynomial $Q(\mathbf{z})$ on $\mathbb{C}^d$, we have*

$$\sum_{\sigma \in \mathcal{S}_n/\mathcal{S}_{n-d}} \frac{Q(\lambda_{\sigma \cdot 1}, \ldots, \lambda_{\sigma \cdot d})}{\prod_{1 \leq m \leq d} \prod_{i=m+1}^{n} (\lambda_{\sigma \cdot i} - \lambda_{\sigma \cdot m})} = \operatorname*{Res}_{\mathbf{z}=\infty} \frac{\prod_{1 \leq m < l \leq d} (z_m - z_l)\, Q(\mathbf{z})\, d\mathbf{z}}{\prod_{l=1}^{d} \prod_{i=1}^{n} (\lambda_i - z_l)} \tag{4.3.7}$$

Proof. We compute the iterated residue (4.3.7) using the Residue Theorem on the projective line $\mathbb{C} \cup \{\infty\}$. The first residue, which is taken with respect to z_d, is a contour integral, whose value is minus the sum of the z_d-residues of the form in (4.3.7). These poles are at $z_d = \lambda_j$, $j = 1, \ldots, n$, and after canceling the signs that arise, we obtain the following expression for the right hand side of (4.3.7):

$$\sum_{j=1}^{n} \frac{\prod_{1 \leq m < l \leq d-1} (z_m - z_l) \prod_{l=1}^{d-1} (z_l - \lambda_j)\, Q(z_1, \ldots, z_{d-1}, \lambda_j)\, dz_1 \ldots dz_{d-1}}{\prod_{l=1}^{d-1} \prod_{i=1}^{n} (\lambda_i - z_l) \prod_{i \neq j}^{n} (\lambda_i - \lambda_j)}.$$

After cancellation and exchanging the sum and the residue operation, at the next step, we have

$$(-1)^{d-1} \sum_{j=1}^{n} \operatorname*{Res}_{z_{d-1}=\infty} \frac{\prod_{1 \leq m < l \leq d-1} (z_m - z_l)\, Q(z_1, \ldots, z_{d-1}, \lambda_j)\, dz_1 \ldots dz_{d-1}}{\prod_{i \neq j}^{n} \Big((\lambda_i - \lambda_j) \prod_{l=1}^{d-1} (\lambda_i - z_l) \Big)}.$$

Now we can repeat our previous trick of applying the Residue Theorem; the only difference is that now the pole $z_{d-1} = \lambda_j$ has been eliminated. As a result, after converting the second residue to a sum, we obtain

$$(-1)^{2d-3} \sum_{j=1}^{n} \sum_{s=1,\, s \neq j}^{n} \frac{\prod_{1 \leq m < l \leq d-2} (z_l - z_m)\, Q(z_1, \ldots, z_{d-2}, \lambda_s, \lambda_j)\, dz_1 \ldots dz_{d-2}}{(\lambda_s - \lambda_j) \prod_{i \neq j,s}^{n} \Big((\lambda_i - \lambda_j)(\lambda_i - \lambda_s) \prod_{l=1}^{d-1} (\lambda_i - z_l) \Big)}.$$

Iterating this process, we arrive at a sum very similar to (4.3.3). The difference between the two sums will be the sign: $(-1)^{d(d-1)/2}$, and that the $d(d-1)/2$ factors of the form $(\lambda_{\sigma(i)} - \lambda_{\sigma(m)})$ with $1 \leq m < i \leq d$ in the denominator will have opposite signs. These two differences cancel each other, and this completes the proof. $\square$

Remark 4.3.5. Changing the order of the variables in iterated residues, usually, changes the result. In this case, however, because all the poles are normal crossing, formula (4.3.7) remains true no matter in what order we take the iterated residues.

4.3.3 Localization in the fiber

Combining Proposition 4.3.3 with Proposition 4.3.4, we arrive at the formula

$$\mathrm{eP}[\Theta_d, \mathcal{J}_d(n,k)] = \operatorname*{Res}_{\mathbf{z}=\infty} \frac{\prod_{1\le m<l\le d}(z_m - z_l)\, Q_{\mathrm{Fl}}(\mathbf{z},\boldsymbol{\theta})\, d\mathbf{z}}{\prod_{l=1}^{d}\prod_{i=1}^{n}(\lambda_i - z_l)}. \tag{4.3.8}$$

The "only" unknown here is the polynomial $Q_{\mathrm{Fl}}(\mathbf{z},\boldsymbol{\theta})$ defined in (4.3.4), and, therefore, we now turn to its computation.

Let us briefly review the construction of $Q_{\mathrm{Fl}}(\mathbf{z},\boldsymbol{\theta})$ (cf. diagram (4.3.1) and Proposition 4.3.3). This polynomial is an equivariant Poincaré dual taken with respect to the group $T_d \times T_k$, which has weights $(z_1,\dots,z_d)$ and $(\theta_1,\dots,\theta_k)$. Consider the $B_L \times B_R$-module $\mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Ym}^{\bullet}\mathbb{C}^d_L)$, and endow it with coordinates $u^l_\tau \in \mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Ym}^{\bullet}\mathbb{C}^d_L)^*$, indexed by pairs $(\tau,l) \in \Pi \times \mathbb{Z}_{>0}$ satisfying $\mathrm{sum}(\tau) \le l \le d$. We will consider the dual space spanned by these coordinates as carrying a *right* action of $T_d \times T_k$; accordingly,

$$\text{the weight of } u^l_\tau \;=\; (z_{i_1} + z_{i_2} + \dots + z_{i_m}, \theta_l), \text{ where } \tau = [i_1, i_2, \dots, i_m]. \tag{4.3.9}$$

For each nondegenerate system $\varepsilon \in \mathcal{E} \subset \mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Ym}^{\bullet}\mathbb{C}^d_L)$ we denote the image $\mathrm{pr}_{\mathcal{E}}(\varepsilon)$ in the quotient $\mathrm{pr}_{\mathcal{E}} : \mathcal{E} \to \widetilde{\mathcal{E}} = \mathcal{E}/B_R$ by $\tilde\varepsilon$; in particular, we have a reference point $\tilde\varepsilon_{\mathrm{ref}} \in \widetilde{\mathcal{E}}$ corresponding to the system $\varepsilon_{\mathrm{ref}}$ given by

$$u^l_\pi(\varepsilon_{\mathrm{ref}}) = \begin{cases} 1, & \text{if } \mathrm{sum}(\pi) = l \\ 0, & \text{otherwise.} \end{cases} \tag{4.3.10}$$

The stabilizer subgroup of $\tilde\varepsilon_{\mathrm{ref}} \in \widetilde{\mathcal{E}}$ under the B_d-action is a d-dimensional subgroup $H_d \subset B_d$, hence the orbit $B_d\tilde\varepsilon_{\mathrm{ref}} \subset \widetilde{\mathcal{E}}$ is a subvariety of dimension $d(d-1)/2$; we denoted the closure of this subvariety by $\mathcal{O}$.

Next consider the vector bundle

$$V = \mathcal{E} \times_{B_R} \mathbb{C}^d_R \longrightarrow \widetilde{\mathcal{E}} = \mathcal{E}/B_R$$

associated to the standard representation of B_R, and the $T_d \times T_k$-equivariant linear bundle map from a trivial bundle

$$s : \widetilde{\mathcal{E}} \times \mathcal{J}_d(n,k) \longrightarrow V^* \otimes \mathbb{C}^k$$

defined by the natural compositions, see (4.2.36). Then, according to Proposition 4.2.17, the polynomial $Q_{\mathrm{Fl}}(\mathbf{z},\boldsymbol{\theta})$ is the *equivariant Poincaré dual of the projection to* $\mathcal{J}_d(n,k)$ *of the restriction* $\ker(s)|\mathcal{O}$ *of the vector bundle* $\ker(s)$. This is, naturally, identical to the variety $\mathrm{ev}_{\widetilde{\mathcal{E}}}(\tau^{-1}_{\widetilde{\mathcal{E}}}(\mathcal{O}))$ which appears in the definition (4.3.4).

While the variety $\mathcal{O}$ is highly singular, the set of T_d-fixed points of $\mathcal{O}$ is finite – as we will see shortly – and hence we can apply here the localization principle based on Rossmann's integration formula: Proposition 3.3.2. The result is:

$$Q_{\mathrm{Fl}}(\mathbf{z},\boldsymbol{\theta}) = \sum_{p \in \mathcal{O}^{T_d}} \frac{\mathrm{Euler}^{T_d\times T_k}(V^*_p \otimes \mathbb{C}^k)\, \mathrm{emult}_p[\mathcal{O}, \widetilde{\mathcal{E}}]}{\mathrm{Euler}^{T_d\times T_k}(\mathrm{T}_p\widetilde{\mathcal{E}})}. \tag{4.3.11}$$

Our task thus has reduced to the identification and computation of the objects in this formula. These are:

- The set $\mathcal{O}^{T_d}$ of T_d-fixed points in $\mathcal{O} \subset \widetilde{\mathcal{E}}$,
- the weights of the T_d-action on the fibers V_p for $p \in \mathcal{O}^{T_d}$,
- the weights of the T_d-action on the tangent spaces $\mathrm{T}_p\widetilde{\mathcal{E}}$ for $p \in \mathcal{O}^{T_d}$,
- the equivariant multiplicities of $\mathcal{O}$ in $\widetilde{\mathcal{E}}$ at each fixed point $p \in \mathcal{O}^{T_d}$.

The most immediate problem we face is that we do not have an effective description of the set $\mathcal{O}^{T_d}$ of T_d-fixed points in $\mathcal{O}$. There is a formal way around this: we replace the fixed point set $\mathcal{O}^{T_d}$ with the larger set $\widetilde{\mathcal{E}}^{T_d}$ and define the equivariant multiplicity $\mathrm{emult}_p[\mathcal{O}, \widetilde{\mathcal{E}}]$ to be zero in the case when $p \in \widetilde{\mathcal{E}}^{T_d} \setminus \mathcal{O}^{T_d}$.

The set of fixed points $\widetilde{\mathcal{E}}^{T_d}$ is fairly easy to determine: these fixed points are given by those nondegenerate systems $\varepsilon \in \mathcal{E} \subset \mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Ym}^{\bullet}\mathbb{C}^d_L)$ for which the tensors $\varepsilon(e_m) \in \mathrm{Ym}^{\bullet}\mathbb{C}^d_L$, $m = 1, \dots, d$ are of pure T_d-weight. These, in turn, may be enumerated as follows.

Definition 4.3.6. We will call a sequence of partitions $\boldsymbol{\pi} = (\pi_1, \dots, \pi_d) \in \Pi^{\times d}$ *admissible* if

1. $\mathrm{sum}(\pi_l) \le l$ for $l = 1, \dots, d$ and
2. $\pi_l \ne \pi_m$ for $1 \le l \ne m \le d$.

We will denote the set of admissible sequences of length d by $\boldsymbol{\Pi}_d$; we also introduce the numerical characteristic:

$$\mathrm{defect}(\boldsymbol{\pi}) = \sum_{l=1}^{d} (l - \mathrm{sum}(\pi_l)).$$

As an example, consider the case $d = 3$. The set of admissible partition sequences of length 3 consists of the following 8 elements:

$$\begin{aligned}\boldsymbol{\Pi}_3 = \{&([1],[2],[3]),\ ([1],[2],[1,2]),\ ([1],[2],[1,1]),\ ([1],[2],[1,1,1])\\ &([1],[1,1],[3]),\ ([1],[1,1],[1,1,1]),\ ([1],[1,1],[2]),\ ([1],[1,1],[1,2])\};\end{aligned}$$

For an admissible $\boldsymbol{\pi} = (\pi_1, \dots, \pi_d) \in \boldsymbol{\Pi}_d$ introduce the system $\varepsilon_{\boldsymbol{\pi}}$ given by

$$u^l_\tau(\varepsilon_{\boldsymbol{\pi}}) = \begin{cases} 1 \text{ if } \tau = \pi_l, \\ 0 \text{ otherwise.} \end{cases} \tag{4.3.12}$$

As usual, the point corresponding to $\varepsilon_{\boldsymbol{\pi}}$ in $\widetilde{\mathcal{E}}$ will be denoted by $\tilde{\varepsilon}_{\boldsymbol{\pi}} = \mathrm{pr}_{\mathcal{E}}(\varepsilon_{\boldsymbol{\pi}})$.

The following statement follows from the definitions.

Lemma 4.3.7. • *The correspondence* $\boldsymbol{\pi} \mapsto \tilde{\varepsilon}_{\boldsymbol{\pi}}$ *establishes a bijection between the set* $\boldsymbol{\Pi}_d$ *of admissible sequences of partitions and the fixed point set* $\widetilde{\mathcal{E}}^{T_d}$.

- *For* $\tau \in \Pi$ *and an integer* i, *denote by* $\mathrm{mult}(i, \tau)$ *the number of times* i *occurs in* τ, *and let* $z_\tau = \sum_{i \in \tau} \mathrm{mult}(i, \tau)\, z_i$. *Then, given an admissible sequence* $\boldsymbol{\pi} \in \boldsymbol{\Pi}_d$, *the weights of the* T_d*-action on the fiber of* V *at the fixed point* $\tilde{\varepsilon}_{\boldsymbol{\pi}}$ *are*

$$z_{\pi_1}, \dots, z_{\pi_d}.$$

Corollary 4.3.8. *The weights of the* $T_d \times T_k$ *action on fiber* $V^*_{\tilde{\varepsilon}_{\boldsymbol{\pi}}} \otimes \mathbb{C}^k$ *are*

$$\{\theta_j - z_{\pi_m};\ m = 1, \dots, d,\ j = 1, \dots, k\}.$$

Next we turn to the 3rd item on our list: the weights of the T_d-action on tangent space of $\widetilde{\mathcal{E}}$ at the fixed points $\tilde{\varepsilon}_{\boldsymbol{\pi}}$; we will use the simplified notation $\mathrm{T}_{\boldsymbol{\pi}}\widetilde{\mathcal{E}}$ for this tangent space. To compute the answer, it will be convenient to linearize the action near $\tilde{\varepsilon}_{\boldsymbol{\pi}}$.

Definition 4.3.9. For each $\boldsymbol{\pi} = (\pi_1, \dots, \pi_d) \in \boldsymbol{\Pi}_d$ introduce the affine-linear subspace $\mathcal{N}_{\boldsymbol{\pi}} \subset \mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Sym}^{\bullet}_d\mathbb{C}^n)$ given by

$$\mathcal{N}_{\boldsymbol{\pi}} = \left\{ \varepsilon \in \mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Sym}^{\bullet}_d\mathbb{C}^n);\ u^m_{\pi_l}(\varepsilon) = \begin{cases} 1 \text{ if } m = l \\ 0 \text{ if } m > l \end{cases} \quad \text{for } 1 \le l \le d \right\};$$

Also, for $\boldsymbol{\pi} \in \boldsymbol{\Pi}_d$ introduce the map

$$\alpha_{\boldsymbol{\pi}} : \mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Sym}^{\bullet}_d\mathbb{C}^n) \to \mathrm{Mat}^{d\times d}$$

which associates to each system ε its $d \times d$ minor corresponding to the sequence of partitions $\boldsymbol{\pi} = (\pi_1, \dots, \pi_d)$.

A few comments are in order. First, we can rewrite the above definition of $\mathcal{N}_{\boldsymbol{\pi}}$ as follows:

$$\mathcal{N}_{\boldsymbol{\pi}} = \left\{ \varepsilon \in \mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Sym}^{\bullet}_d\mathbb{C}^n);\ \alpha_{\boldsymbol{\pi}}(\varepsilon) \in U_- \right\} \tag{4.3.13}$$

where U_- is the subgroup of lower-triangular $d \times d$ matrices with 1s on the diagonal; this way it is apparent that $\mathcal{N}_{\boldsymbol{\pi}} \subset \mathcal{E}$.

Also, observe that $\varepsilon_{\boldsymbol{\pi}} \in \mathcal{N}_{\boldsymbol{\pi}}$, and considering this special point to be the origin, we will think of $\mathcal{N}_{\boldsymbol{\pi}}$ as a *linear* space. Then $\mathcal{N}_{\boldsymbol{\pi}}$ is endowed with a natural set of coordinates:

$$\hat{u}^l_{\tau|\boldsymbol{\pi}} = u^l_\tau | \mathcal{N}_{\boldsymbol{\pi}},\ \mathrm{sum}(\tau) \le l \le d,\ \tau \ne \pi_1, \dots, \pi_l. \tag{4.3.14}$$

Proposition 4.3.10. *Let* $\boldsymbol{\pi} \in \boldsymbol{\Pi}_d$ *be an admissible sequence of partitions. Then*

1. *the restriction of the projection* $\mathrm{pr}_{\mathcal{E}} : \mathcal{E} \to \widetilde{\mathcal{E}}$ *to* $\mathcal{N}_{\boldsymbol{\pi}}$ *is an embedding and the collection* $\{\mathrm{pr}_{\mathcal{E}}(\mathcal{N}_{\boldsymbol{\pi}});\ \boldsymbol{\pi} \in \boldsymbol{\Pi}_d\}$ *forms an open cover of* $\widetilde{\mathcal{E}}$.

2. *for* any $\boldsymbol{\pi} \in \mathbf{\Pi}_d$*, the image* $\mathrm{pr}_{\mathcal{E}}(\mathcal{N}_{\boldsymbol{\pi}}) \subset \widetilde{\mathcal{E}}$ *is* T_d*-invariant, and the induced* T_d*-action on* $\mathcal{N}_{\boldsymbol{\pi}}$ *is linear, and diagonal with respect to the coordinates* (4.3.14)*. Considering* T_d *as acting on the* right *on these coordinates,*

$$\text{the weight of } \hat{u}^l_{\tau|\boldsymbol{\pi}} = z_\tau - z_{\pi_l}. \tag{4.3.15}$$

3. *If* $\mathrm{defect}(\boldsymbol{\pi}) = 0$*, then* $\mathrm{pr}_{\mathcal{E}}(\mathcal{N}_{\boldsymbol{\pi}}) \subset \widetilde{\mathcal{E}}$ *is* B_d*- invariant.*

Remark 4.3.11. We will denote by $T_{\boldsymbol{\pi}}$ and $B_{\boldsymbol{\pi}}$ the actions of T_d and B_d induced on $N_{\boldsymbol{\pi}}$ by the embedding $\mathrm{pr}_{\mathcal{E}}$.

Proof. We first show that $\cup\{\mathrm{pr}_{\mathcal{E}}(\mathcal{N}_{\boldsymbol{\pi}});\ \boldsymbol{\pi} \in \mathbf{\Pi}_d\} = \widetilde{\mathcal{E}}$. This means that for an arbitrary element $\varepsilon \in \mathcal{E}$, we have to find an admissible partition $\boldsymbol{\pi} \in \mathbf{\Pi}_d$ and an upper-triangular matrix $b_R = b_R(\varepsilon, \boldsymbol{\pi}) \in B_R$ such that $\varepsilon \cdot b_R \in \mathcal{N}_{\boldsymbol{\pi}}$. This can be done by elementary column operations: consider ε as a $\dim(\mathrm{Ym}^{\bullet}\mathbb{C}^d_L) \times d$ matrix whose columns are linearly independent, and whose rows are indexed by partitions. The only nonzero entry in the first column corresponds to the trivial partition [1]; then we can multiply the first column by a constant to rescale this entry to 1, and then annihilate all other entries in the same row by adding multiples of the first column to the others. Next, since ε is nonsingular, we can pick a nonzero entry in the second column of the resulting matrix – this entry will correspond to a partition π_2 – and, again, using column operations, we annihilate all entries in this row starting form column 3 and so on. Continuing this process, we obtain an admissible $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_d)$, and the described sequence of column operations produces an upper-triangular $b_R \in B_R$ such that $\varepsilon \cdot b_R \in \mathcal{N}_{\boldsymbol{\pi}}$.

The process described above finds an appropriate $\boldsymbol{\pi} \in \mathbf{\Pi}_d$ for each ε, and brings $\alpha_{\boldsymbol{\pi}}(\varepsilon)$ to lower-triangular form. Moreover, if $\mathrm{pr}_{\mathcal{E}}(\varepsilon_1) = \mathrm{pr}_{\mathcal{E}}(\varepsilon_2)$ for $\varepsilon_1, \varepsilon_2 \in \mathcal{N}_{\boldsymbol{\pi}}$, then $\varepsilon_1 \cdot b_R = \varepsilon_2$ for some $b_R \in B_R$, and therefore $\alpha_{\boldsymbol{\pi}}(\varepsilon_1) \cdot b_R = \alpha_{\boldsymbol{\pi}}(\varepsilon_2)$. Since $\alpha_{\boldsymbol{\pi}}(\varepsilon_1), \alpha_{\boldsymbol{\pi}}(\varepsilon_2)$ are lower-triangular with 1s on the diagonal and B_R is upper-triangular, this can only happen when b_R is the unit matrix, so $\varepsilon_1 = \varepsilon_2$. This proves that $\mathrm{pr}_{\mathcal{E}}$ is injective on $\mathcal{N}_{\boldsymbol{\pi}}$, hence the restriction $\mathrm{pr}_{\mathcal{E}}|\mathcal{N}_{\boldsymbol{\pi}}$ is an embedding.

To approach statements (2) and (3), next, we write down the action of B_d on $\widetilde{\mathcal{E}}$ in the chart $\mathcal{N}_{\boldsymbol{\pi}}$. Recall that the multiplication map $U_- \times B_d \to \mathrm{GL}_d$ is injective. This allows us to define the B_d-component a^B for an element $a \in U_- B_d$; in particular, for any such a, we have $a \cdot (a^B)^{-1} \in U_-$. Then, for $b \in B_d$ and $\varepsilon \in \mathcal{N}_{\boldsymbol{\pi}}$ we can define the partial action:

$$(b, \varepsilon) \mapsto b_{\boldsymbol{\pi}}\varepsilon = b_L \cdot \varepsilon \cdot (\alpha_{\boldsymbol{\pi}}(b_L \cdot \varepsilon)^B)^{-1}, \tag{4.3.16}$$

which is valid if $\alpha_{\boldsymbol{\pi}}(b_L \cdot \varepsilon) \in U_- B_d$.

Now consider the case when $b = t \in T_d$ is a diagonal matrix. In this case, $\alpha_{\boldsymbol{\pi}}(b_L \cdot \varepsilon)$ remains lower-triangular, with the numbers $(t^{\pi_1}, \ldots, t^{\pi_d})$ on the diagonal, where t^τ is the character of T_d corresponding to the weight z_τ. This means that $\alpha_{\boldsymbol{\pi}}(b_L \cdot \varepsilon) \in U_- B_d$, and the Borel factor $\alpha_{\boldsymbol{\pi}}(b_L \cdot \varepsilon)^B$ is the diagonal matrix with these same entries:

$$\alpha_{\boldsymbol{\pi}}(b_L \cdot \varepsilon)^B = \mathrm{diag}[t^{\pi_1}, \ldots, t^{\pi_d}]. \tag{4.3.17}$$

Note that this matrix is independent of ε. Now statement (2) follows easily.

Finally, to prove (3), observe that if $\mathrm{defect}(\boldsymbol{\pi}) = 0$, then the filtration-preserving property implies that $\alpha_{\boldsymbol{\pi}}(\varepsilon)$ is upper-triangular for any $\varepsilon \in \mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Sym}^{\bullet}_d\mathbb{C}^n)$. Hence for $\varepsilon \in \mathcal{N}_{\boldsymbol{\pi}}$, the matrix $\alpha_{\boldsymbol{\pi}}(\varepsilon)$ is the identity matrix, and then, using the condition $\mathrm{defect}(\boldsymbol{\pi}) = 0$ once again, we can conclude that $\alpha_{\boldsymbol{\pi}}(b_L \cdot \varepsilon)$ is upper-triangular with the numbers $(t^{\pi_1}, \dots, t^{\pi_d})$ on the diagonal, where t is the diagonal part of b. This means that $\alpha_{\boldsymbol{\pi}}(b_L \cdot \varepsilon)^B = \alpha_{\boldsymbol{\pi}}(b_L \cdot \varepsilon) \in B_d$, which implies statement (3). □

Remark 4.3.12. Clearly, $\alpha_{\boldsymbol{\pi}}(b_L \cdot \varepsilon)$ depends linearly on ε. In the case $\mathrm{defect}(\boldsymbol{\pi}) = 0$, we have $\alpha_{\boldsymbol{\pi}}(b_L \cdot \varepsilon)^B = \alpha_{\boldsymbol{\pi}}(b_L \cdot \varepsilon)$, and hence the action (4.3.16) of $B_{\boldsymbol{\pi}}$ on $\mathcal{N}_{\boldsymbol{\pi}}$ is quadratic, not linear as the $T_{\boldsymbol{\pi}}$-action. When $\mathrm{defect}(\boldsymbol{\pi}) > 0$, the action of $B_{\boldsymbol{\pi}}$ is not defined on the whole of $\mathcal{N}_{\boldsymbol{\pi}}$.

Proposition 4.3.10 provides us with a linearization of the T_d-action on $\widetilde{\mathcal{E}}$ near every fixed point. This allows us to compute equivariant multiplicities in (4.3.11) using (2.2.12). Indeed, if we introduce the notation

$$\mathcal{O}_{\boldsymbol{\pi}} \overset{\mathrm{def}}{=} (\mathrm{pr}_{\mathcal{E}}|\mathcal{N}_{\boldsymbol{\pi}})^{-1}(\mathcal{O}) \tag{4.3.18}$$

for the part of $\mathcal{O}$ in the local chart $\mathcal{N}_{\boldsymbol{\pi}}$, then we can write

$$\mathrm{emult}_{\tilde{\varepsilon}_{\boldsymbol{\pi}}}[\mathcal{O}, \widetilde{\mathcal{E}}] = \mathrm{eP}[\mathcal{O}_{\boldsymbol{\pi}}, \mathcal{N}_{\boldsymbol{\pi}}]. \tag{4.3.19}$$

Next, we take a closer look at the set $\mathcal{O}_{\boldsymbol{\pi}}$.

Lemma 4.3.13. *For every $\boldsymbol{\pi} \in \boldsymbol{\Pi}_d$, we have*

$$\mathcal{O}_{\boldsymbol{\pi}} = \overline{B_L \varepsilon_{\mathrm{ref}} B_R} \cap \mathcal{N}_{\boldsymbol{\pi}}. \tag{4.3.20}$$

Moreover, $\varepsilon_{\mathrm{ref}} \in \mathcal{N}_{\boldsymbol{\pi}}$ if and only if $\mathrm{defect}(\boldsymbol{\pi}) = 0$, *and in this case $\mathcal{O}_{\boldsymbol{\pi}} = \overline{B_{\boldsymbol{\pi}} \varepsilon_{\mathrm{ref}}}$, where $B_{\boldsymbol{\pi}}$ stands for the action* (4.3.16).

Proof. By definition, $\mathcal{O}_{\boldsymbol{\pi}} = \overline{B_L \varepsilon_{\mathrm{ref}} B_R \cap N_{\boldsymbol{\pi}}}$, and hence (4.3.20) follows from the fact that B_d acts properly on the right on $U_{-}B_d \subset \mathrm{GL}_d$. The second statement then immediately follows from the comparison of (4.3.10) and Definition 4.3.9. □

We are finally ready to take stock of our results so far. Substituting the weights from Corollary 4.3.8 and (4.3.15) into (4.3.11), and taking into consideration (4.3.19), we obtain:

$$Q_{\mathrm{Fl}}(\boldsymbol{\lambda}, \boldsymbol{\theta}) = \sum_{\boldsymbol{\pi} \in \boldsymbol{\Pi}_d} \frac{\prod_{m=1}^{d} \prod_{j=1}^{k} (\theta_j - z_{\pi_m})\, Q_{\boldsymbol{\pi}}(z_1, \dots, z_d)}{\displaystyle\prod_{l=1}^{d} \prod_{\substack{\tau \neq \pi_1, \dots, \pi_l \\ \mathrm{sum}(\tau) \leq l}} (z_\tau - z_{\pi_l})}, \tag{4.3.21}$$

where

$$Q_{\boldsymbol{\pi}} = \begin{cases} \mathrm{eP}[(\mathcal{O}_{\boldsymbol{\pi}}, \mathcal{N}_{\boldsymbol{\pi}}] \text{ if } \tilde{\varepsilon}_{\boldsymbol{\pi}} \in \mathcal{O}, \\ 0 \text{ if } \tilde{\varepsilon}_{\boldsymbol{\pi}} \notin \mathcal{O}. \end{cases} \tag{4.3.22}$$

Combining this formula with (4.3.7), and arrive at our first formula for $\mathrm{eP}[\overline{\Theta}_d]$:

$$\mathrm{eP}[\overline{\Theta}_d] = \operatorname*{Res}_{\mathbf{z}=\infty} \frac{\prod_{m<l}(z_m - z_l)\, dz_1 \dots dz_d}{\prod_{l=1}^d \prod_{i=1}^n (\lambda_i - z_l)} \sum_{\boldsymbol{\pi}\in\boldsymbol{\Pi}_d} \frac{\prod_{m=1}^d \prod_{j=1}^k (\theta_j - z_{\pi_m})\, Q_{\boldsymbol{\pi}}(\mathbf{z})}{\prod_{l=1}^d \prod\{(z_\tau - z_{\pi_l});\ \mathrm{sum}(\tau) \le l,\ \tau \ne \pi_1, \dots, \pi_l\}} \tag{4.3.23}$$

Now observe that the sum here is finite, hence we are free to exchange the summation with the residues. Rearranging our the formula accordingly, we arrive at the following statement.

Proposition 4.3.14. *For each admissible series $\boldsymbol{\pi} = (\pi_1, \dots, \pi_d)$ of d partitions, introduce the polynomial $Q_{\boldsymbol{\pi}}(\mathbf{z})$ defined by* (4.3.22)*, then*

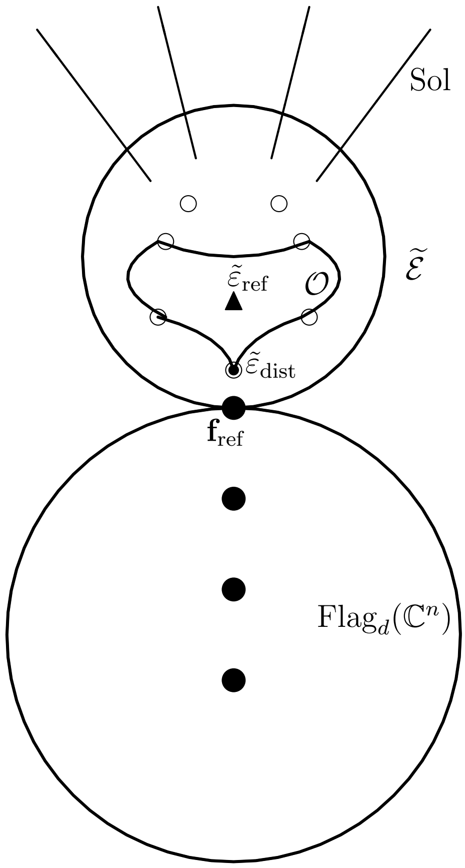
$$\mathrm{eP}[\Theta_d] = \sum_{\boldsymbol{\pi}\in\boldsymbol{\Pi}_d} \operatorname*{Res}_{\mathbf{z}=\infty} \frac{Q_{\boldsymbol{\pi}}(\mathbf{z}) \prod\limits_{m<l}(z_m - z_l)}{\prod\limits_{l=1}^{d} \prod\limits_{\substack{\tau\ne\pi_1,\dots,\pi_l \\ \mathrm{sum}(\tau)\le l}}} \frac{\prod\limits_{m=1}^{d}\prod\limits_{j=1}^{k}(\theta_j - z_{\pi_m})}{\prod\limits_{l=1}^{d}\prod\limits_{i=1}^{n}(\lambda_i - z_l)}\, d\mathbf{z}. \tag{4.3.24}$$

This formula has the pleasant feature that the three parameters of our problem, n, k and d, enter in it in a separate manner. The first fraction here only depends on d, the denominator of the second only depends on n, and the numerator of this latter fraction controls the k-dependence, with some interference from the sequence $\boldsymbol{\pi}$.

While this formula is a step forward, it is rather difficult to use in practice, since the number of terms and factors in it grows with d as the the number of elements in $\boldsymbol{\Pi}_d$. Also, the known properties of Thom polynomials listed in Proposition 2.2.13 are not manifest in (4.3.24).

In the next section, we will see that this formula goes through two dramatic simplifications, which make it easy to compute for small values of d.

Before proceeding, we present a schematic diagram of the main objects of our constructions. We hope this will help the reader to navigate among the various spaces we have introduced.



Explanations:

- The lower circle is the flag variety $\mathrm{Flag}_d(\mathbb{C}^n)$; the fat dots inside represent the T_n-fixed flags in $\mathrm{Flag}_d(\mathbb{C}^n)$.

- The upper circle is $\widetilde{\mathcal{E}}$, the fiber of the bundle $\mathrm{Ind}(\widetilde{\mathcal{E}})$ over the reference flag $\mathbf{f}_{\mathrm{ref}}$. The small circles inside represent the T_d-fixed points in $\widetilde{\mathcal{E}}$. One of these fixed points, $\tilde{\varepsilon}_{\mathrm{dist}} \in \widetilde{\mathcal{E}}$ will play an important role in what follows.

- The region bounded by the curvylinear pentagon represents the B_d-orbit of the reference point $\tilde{\varepsilon}_{\mathrm{ref}}$, which is marked by a triangle. The closure of the orbit is $\mathcal{O}$; this is a singular subvariety of $\widetilde{\mathcal{E}}$, which contains some of the fixed points of $\widetilde{\mathcal{E}}$, but not all of them.

- The straight lines on top are the linear solution spaces of the corresponding systems of equations in $\widetilde{\mathcal{E}}$. The union of these solution spaces lying over those points of the fiber bundle $\mathrm{Ind}(\widetilde{\mathcal{E}})$ which correspond to $\mathcal{O}$ form the closure of our singularity locus Θ_d.

4.4 Vanishing residues and the main result

The terms on the right hand side of formula (4.3.24) are enumerated by admissible sequences. There is a simplest one among these:

$$\boldsymbol{\pi}_{\mathrm{dist}} = ([1],[2],\dots,[d]), \tag{4.4.1}$$

which we will call *distinguished* . To avoid double indices, below, we will use the simplified notation Q_{dist} instead of $Q_{\boldsymbol{\pi}_{\mathrm{dist}}}$, and similarly $\tilde{\varepsilon}_{\mathrm{dist}}, \mathcal{N}_{\mathrm{dist}}, \mathcal{O}_{\mathrm{dist}}$, etc.

The following remarkable vanishing result holds.

Proposition 4.4.1. *Assume that $d \ll n \le k$. Then all terms of the sum in* (4.3.24) *vanish except for the term corresponding to the sequence of partitions $\boldsymbol{\pi}_{\mathrm{dist}} = ([1],[2],\dots,[d])$. Hence, formula* (4.3.24) *reduces to*

$$\mathrm{eP}[\Theta_d] = \operatorname*{Res}_{\mathbf{z}=\infty} \frac{Q_{\mathrm{dist}}(z_1,\dots,z_n)\ \prod_{m<l}(z_m - z_l)\ d\mathbf{z}}{\prod_{l=1}^{d}\prod\{(z_\tau - z_l);\ \mathrm{sum}(\tau)\le l,\ |\tau|>1\}} \frac{\prod_{l=1}^{d}\prod_{j=1}^{k}(\theta_j - z_l)}{\prod_{l=1}^{d}\prod_{i=1}^{n}(\lambda_i - z_l)}, \tag{4.4.2}$$

where $Q_{\mathrm{dist}} = \mathrm{eP}[\mathcal{O}_{\mathrm{dist}}, \mathcal{N}_{\mathrm{dist}}]$.

Before turning to the proof, we make a few remarks. First, note that this simplification is dramatic: the number of terms in (4.3.24) grows exponentially with d, and of this sum now a single term survives. This is fortunate, because computing all the polynomials $Q_{\boldsymbol{\pi}}$, $\boldsymbol{\pi} \in \boldsymbol{\Pi}_d$ seems to be an insurmountable task; at the moment, we do not even have an algorithm to determine when $Q_{\boldsymbol{\pi}} = 0$, i.e. when $\tilde{\varepsilon}_{\boldsymbol{\pi}} \in \mathcal{O}$.

Our second observation is that after replacing in (4.4.2) z_l by $-z_l$, $l = 1,\dots,d$, we can rewrite (4.4.2) as

$$\mathrm{eP}[\Theta_d] = \operatorname*{Res}_{\mathbf{z}=\infty} \frac{(-1)^d \prod_{m<l}(z_m - z_l)\, Q_{\mathrm{dist}}(z_1,\dots,z_n)}{\prod_{l=1}^{d}\prod\{(z_\tau - z_l);\ \mathrm{sum}(\tau)\le l,\ |\tau|>1\}} \prod_{l=1}^{d} \mathrm{RC}\left(\frac{1}{z_l}\right) z_l^{k-n}\, dz_l, \tag{4.4.3}$$

where $\mathrm{RC}(z)$ is the generating series of the relative Chern classes introduced in (2.2.20). The denominator and numerator of the fraction in (4.4.3) are homogeneous polynomials of the same degree, so this substitution does not change it.

This means that our formulas explicitly conform to the framework of Thom-Damon, Proposition 2.2.13 (3): we have obtained an explicit of the Thom polynomial of the A_d-singularity in terms of the relative Chern classes.

Most of the present section will be taken up by the proof of Proposition 4.4.1. In § 4.4.2, we derive a criterion for the vanishing of iterated residues of the form (4.3.5). Applying this criterion to the right hand side of (4.3.24) reduces Proposition 4.4.1 to a statement about the factors of the polynomials $Q_{\boldsymbol{\pi}}$, $\boldsymbol{\pi} \in \boldsymbol{\Pi}_d$: Proposition 4.4.4. According to Lemma 2.2.2, such divisibility properties follow from the existence of relations of a certain form in the ideal of the subvariety $\mathcal{O}_{\boldsymbol{\pi}} \subset \mathcal{N}_{\boldsymbol{\pi}}$. We find a family of such relations in § 4.4.3 (see (4.4.17)), and then convert the condition in Lemma

2.2.2 into a combinatorial condition on $\boldsymbol{\pi}$ (cf. Lemma 4.4.12). At the end of § 4.4.3, we show that if a sequence $\boldsymbol{\pi}$ does not satisfy this combinatorial condition, then it is either $\boldsymbol{\pi}_{\text{dist}}$ or $\tilde{\varepsilon}_{\boldsymbol{\pi}} \notin \mathcal{O}$, thus completing the proof of Proposition 4.4.1.

Introduce the subset $\boldsymbol{\Pi}_{\mathcal{O}} \subset \boldsymbol{\Pi}_d$ defined by

$$\boldsymbol{\Pi}_{\mathcal{O}} = \{\boldsymbol{\pi} \in \boldsymbol{\Pi}_d;\ \tilde{\varepsilon}_{\boldsymbol{\pi}} \in \mathcal{O}\}\,. \tag{4.4.4}$$

As we mentioned earlier, at the moment, we do not have an explicit description of this set. In the course of this proof, however, we obtain a rather efficient, albeit incomplete criterion for a sequence $\boldsymbol{\pi} \in \boldsymbol{\Pi}_d$ *not* to belong to $\boldsymbol{\Pi}_{\mathcal{O}}$; we explain this criterion in § 4.4.4. Finally, in §4.4.5, we further simplify (4.4.3), and formulate our main result, Theorem 4.4.16.

Before embarking on this rather tortuous route, we give a few examples below in §4.4.1, which demonstrate the localization formulas and the vanishing property explicitly. Note that we devote the last chapter of the thesis to the detailed study of (4.4.3) for small values of d, and hence the proofs in §4.4.1 will be omitted.

4.4.1 The localization formulas for $d = 2, 3$

The situation for $d = 2$ and 3 is simplified by the fact, that in these cases the closure of the Borel-orbit $\mathcal{O} = \overline{B_d\tilde{\varepsilon}_{\text{ref}}} \subset \widetilde{\mathcal{E}}$ is smooth. Hence we can use the Berline-Vergne localization formula (2.2.14) instead of Rossmann's formula, and instead of (4.3.21) we can work with an explicit expression, not containing equivariant multiplicities which need to be computed. This allows us to write down the fixed point formula for $\mathrm{eP}[\Theta_d]$ obtained by substituting a simplified version of(4.3.21) into (4.3.8), and then compare it to the residue formula (4.4.2). In these cases we can describe the set $\boldsymbol{\Pi}_{\mathcal{O}}$ easily as well. The formulas below are justified in §5.1.

For $d = 2$, we have $\mathcal{O} = \widetilde{\mathcal{E}} \cong \mathbb{P}^1$. There are two fixed points in $\widetilde{\mathcal{E}}$:

$$\boldsymbol{\Pi}_{\mathcal{O}} = \boldsymbol{\Pi}_2 = \{([1],[2]),\quad ([1],[1,1])\}.$$

Then our fixed point formula reads as follows:

$$\mathrm{eP}[\Theta_2] = \sum_{s=1}^{n}\sum_{t\neq s}^{n} \frac{1}{\prod_{i\neq s}^{n}(\lambda_i-\lambda_s)\prod_{i\neq s,t}^{n}(\lambda_i-\lambda_t)} \times \left(\frac{\prod_{j=1}^{k}(\theta_j-\lambda_s)\prod_{j=1}^{k}(\theta_j-\lambda_t)}{2\lambda_s-\lambda_t} + \frac{\prod_{j=1}^{k}(\theta_j-\lambda_s)\prod_{j=1}^{k}(\theta_j-2\lambda_s)}{\lambda_t-2\lambda_s}\right).$$

This is equal to the residue (4.3.24):

$$\operatorname*{Res}_{z_1=\infty}\operatorname*{Res}_{z_2=\infty} \frac{z_1-z_2}{\prod_{i=1}^{n}(\lambda_i-z_1)\prod_{i=1}^{n}(\lambda_i-z_2)} \times \left(\frac{\prod_{j=1}^{k}(\theta_j-z_1)\prod_{j=1}^{k}(\theta_j-z_2)}{2z_1-z_2} + \frac{\prod_{j=1}^{k}(\theta_j-z_1)\prod_{j=1}^{k}(\theta_j-2z_1)}{z_2-2z_1}\right).$$

Proposition 4.4.1 states that the residue of the second term vanishes; this is easy to check by hand.

For $d = 3$, the orbit closure $\mathcal{O}$ is a smooth 3-dimensional hypersurface in $\widetilde{\mathcal{E}}$. There are 6 fixed points in $\mathcal{O}$, namely

$$\mathbf{\Pi}_{\mathcal{O}} = \{([1],[2],[3]),\ ([1],[2],[1,2]),\ ([1],[2],[1,1]),\\ ([1],[1,1],[3]),\ ([1],[1,1],[1,1,1]),\ ([1],[1,1],[2])\};$$

the remaining 2 fixed points in $\widetilde{\mathcal{E}}$ do not belong to $\mathcal{O}$ (see Proposition 4.4.14):

$$([1],[2],[1,1,1]),\ ([1],[1,1],[1,2]) \notin \mathbf{\Pi}_{\mathcal{O}}.$$

Hence the corresponding fixed point formula has 6 terms:

$$\mathrm{eP}[\Theta_3] = \sum_{s=1}^{n}\sum_{t\neq s}^{n}\sum_{u\neq s,t}^{n} \frac{\prod_{j=1}^{k}(\theta_j-\lambda_s)}{\prod_{i\neq s}^{n}(\lambda_i-\lambda_s)\prod_{i\neq s,t}^{n}(\lambda_i-\lambda_t)\prod_{i\neq s,t,u}^{n}(\lambda_i-\lambda_u)} \cdot \left[\frac{\prod_{j=1}^{k}(\theta_j-\lambda_t)}{2\lambda_s-\lambda_t}\cdot\right.$$
$$\left(\frac{\prod_{j=1}^{k}(\theta_j-\lambda_u)}{(2\lambda_s-\lambda_u)(\lambda_s+\lambda_t-\lambda_u)} + \frac{\prod_{j=1}^{k}(\theta_j-\lambda_s-\lambda_t)}{(\lambda_u-\lambda_s-\lambda_t)(2\lambda_s-\lambda_s-\lambda_t)} + \frac{\prod_{j=1}^{k}(\theta_j-2\lambda_s)}{(\lambda_u-2\lambda_s)(\lambda_s+\lambda_t-2\lambda_s)}\right) +$$
$$\left.\frac{\prod_{j=1}^{k}(\theta_j-2\lambda_s)}{\lambda_t-2\lambda_s}\cdot\left(\frac{\prod_{j=1}^{k}(\theta_j-\lambda_u)}{(\lambda_t-\lambda_u)(3\lambda_s-\lambda_u)} + \frac{\prod_{j=1}^{k}(\theta_j-3\lambda_s)}{(\lambda_u-3\lambda_s)(\lambda_t-3\lambda_s)} + \frac{\prod_{j=1}^{k}(\theta_j-\lambda_t)}{(\lambda_u-\lambda_t)(3\lambda_s-\lambda_t)}\right)\right].$$

The corresponding residue formula (4.3.24) also has 6 terms:

$$\mathrm{eP}[\Theta_3] = \operatorname*{Res}_{z_1=\infty}\operatorname*{Res}_{z_2=\infty}\operatorname*{Res}_{z_3=\infty} \frac{(z_1-z_2)(z_1-z_3)(z_2-z_3)\prod_{j=1}^{k}(\theta_j-z_1)}{\prod_{i=1}^{n}(\lambda_i-z_1)\prod_{i=1}^{n}(\lambda_i-z_2)\prod_{i=1}^{n}(\lambda_i-z_3)} \times \left[\frac{\prod_{j=1}^{k}(\theta_j-z_2)}{2z_1-z_2}\cdot\right.$$
$$\left(\frac{\prod_{j=1}^{k}(\theta_j-z_3)}{(2z_1-z_3)(z_1+z_2-z_3)} + \frac{\prod_{j=1}^{k}(\theta_j-z_1-z_2)}{(z_3-z_1-z_2)(2z_1-z_1-z_2)} + \frac{\prod_{j=1}^{k}(\theta_j-2z_1)}{(z_3-2z_1)(z_1+z_2-2z_1)}\right) +$$
$$\left.\frac{\prod_{j=1}^{k}(\theta_j-2z_1)}{z_2-2z_1}\cdot\left(\frac{\prod_{j=1}^{k}(\theta_j-z_3)}{(z_2-z_3)(3z_1-z_3)} + \frac{\prod_{j=1}^{k}(\theta_j-3z_1)}{(z_3-3z_1)(z_2-3z_1)} + \frac{\prod_{j=1}^{k}(\theta_j-z_2)}{(z_3-z_2)(3z_1-z_2)}\right)\right].$$

Here, again, the last 5 terms vanish, and only the one corresponding to the distinguished fixed point $([1],[2],[3])$ remains, leaving us with (4.4.2).

For $d > 3$, the variety $\mathcal{O}_d \subset \widetilde{\mathcal{E}}_d$ is singular. This means that the analogs of these formulas involve calculation of equivariant multiplicities, which is a rather difficult problem. We present some of these computations in § 5.1.

4.4.2 The vanishing of residues

In this paragraph, we describe the conditions under which iterated residues of the type appearing in the sum in (4.3.24) vanish.

We start with the 1-dimensional case, where the residue at infinity is defined by (4.3.5) with $d = 1$. By bounding the integral representation along a contour $|z| = R$ with R large, one can easily prove

Lemma 4.4.2. *Let $p(z), q(z)$ be polynomials of one variable. Then*

$$\operatorname*{Res}_{z=\infty} \frac{p(z)\,dz}{q(z)} = 0 \quad \text{if } \deg(p(z)) + 1 < \deg(q).$$

Consider now the multidimensional situation. Let $p(\mathbf{z}), q(\mathbf{z})$ be polynomials in the d variables $z_1, \ldots, z_d$, and assume that $q(\mathbf{z})$ is the product of linear factors $q = \prod_{i=1}^{N} L_i$, as in (4.4.2). Introduce the notation $d\mathbf{z} = dz_1 \ldots dz_d$. We would like to formulate conditions under which the iterated residue

$$\operatorname*{Res}_{z_1=\infty} \operatorname*{Res}_{z_2=\infty} \ldots \operatorname*{Res}_{z_d=\infty} \frac{p(\mathbf{z})\ d\mathbf{z}}{q(\mathbf{z})} \tag{4.4.5}$$

vanishes. Introduce the following notation:

- For a set of indices $S \subset \{1, \ldots, d\}$, denote by $\deg(p(\mathbf{z}); S)$ the degree of the one-variable polynomial $p_S(t)$ obtained from p via the substitution $z_m \to \begin{cases} t \text{ if } m \in S, \\ 1 \text{ if } m \notin S. \end{cases}$

- For a nonzero linear function $L = a_0 + a_1 z_1 + \ldots + a_d z_d$, denote by $\operatorname{coeff}(L, z_l)$ the coefficient a_l;

- finally, for $1 \le m \le d$, set

  $$\operatorname{lead}(q(\mathbf{z}); m) = \#\{i;\ \max\{l;\ \operatorname{coeff}(L_i, z_l) \neq 0\} = m\},$$

  which is the number of those factors L_i in which the coefficient of z_m does not vanish, but the coefficients of $z_{m+1}, \ldots, z_d$ are 0.

Thus we group the N linear factors of $q(\mathbf{z})$ according to the nonvanishing coefficient with the largest index; in particular, for $1 \le m \le d$ we have

$$\deg(q(\mathbf{z}); m) \ge \operatorname{lead}(q(\mathbf{z}); m), \ \text{ and } \sum_{m=1}^{d} \operatorname{lead}(q(\mathbf{z}); m) = N.$$

Now applying Lemma 4.4.2 to the first residue in (4.4.5), we see that

$$\operatorname*{Res}_{z_d=\infty} \frac{p(z_1, , \ldots, , z_{d-1}, z_d)\,d\mathbf{z}}{q(z_1, , \ldots, , z_{d-1}, z_d)} = 0$$

whenever $\deg(p(\mathbf{z}); d) + 1 < \deg(q(\mathbf{z}), d)$; in this case, of course, the entire iterated residue (4.4.5) vanishes.

Now we suppose the residue with respect to z_d does not vanish, and we look for conditions of vanishing of the next residue:

$$\operatorname*{Res}_{z_{d-1}=\infty}\operatorname*{Res}_{z_d=\infty}\frac{p(z_1,,\ldots,,z_{d-2},z_{d-1},z_d)\,d\mathbf{z}}{q(z_1,,\ldots,,z_{d-2},z_{d-1},z_d)}. \tag{4.4.6}$$

Note that now the condition $\deg(p(\mathbf{z});d-1)+1<\deg(q(\mathbf{z}),d-1)$ is *insufficient*; indeed,

$$\operatorname*{Res}_{z_{d-1}=\infty}\operatorname*{Res}_{z_d=\infty}\frac{1}{z_{d-1}(z_{d-1}+z_d)}\,d\mathbf{z}=\operatorname*{Res}_{z_{d-1}=\infty}\operatorname*{Res}_{z_d=\infty}\Big(\frac{1}{z_{d-1}z_d}+\ldots\Big)=1 \tag{4.4.7}$$

In fact, we need to require

$$\deg(p(\mathbf{z});d-1,d)+2<\deg(q(\mathbf{z});d-1,d). \tag{4.4.8}$$

To see that (4.4.8) is sufficient, let $q=\prod_{i=1}^{N}L_i$ be the product of linear forms. If

$$L=a_0+a_1z_1+\ldots+a_dz_d$$

is a linear form, then by (4.3.6):

- If $a_d\neq 0$, then
$$\frac{1}{L}=\sum_{j=0}^{\infty}\frac{(a_0+a_1z_1+\ldots+a_{d-1}z_{d-1})^j}{(a_dz_d)^{j+1}},$$
- If $a_d=0$ but $a_{d-1}\neq 0$, then
$$\frac{1}{L}=\sum_{j=0}^{\infty}\frac{(a_0+a_1z_1+\ldots+a_{d-2}z_{d-2})^j}{(a_{d-1}z_{d-1})^{j+1}},$$

  The case $a_d=a_{d-1}=0$ is irrevelant in the computation of $\mathrm{Res}_{z_{d-1}}\mathrm{Res}_{z_d}$.

When $a_{d-1}\neq 0, a_d\neq 0$, by expanding $\frac{1}{L}$, we can increase the z_{d-1}-degree in the numerator at the expense of increasing the z_d-degree in the denominator. So the Laurent series of $\frac{p(z)}{q(z)}$ has terms $z_1^{-i_1}\ldots z_d^{-i_d}$ with $i_{d-1}+i_d$ not less then $\deg(q(z);d-1,d)$. Therefore, for $\deg(p(z);d-1,d)+2<\deg(q(z),d-1,d)$ the following terms are missing from the Laurent series:

$$\frac{1}{z_1z_2},\frac{1}{z_1^2},\frac{1}{z_2^2}.$$

There is, also an another way to ensure the vanishing of (4.4.6): suppose that for $i=1,\ldots,N$, every time we have $\mathrm{coeff}(L_i,z_{d-1})\neq 0$, we also have $\mathrm{coeff}(L_i,z_d)=0$. This is equivalent to the condition $\deg(q(\mathbf{z}),d-1)=\mathrm{lead}(q(\mathbf{z});d-1)$. If this holds, then the compensating effect we described above cannot take place, and we can conclude the vanishing of (4.4.6) as long as

$$\deg(p(\mathbf{z}),d-1)+1<\deg(q(\mathbf{z}),d-1)$$

holds. This argument generalizes to the following statement.

Proposition 4.4.3. *Let $p(\mathbf{z})$ and $q(\mathbf{z})$ be polynomials in the variables $z_1, \ldots, z_d$, and assume that $q(\mathbf{z})$ is a product of linear factors: $q(\mathbf{z}) = \prod_{i=1}^{N} L_i$; set $d\mathbf{z} = dz_1 \ldots dz_d$. Then*

$$\operatorname*{Res}_{z_1=\infty} \operatorname*{Res}_{z_2=\infty} \ldots \operatorname*{Res}_{z_d=\infty} \frac{p(\mathbf{z})\, d\mathbf{z}}{q(\mathbf{z})} = 0$$

if for some $l \leq d$, either of the following two options hold:

- $\deg(p(\mathbf{z}); d, d-1, \ldots, l) + d - l + 1 < \deg(q(\mathbf{z}); d, d-1, \ldots, l)$,
  or

- $\deg(p(\mathbf{z}); l) + 1 < \deg(q(\mathbf{z}); l) = \operatorname{lead}(q(\mathbf{z}); l)$.

For us the second option will be more important. In terms of the linear factors of $q(\mathbf{z})$, the equality $\deg(q(\mathbf{z}); l) = \operatorname{lead}(q(\mathbf{z}); l)$ means that

$$\text{for each } i = 1, \ldots, N \text{ and } m > l,\ \operatorname{coeff}(L_i, z_l) \neq 0 \text{ implies } \operatorname{coeff}(L_i, z_m) = 0. \tag{4.4.9}$$

Recall that our goal is to show that all the terms of the sum in (4.3.24) vanish except for the one corresponding to $\boldsymbol{\pi}_{\text{dist}} = ([1], \ldots, [d])$. Let us apply our new-found tool, Proposition 4.4.3, to the terms of this sum, and see what happens.

Fix a sequence $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_d) \in \boldsymbol{\Pi}_d$, and consider the iterated residue corresponding to it on the right hand side of (4.3.24). The expression under the residue is the product of two fractions:

$$\frac{p(\mathbf{z})}{q(\mathbf{z})} = \frac{p_1(\mathbf{z})}{q_1(\mathbf{z})} \cdot \frac{p_2(\mathbf{z})}{q_2(\mathbf{z})},$$

where

$$\frac{p_1(\mathbf{z})}{q_1(\mathbf{z})} = \frac{Q_{\boldsymbol{\pi}}(\mathbf{z}) \prod_{m<l} (z_m - z_l)}{\prod_{l=1}^{d} \prod_{\operatorname{sum}(\tau) \leq l}^{\tau \neq \pi_1, \ldots, \pi_l} (z_\tau - z_{\pi_l})} \quad \text{and} \quad \frac{p_2(\mathbf{z})}{q_2(\mathbf{z})} = \frac{\prod_{m=1}^{d} \prod_{j=1}^{k} (\theta_j - z_{\pi_m})}{\prod_{l=1}^{d} \prod_{i=1}^{n} (\lambda_i - z_l)}. \tag{4.4.10}$$

Note that $p(\mathbf{z})$ is a polynomial, while $q(\mathbf{z})$ is a product of linear forms, and that $p_1(\mathbf{z})$ and $q_1(\mathbf{z})$ are independent of n and k, and depend on d only.

As a warm-up, we show that if the last element of the sequence is not the trivial partition, i.e. if $\pi_d \neq [d]$, then already the first residue in the corresponding term on the right hand side of (4.3.24) – the one with respect to z_d – vanishes. Indeed, if $\pi_d \neq [d]$, then $\deg(q_2(\mathbf{z}); d) \geq n$, while z_d does not appear in $p_2(\mathbf{z})$. Then, assuming that $d \ll n$, we have $\deg(p(\mathbf{z}); d) \ll \deg(q(\mathbf{z}); d)$, and this, in turn, implies the vanishing of the residue with respect to z_d (see Proposition 4.4.3).

We can thus assume that $\pi_d = [d]$, and proceed to the study of the next residue, the one taken with respect to z_{d-1}. Again, assume that $\pi_{d-1} \neq [d-1]$. As in the case of z_d above, $d \ll n$ implies $\deg(p(\mathbf{z}); d-1) \ll \deg(q(\mathbf{z}); d-1)$. However, now we cannot use the first option in Proposition 4.4.3, because $\deg(p_2(\mathbf{z}); d-1, d) = k \geq n$. In order to apply the second option, we have to exclude all linear factors from $q_1(\mathbf{z})$ which have nonzero coefficients in front of both z_{d-1} and z_d. The fact that $\pi_d = [d]$, and the restrictions $\mathrm{sum}(\pi_l) \leq l$, $l = 1, \ldots, d$, tell us that there are two troublesome factors: $(z_d - z_{d-1})$ and $(z_d - z_{d-1} - z_1)$ which come from the two partitions: $\tau = [d-1]$ and $\tau = [d-1, 1]$ in the $l = d$ part of $q_1(\mathbf{z})$. The first of the two fortunately cancels with a factor in the Vandermonde determinant in the numerator; as for the second factor: our only hope is to find it as a factor in the polynomial $Q_{\boldsymbol{\pi}}$.

Continuing this argument by induction, we can reduce Proposition 4.4.1 to the following statement about the equivariant multiplicities $Q_{\boldsymbol{\pi}}$, $\boldsymbol{\pi} \in \boldsymbol{\Pi}_d$.

Proposition 4.4.4. *Let $l \geq 1$, and let $\boldsymbol{\pi}$ be an admissible sequence of partitions of the form (4.4.12), where $\pi_l \neq [l]$. Then for $m > l$, and every partition τ such that $l \in \tau$, $\mathrm{sum}(\tau) \leq m$, and $|\tau| > 1$, we have*

$$(z_\tau - z_m) | Q_{\boldsymbol{\pi}}. \tag{4.4.11}$$

This statement will be proved in the next paragraph: §4.4.3. For now, we will assume that it is true, and give a quick proof of the result with which we started this section.

Proof of Proposition 4.4.1: Let $\boldsymbol{\pi} \neq \boldsymbol{\pi}_{\mathrm{dist}}$ be an admissible sequence of partitions. This means that there is $l > 1$ such that $\pi_l \neq [l]$, but $\pi_m = [m]$ for $m > l$:

$$\boldsymbol{\pi} = (\pi_1, \ldots \pi_l, [l+1], [l+2], \ldots, [d]). \tag{4.4.12}$$

Note that l does not appear anywhere in $\boldsymbol{\pi}$, and thus we can conclude $\deg(p(\mathbf{z}); l) \ll \deg(q(\mathbf{z}); l)$ from $d \ll n$, as usual. This allows us to apply the second option of Proposition 4.4.3 to the residue taken with respect to z_l as long as we can cancel from $q_2(\mathbf{z})$ all factors which do not satisfy condition (4.4.9).

These factors are of the form $z_\tau - z_m$, where $m > l$ and $l \in \tau$. If $|\tau| = 1$, i.e. if $\tau = [l]$, then we can find this factor in the Vandermonde determinant in the numerator. We can use Proposition 4.4.4 to cancel the rest of the factors, as long as we make sure that such factors occur in $q_1(\mathbf{z})$ with multiplicity 1. This is straightforward in our case, since the variable z_m with $m \geq l$ may appear only in the mth factor of $q_1(\mathbf{z})$. □

4.4.3 The homogeneous ring of $\widetilde{\mathcal{E}}$ and factorization of $Q_{\boldsymbol{\pi}}$

Now we turn to the proof of Proposition 4.4.4. Let $\boldsymbol{\pi} \in \boldsymbol{\Pi}_d$ be an admissible sequence of partitions. Recall (cf. (4.3.22)) that $Q_{\boldsymbol{\pi}}$ is the T_d-equivariant Poincaré dual of the part $\mathcal{O}_{\boldsymbol{\pi}} = \mathrm{pr}_{\mathcal{E}}^{-1}(\mathcal{O}) \cap \mathcal{N}_{\boldsymbol{\pi}}$ of the orbit closure $\mathcal{O}$ in the linear chart $\mathcal{N}_{\boldsymbol{\pi}}$ (cf. (4.3.19)); this latter linear space is endowed with coordinates $\hat{u}^l_{\tau|\boldsymbol{\pi}}$ defined in (4.3.14).

Our plan is to use Lemma 2.2.2, which, when applied to our situation, says that the divisibility relation (4.4.11) follows if we find a relation in the ideal of the subvariety $\mathcal{O}_{\boldsymbol{\pi}} \subset \mathcal{N}_{\boldsymbol{\pi}}$ expressing the appropriate variable $\hat{u}^m_{\tau|\boldsymbol{\pi}}$ as a polynomial of the rest of the variables.

We will lift the calculation from $\widetilde{\mathcal{E}}$ to the vector space $\mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Ym}^{\bullet}\mathbb{C}^d_L)$. Denote by $\mathbb{C}[u^\bullet]$ the ring of polynomial functions on $\mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Ym}^{\bullet}\mathbb{C}^d_L)$, i.e. the space of polynomials in the variables u^l_τ, $1 \le l \le d$, $\mathrm{sum}(\tau) \le l$. As one can see from Definition 4.3.9, and (4.3.14), the relations on the two spaces are connected as follows:

Lemma 4.4.5. *Let $Z \in \mathbb{C}[u^\bullet]$ be a polynomial on $\mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Sym}^{\bullet}_d\mathbb{C}^n)$, and let $M \subset \mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Sym}^{\bullet}_d\mathbb{C}^n)$ be a closed subvariety, such that $Z|M$ vanishes. Then the restricted polynomial $\hat{Z} = Z|\mathcal{N}_{\boldsymbol{\pi}}$, written in terms of the coordinates $\hat{u}_{\cdot|\boldsymbol{\pi}}$, may be obtained from Z as follows:*

- *setting $u^l_{\pi_l}$ to 1, for $l = 1, \ldots, d$,*
- *setting $u^m_{\pi_l}$ to 0, for $1 \le l \le m \le d$,*
- *replacing the remaining variables u^l_τ by $\hat{u}^l_{\tau|\boldsymbol{\pi}}$.*

In addition, $\hat{Z}$ vanishes on $M \cap \mathcal{N}_{\boldsymbol{\pi}}$.

Eventually, using this lemma with $M = \overline{B_L \varepsilon_{\mathrm{ref}} B_R}$ and $M \cap \mathcal{N}_{\boldsymbol{\pi}} = \mathcal{O}_{\boldsymbol{\pi}}$, we will be able to produce the necessary relations in the defining ideal of $\mathcal{O}_{\boldsymbol{\pi}} \subset \mathcal{N}_{\boldsymbol{\pi}}$. As most of the action will take space in $\mathbb{C}[u^\bullet]$, our next task is to set up some convenient notation for this ring.

The ring $\mathbb{C}[u^\bullet]$ carries a right action of the group B_L, and a left action of the group B_R. In particular, it has two multigradings induced from the T_L and T_R actions: the L-multigrading is the vector of multiplicities $(\mathrm{mult}(i, \pi),\ i = 1, \ldots, d)$, while the R-multigrading is the lth basis vector in $\mathbb{Z}^d$. A combination of these gradings will be particularly important for us (cf. Definition 4.3.6):

$$\mathrm{defect}(u^l_\pi) = l - \mathrm{sum}(\pi); \tag{4.4.13}$$

this induces a $\mathbb{Z}^{\ge 0}$-grading on $\mathbb{C}[u^\bullet]$.

Recall that the projection $B_d \to T_d$ is a group homomorphism, whose kernel is the subgroup of unipotent matrices. We denote the corresponding nilpotent Lie algebras of strictly upper-triangular matrices by $\mathfrak{n}_R$ and $\mathfrak{n}_L$ for B_R and B_L, respectively..

The two Lie algebras, $\mathfrak{n}_L$ and $\mathfrak{n}_R$ are generated by the simple root vectors

$$\Delta_L = \{E^L_{l,l+1};\ l = 1, \ldots, d-1\}, \text{ and } \Delta_R = \{E^R_{l,l+1};\ l = 1, \ldots, d-1\},$$

respectively, where $E_{l,l+1}$ is the matrix whose only nonvanishing entry is a 1 in the lth row and $l+1$st column. Let us write down the action of these root vectors on $\mathbb{C}[u^\bullet]$ in the coordinates u^l_τ, $|\tau| \le l \le d$. We first define certain operations on partitions:

- given a positive integer m and a partition $\tau \in \Pi$, denote by $\tau \cup m$ the partition with m added to τ, e.g. $[2,3,4] \cup 3 = [2,3,3,4]$
- if $m \in \tau$, then denote by $\tau - m$ the partition τ with one of the ms deleted, e.g. $[2,4,4,5,5,5,6] - 5 = [2,4,4,5,5,6]$;
- more generally, we will write $[2,4,5,5] \cup [3,4] = [2,3,4,4,5,5]$, and $[2,4,5,5] - [4,5] = [2,5]$.

Returning to the Lie algebra actions, we have

$$\begin{cases} \mathfrak{n}_R u^l_\tau = u^l_\tau \mathfrak{n}_L = 0, \text{ if } \operatorname{sum}(\tau) = l, \\ E^R_{m,m+1} u^l_\tau = \delta_{l,m+1} u^{l-1}_\tau, \ u^l_\tau E^L_{m,m+1} = \operatorname{mult}(m,\tau)\, u^l_{\tau - m \cup m+1}, \text{ if } \operatorname{sum}(\tau) < l. \end{cases} \tag{4.4.14}$$

where $\delta_{a,b}$ is the Kronecker delta. Observe that both $\mathfrak{n}_R$ and $\mathfrak{n}_L$ act compatibly with the $T_R \times T_L$-multigrading, and they both decrease the defect (4.4.13).

The following subspace will play a key role in our calculations:

$$I_{\mathcal{O}} = \left\{ Z \in \mathbb{C}[u^\bullet];\ \mathfrak{n}_R Z = 0 \text{ and } [Z\mathfrak{n}^N_L](\varepsilon_{\text{ref}}) = 0 \text{ for } N = 0,1,2,\dots \right\}, \tag{4.4.15}$$

where $\mathfrak{n}^N_L$ is the subset $\{X_1 \cdot \dots \cdot X_N;\ X_i \in \mathfrak{n}_L,\ i = 1,\dots,N\}$ of the universal enveloping algebra of $\mathfrak{n}_L$.

Proposition 4.4.6. *If $Z \in I_{\mathcal{O}}$, then $Z(\varepsilon) = 0$ for every $\varepsilon \in B_L \varepsilon_{\text{ref}} B_R$.*

Proof. First, observe that the actions of $\mathfrak{n}_R$ and $\mathfrak{n}_L$ described in (4.4.14) are compatible with the multigrading induced by the $T_R \times T_L$-action, and hence, if Z is in $I_{\mathcal{O}}$, then so are all of its $T_R \times T_L$-homogeneous components. This means that without loss of generality we may assume that Z is a homogeneous element of $I_{\mathcal{O}}$.

For such Z, clearly, $Z(\varepsilon) = 0 \Leftrightarrow t_R Z t_L(\varepsilon) = 0$ for any $t_L \in T_L$, $t_R \in T_R$. Combining this with the condition $\mathfrak{n}_R Z = 0$ we can conclude that the zero set of Z is B_R-invariant, hence it is sufficient to show $Z(\varepsilon) = 0$ for $B_L \varepsilon_{\text{ref}}$. Now, since $\ker(B_L \to T_L) = \exp(\mathfrak{n}_L)$, the definition of $I_{\mathcal{O}}$ also implies $Z(b\varepsilon_{\text{ref}}) = 0$ for all $b \in B_L$., and this completes the proof. □

Remark 4.4.7. Before we proceed, we make a comment on the geometric meaning of $I_{\mathcal{O}}$. The space $\{Z \in \mathbb{C}[u^\bullet];\ \mathfrak{n}_R Z = 0\}$ is the homogeneous coordinate ring of $\widetilde{\mathcal{E}}$, corresponding to the line bundles induced by the characters of T_R. Then Proposition 4.4.6 may be interpreted as saying that $I_{\mathcal{O}}$ is contained in the ideal of functions vanishing on $\mathcal{O}$. In fact, is not difficult to show that $I_{\mathcal{O}}$ is exactly this ideal.

We will be looking for polynomials $Z \in I_{\mathcal{O}}$ in a particular subspace of $\mathbb{C}[u^\bullet]$. To describe this space, introduce for each $\boldsymbol{\pi} \in \boldsymbol{\Pi}_d$ the monomial

$$\mathbf{u}^{\boldsymbol{\pi}} = \prod_{l=1}^{d} u^l_{\pi_l};\ \text{these satisfy } \mathbf{u}_{\boldsymbol{\pi}}(\varepsilon_{\boldsymbol{\pi}'}) = \begin{cases} 1, \text{ if } \boldsymbol{\pi} = \boldsymbol{\pi}' \\ 0, \text{ otherwise.} \end{cases} \tag{4.4.16}$$

Now consider the linear span of these monomials:

$$\Lambda = \left\{ \sum_{\boldsymbol{\pi} \in \boldsymbol{\Pi}_d} \alpha_{\boldsymbol{\pi}} \mathbf{u}^{\boldsymbol{\pi}} \in \mathbb{C}[u^\bullet];\ \alpha_{\boldsymbol{\pi}} \in \mathbb{C} \right\}.$$

In order to write down our formulas for certain elements of $\Lambda \cap I_{\mathcal{O}}$, we need to introduce two operations on $\boldsymbol{\Pi}_d$. For a sequence of partitions $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_d)$ and a permutation $\sigma \in \mathcal{S}_d$ define the the permuted sequence

$$\boldsymbol{\pi} \cdot \sigma = (\pi_{\sigma(1)}, \ldots, \pi_{\sigma(d)});$$

this defines a natural right action of $\mathcal{S}_d$ on $\Pi^{\times d}$. Note that permuting an admissible sequence $\boldsymbol{\pi} \in \boldsymbol{\Pi}_d$ does not necessarily result in an admissible sequence.

The second operation modifies just one entry of $\boldsymbol{\pi}$: for $\boldsymbol{\pi} \in \boldsymbol{\Pi}_d$ and $\tau \in \Pi$, define

$$\boldsymbol{\pi} \cup_m \tau = (\pi_1, \ldots, , \pi_{m-1}, \pi_m \cup \tau, \pi_{m+1}, \ldots, \pi_d).$$

Now we are ready to write down our relations.

Proposition 4.4.8. *Let $\boldsymbol{\pi} \in \boldsymbol{\Pi}_d$ be an admissible sequence of partitions and let $\tau \in \Pi$ be any partition. Then following polynomial is an element of $I_{\mathcal{O}}$:*

$$\mathrm{Rel}(\boldsymbol{\pi}, \tau) = \sum \mathrm{sign}(\sigma)\, \mathbf{u}^{\boldsymbol{\pi}\cdot\sigma\cup_m\tau}, 1 \le m \le d,\ \sigma \in \mathcal{S}_d,\ \boldsymbol{\pi} \cdot \sigma \cup_m \tau \in \boldsymbol{\Pi}_d, \qquad (4.4.17)$$

Remark 4.4.9. The sum in (4.4.17) may be empty. This happens when there are no pairs (σ, m) satisfying the conditions in (4.4.17). Note, however, that no two terms of this sum may cancel each other.

Proof. We begin by noting that $\mathrm{Rel}(\boldsymbol{\pi}, \tau)$ is of pure $T_R \times T_L$ weight. Indeed, the torus T_R acts on the whole space Λ with the same weight $(1, 1, \ldots, 1)$, while the lth component of the T_L-weight of a term of $\mathrm{Rel}(\boldsymbol{\pi}, \tau)$ is equal to $\mathrm{mult}(l, \tau) + \sum_{m=1}^{d} \mathrm{mult}(l, \pi_m)$.

Next, we show that

$$E^R_{l,l+1} \mathrm{Rel}(\boldsymbol{\pi}, \tau) = 0,\ l = 1, \ldots, d-1, \qquad (4.4.18)$$

which implies that $\mathfrak{n}_R \mathrm{Rel}(\boldsymbol{\pi}, \tau) = 0$. Let us fix l; the terms of $\mathrm{Rel}(\boldsymbol{\pi}, \tau)$ in (4.4.17) are indexed by pairs (σ, m), and we can ignore those pairs for which $\mathrm{sum}(\pi_{l+1}) + \delta_{m,l+1}\mathrm{sum}(\tau) \ge l + 1$, since in this case $E^R_{l,l+1} \mathbf{u}^{\boldsymbol{\pi}\cdot\sigma\cup_m\tau} = 0$. Then the vanishing (4.4.18) clearly follows if, on the set of the remaining pairs contributing to (4.4.17), we find an involution $(\sigma, m) \mapsto (\sigma', m')$ such that

$$E^R_{l,l+1} \mathbf{u}^{\boldsymbol{\pi}\cdot\sigma\cup_m\tau} = E^R_{l,l+1} \mathbf{u}^{\boldsymbol{\pi}\cdot\sigma'\cup_{m'}\tau} \text{ and } \mathrm{sign}(\sigma') = -\mathrm{sign}(\sigma).$$

Indeed, it is easy to check that this holds for the involution

$$(\sigma', m') = (\sigma \cdot \langle l \leftrightarrow l+1 \rangle, \langle l \leftrightarrow l+1 \rangle(m)),$$

where $\langle l \leftrightarrow l+1\rangle \in \mathcal{S}_d$ is the transposition of l and $l+1$. This proves (4.4.18).

Our second task is to show that $\mathrm{Rel}(\boldsymbol{\pi},\tau)$ is in the linear space

$$I'_{\mathcal{O}} = \left\{ Z \in \mathbb{C}[u^\bullet];\ \left[Z \mathfrak{n}_L^N \right](\varepsilon_{\mathrm{ref}}) = 0 \text{ for } N = 0,1,\dots \right\}.$$

Using the Leibniz rule, it is easy to see see that $I'_{\mathcal{O}} \subset \mathbb{C}[u^\bullet]$ is an ideal.

First we show that for partitions $\rho,\tau \in \Pi$ and $m \geq \mathrm{sum}(\rho)+\mathrm{sum}(\tau)$ the polynomial

$$Z^m_{\rho\tau} = u^m_{\rho\cup\tau} - \sum u^t_\rho u^r_\tau,\ t+r=m,\ t \geq \mathrm{sum}(\rho),\ r \geq \mathrm{sum}(\tau) \tag{4.4.19}$$

is in $I'_{\mathcal{O}}$. Indeed, a quick computation produces the equality

$$Z^m_{\rho\tau} E^L_{l,l+1} = \mathrm{mult}(l,\rho) Z^m_{\rho'\tau} + \mathrm{mult}(l,\tau) Z^m_{\rho\tau'},\ \text{where } \rho' = \rho - l \cup [l+1],\ \tau' = \tau - l \cup [l+1].$$

This equality implies that it is sufficient for us to prove $Z^m_{\rho\tau}(\varepsilon_{\mathrm{ref}}) = 0$ for the case $m = \mathrm{sum}(\rho)+\mathrm{sum}(\tau)$. In this case we have

$$Z^m_{\rho\tau} = u^m_{\rho\cup\tau} - u^{\mathrm{sum}(\rho)}_\rho u^{\mathrm{sum}(\tau)}_\tau, \tag{4.4.20}$$

and this polynomial clearly vanishes on $\varepsilon_{\mathrm{ref}}$, because all three coordinates appearing in this relation are equal to 1 according to (4.3.10).

Now we return to the proof of $\mathrm{Rel}(\boldsymbol{\pi},\tau) \in I'_{\mathcal{O}}$. Using the fact that $Z^m_{\rho\tau}$ is in the ideal $I'_{\mathcal{O}}$, modulo the $I'_{\mathcal{O}}$, we can replace all the factors of the form $u^m_{\pi_{\sigma(m)}\cup\tau}$ in all the terms of $\mathrm{Rel}(\boldsymbol{\pi},\tau)$ by the appropriate sum of quadratic terms in (4.4.19). Our claim is that the resulting polynomial is identically zero, which implies that $\mathrm{Rel}(\boldsymbol{\pi},\tau) \in I'_{\mathcal{O}}$.

Indeed, let us perform this substitution; the terms of the resulting sum are parametrized by a triple (σ,m,r), which is obtained by applying (4.4.19) to the term of $\mathrm{Rel}(\boldsymbol{\pi},\tau)$ indexed by (σ,m) and taking the term corresponding to r in (4.4.19). The correspondence is thus

$$(\sigma,m,r) \longrightarrow u^1_{\pi_{\sigma(1)}} \dots u^{m-1}_{\pi_{\sigma(m-1)}} u^{m-r}_{\pi_{\sigma(m)}} u^r_\tau u^{m+1}_{\pi_{\sigma(m+1)}} \dots u^d_{\pi_{\sigma(d)}}. \tag{4.4.21}$$

Just as above, we can see that the involution $(\sigma,m,r) \mapsto (\sigma\cdot\langle m \leftrightarrow m-r\rangle, m, r)$ provides us with a complete pairing of the terms of the sum described above; each pair consists of identical monomials with opposite signs. This implies that indeed, the result is zero, hence $\mathrm{Rel}(\boldsymbol{\pi},\tau)$ vanishes modulo $I'_{\mathcal{O}}$, i.e. $\mathrm{Rel}(\boldsymbol{\pi},\tau) \in I'_{\mathcal{O}}$. □

Armed with these relations, we are ready to *prove Proposition* 4.4.4. Recall that according to the strategy described at the beginning of this paragraph, given $\boldsymbol{\pi} \in \boldsymbol{\Pi}_d$, l and τ as in Proposition 4.4.4, we need to find a relation of the form $\mathrm{Rel}(\cdot,\cdot)$, which, when restricted to $\mathcal{N}_{\boldsymbol{\pi}}$ expresses the variable $\hat{u}^l_{\tau|\boldsymbol{\pi}}$ in terms of the rest of the variables.

Thus the first thing is to study the conditions under which $\hat{u}^l_{\tau|\boldsymbol{\pi}}$ appears the restriction of a monomial of the form $\mathbf{u}^{\boldsymbol{\pi}'}$. The following statement immediately follows form the prescription (4.4.5).

Lemma 4.4.10. *Given and positive integer $l \leq d$, a partition $\tau \in \Pi \setminus \{\pi_1, \ldots, \pi_d\}$ satisfying* $\mathrm{sum}(\tau) \leq l$, *and $\boldsymbol{\pi} \in \boldsymbol{\Pi}_d$, we have $\mathbf{u}^{\boldsymbol{\pi}'}|\mathcal{N}_{\boldsymbol{\pi}} = \hat{u}^l_{\tau|\boldsymbol{\pi}}$ for some $\boldsymbol{\pi}' \in \boldsymbol{\Pi}_d$ if and only if*

$$\boldsymbol{\pi}' = (\pi_1, \ldots, \pi_{l-1}, \tau, \pi_{l+1}, \ldots, \pi_d).$$

The next step is to find out for which $\boldsymbol{\pi} \in \boldsymbol{\Pi}_d$ the monomial $\mathbf{u}^{\boldsymbol{\pi}}$ appears in one of the relations (4.4.17). Taking into account Remark 4.4.9, the criterion is easy to find:

Definition 4.4.11. We will call an admissible sequence of partitions $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_d)$ *complete* if for every $l \in \{1, \ldots, d\}$ and every nontrivial subpartition $\tau \subset \pi_l$, there is $m \in \{1, \ldots, d\}$ such that $\pi_m = \tau$.

Lemma 4.4.12. *A monomial $\mathbf{u}^{\boldsymbol{\pi}}$ appears in a relation* $\mathrm{Rel}(\boldsymbol{\rho}, \tau)$ *for some $\boldsymbol{\rho} \in \boldsymbol{\Pi}_d$ and $\tau \in \Pi$ if and only if $\boldsymbol{\pi}$ is* not *complete.*

Comparing Lemmas 4.4.10 and 4.4.12 to the conditions of Proposition 4.4.4, and keeping in mind our strategy, we see that the desired divisibility property (4.4.11) is reduced to the following statement: given $1 \leq l < m \leq d$ and $\tau \in \Pi$ satisfying

$$\mathrm{sum}(\tau) \leq m,\ l \in \tau \text{ and } |\tau| > 1,$$

and a sequence $\boldsymbol{\pi}$ of the form (4.4.12) with $\pi_l \neq [l]$, we need to show that the sequence

$$(\pi_1, \ldots, , \pi_l, [l+1], [l+2], \ldots, , [m-1], \tau, [m+1], , \ldots, , [d-1], [d])$$

is admissible but not complete. This immediately follows, however, from the fact that $[l]$ is a proper subpartition of τ, which cannot be equal to any of the other partitions in this sequence. This completes the proof of Proposition 4.4.4 and thus also the proof of Proposition 4.4.1. □

4.4.4 The fixed points of the T_L-action on $\mathcal{O}$

As a small detour, based on the results of the previous paragraph, we obtain a rather powerful criterion for $\boldsymbol{\pi} \in \boldsymbol{\Pi}_d$ *not* to belong to $\boldsymbol{\Pi}_{\mathcal{O}}$, i.e. we will construct a large number of T_L-fixed points which do not lie in $\mathcal{O}$. We note, however, that describing the set $\boldsymbol{\Pi}_{\mathcal{O}}$ remains an interesting open problem. Our starting point is (4.4.16).

Lemma 4.4.13. *If the monomial $\mathbf{u}^{\boldsymbol{\pi}}$ appears with nonzero coefficient in a polynomial from $\Lambda \cap I_{\mathcal{O}}$, then the fixed point $\tilde{\varepsilon}_{\boldsymbol{\pi}} \notin \mathcal{O}$, i.e. $\boldsymbol{\pi} \notin \boldsymbol{\Pi}_{\mathcal{O}}$.*

Proof. Indeed, let Z be such a polynomial. According to Proposition 4.4.6, a polynomial in $I_{\mathcal{O}}$ vanishes at all points of $\mathcal{O}$. On the other hand, it is clear from (4.4.16) that all but exactly one of the terms of Z vanishes at $\varepsilon_{\boldsymbol{\pi}}$, and hence $Z(\varepsilon_{\boldsymbol{\pi}}) \neq 0$. □

Combining this statement with Lemma 4.4.12 we have the following.

Proposition 4.4.14. *If $\boldsymbol{\pi} \in \boldsymbol{\Pi}_{\mathcal{O}}$. i.e. if $\tilde{\varepsilon}_{\boldsymbol{\pi}} \in \mathcal{O}$, then the sequence $\boldsymbol{\pi}$ is complete.*

This Proposition provides us a rather strict necessary , although, as an example below shows, not sufficient condition for $\boldsymbol{\pi}$ to be in $\boldsymbol{\Pi}_{\mathcal{O}}$.

Example 4.4.15.

1. The sequence

$$([1],[2],\ldots,[d-1],[l,m]), \quad \text{where } l+m \le d.$$

   is complete, and, in fact, it corresponds to a fixed point.

2. For $d = 3, 4$, the reverse of Proposition 4.4.14 holds: if $\boldsymbol{\pi}$ is complete then the fixed point $\tilde{\varepsilon}_{\boldsymbol{\pi}}$ lies in the orbit closure $\mathcal{O}_d$, see section §5.1.

3. The completeness of $\boldsymbol{\pi}$ is a necessary but not sufficient condition for $\boldsymbol{\pi}$ to be in $\boldsymbol{\Pi}_{\mathcal{O}}$. An example is the following zero-defect sequence of partitions: let $d = 60$, $\tau = [1, 12, 12, 15, 20]$ and set

$$\pi_l = \begin{cases} \rho, & \text{if } \rho \subset \tau \text{ and } \operatorname{sum}(\rho) = l, \\ [l], & \text{otherwise.} \end{cases}$$

   By definition, this is a complete sequence of partitions, but it can be checked that it is not in $\mathcal{O}$.

4.4.5 The distinguished fixed point and the main result

Now we turn our attention to our much simplified formula (4.4.2) for the Thom polynomial of the A_d-singularity.

The proof of the vanishing of the contributions to (4.3.24), naturally, fails at the fixed point $\tilde{\varepsilon}_{\text{dist}}$. Indeed, for the for the factors (4.4.10) in the case of the distinguished sequence $\boldsymbol{\pi}_{\text{dist}}$, we have $\deg(p_2(\mathbf{z}); l) > \deg(q_2(\mathbf{z}); l)$ for $l = 1, \ldots, d$, and hence we cannot apply Proposition 4.4.3.

The factorization arguments of §4.4.3 may be partially saved, however. Indeed, for the case of the distinguished partition $\boldsymbol{\pi}_{\text{dist}}$, each T_L-weight $z_\tau - z_l$ of $\mathcal{N}_{\text{dist}}$ appears with multiplicity one (cf. end of §4.4.2). Hence, again, we can apply Lemmas 2.2.2, 4.4.10 and 4.4.12 to conclude that for $|\tau| > 1$,

$$(z_\tau - z_l) \,|\, Q_{\text{dist}} \;\text{ if }\; ([1],[2],\ldots,[l-1],\tau,[l+1],\ldots,[d-1],[d]) \text{ is not complete.}$$

Clearly, such a sequence is complete if and only if $|\tau| = 2$, and this means that in the fraction on the right hand side of (4.4.3), we can cancel all factors between the numerator and the denominator corresponding to partitions τ with $|\tau| > 2$. This reduces the denominator to the product of the factors with $|\tau| = 2$:

$$\prod (z_m + z_r - z_l), \; 1 \le m \le r, \; m + r \le l \le d,$$

while Q_{dist} is replaced by a polynomial $\widehat{Q}_d$, whose degree is much smaller than that of Q_{dist}. Note that in this case no factors of the Vandermonde in the numerator are canceled; the fraction in (4.4.3) thus simplifies to

$$\frac{(-1)^d \prod_{m<l}(z_m - z_l)\, \widehat{Q}_d(z_1,\dots,z_d)}{\prod_{l=1}^{d}\prod_{m=1}^{l-1}\prod_{r=1}^{\min(m,l-m)}(z_m+z_r-z_l)}$$

The polynomial $\widehat{Q}_d$, just as Q_{dist}, only depends on d; we mark its d-dependence explicitly.

All that remains to do before we can formulate our final result, is to describe the geometric meaning of this cancellation, and that of the polynomial $\widehat{Q}_d$ itself.

First, note that $\boldsymbol{\pi}_{\mathrm{dist}}$ is of the defect-0 type, hence, according to Proposition 4.3.10 (3) and Lemma 4.3.13, we have an action of the upper-triangular group B_{dist} on $\mathcal{N}_{\mathrm{dist}}$ given by (4.3.16); moreover, $\varepsilon_{\mathrm{ref}} \in \mathcal{N}_{\mathrm{dist}}$ and $\mathcal{O}_{\mathrm{dist}} = \overline{B_{\mathrm{dist}} \cdot \varepsilon_{\mathrm{ref}}}$. Remarkably, this action is also linear (cf. Remark 4.3.12), because the $B_L \times B_R$-action on $\mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Sym}^{\bullet}_d\mathbb{C}^n)$ preserves the length of the partitions, and $\boldsymbol{\pi}_{\mathrm{dist}}$ contains all the partitions of length 1.

Next, define the linear subspace $\widehat{\mathcal{N}}_d \subset \mathcal{N}_{\mathrm{dist}}$:

$$\widehat{\mathcal{N}}_d = \{\varepsilon \in \mathcal{N}_{\mathrm{dist}};\ \hat{u}^m_{\tau|\mathrm{dist}}(\varepsilon) = 0 \text{ for } |\tau| > 2\} \subset \mathrm{Hom}(\mathbb{C}^d, \mathrm{Sym}^2\mathbb{C}^d), \qquad (4.4.22)$$

and let $\widehat{\mathrm{pr}} : \mathcal{N}_{\mathrm{dist}} \to \widehat{\mathcal{N}}_d$ be the natural projection. Then (cf. Remark 2.2.3) we can conclude that

$$\widehat{Q}_d = \mathrm{eP}[\widehat{\mathcal{O}}_d, \widehat{\mathcal{N}}_d] \ \text{ where } \ \widehat{\mathcal{O}}_d = \widehat{\mathrm{pr}}(\widehat{\mathcal{O}}_{\mathrm{dist}}). \qquad (4.4.23)$$

In addition, it is easy to see that $\widehat{\mathcal{N}}_d \subset \mathcal{N}_{\mathrm{dist}}$ is B_{dist}-invariant, and the projection $\widehat{\mathrm{pr}}$ commutes with this of the group of upper-triangular matrices. This implies that

$$\widehat{\mathcal{O}}_d = \overline{B_d \hat{\varepsilon}_{\mathrm{ref}}}, \text{ where } \hat{\varepsilon}_{\mathrm{ref}} = \widehat{\mathrm{pr}}(\varepsilon_{\mathrm{ref}}).$$

Stripping our formulas from extraneous notation, we can formulate our main result in a self-contained manner as follows:

Theorem 4.4.16. *Let $T_d \subset B_d \subset \mathrm{GL}_d$ be the subgroups of invertible diagonal and upper-triangular matrices, respectively; denote the diagonal weights of T_d by $z_1, \dots, z_d$. Consider the GL_d-module of 3-tensors $\mathrm{Hom}(\mathbb{C}^d, \mathrm{Sym}^2\mathbb{C}^d)$; identifying the weight-$(z_m + z_r - z_l)$ symbols q_l^{mr} and q_l^{rm}, we can write a basis for this space as follows:*

$$\mathrm{Hom}(\mathbb{C}^d, \mathrm{Sym}^2\mathbb{C}^d) = \bigoplus \mathbb{C}q_l^{mr},\ 1 \le m, r, l \le d.$$

Consider the reference element

$$\hat{\varepsilon}_{\mathrm{ref}} = \sum_{m=1}^{d}\sum_{r=1}^{d-m} q_{mr}^{m+r},$$

in the B_d-invariant subspace

$$\widehat{\mathcal{N}}_d = \bigoplus_{1\le m+r\le l\le d} \mathbb{C}q_l^{mr} \subset \mathrm{Hom}(\mathbb{C}^d, \mathrm{Sym}^2\mathbb{C}^d). \tag{4.4.24}$$

Set the notation $\widehat{\mathcal{O}}_d$ for the orbit closure $\overline{B_d\hat{\varepsilon}_{\mathrm{ref}}} \subset \widehat{\mathcal{N}}_d$, and consider its T_d-equivariant Poincaré dual

$$\widehat{Q}_d(z_1,\dots,z_d) = \mathrm{eP}[\widehat{\mathcal{O}}_d, \widehat{\mathcal{N}}_d]_{T_d},$$

which is a homogeneous polynomial of degree $\dim(\widehat{\mathcal{N}}_d) - \dim(\widehat{\mathcal{O}}_d)$.

Then for arbitrary integers $n \le k$, the Thom polynomial for the A_d-singularity with n-dimensional source space and k-dimensional target space is given by the following iterated reside formula:

$$\mathrm{eP}[\Theta_d] = \operatorname*{Res}_{\mathbf{z}=\infty} \frac{(-1)^d \prod_{m<l}(z_m - z_l)\, \widehat{Q}_d(z_1,\dots,z_d)}{\prod_{l=1}^{d}\prod_{m=1}^{l-1}\prod_{r=1}^{\min(m,l-m)}(z_m+z_r-z_l)} \prod_{l=1}^{d} \mathrm{RC}\left(\frac{1}{z_l}\right) z_l^{k-n}\, dz_l, \tag{4.4.25}$$

where $\mathrm{RC}(\cdot)$ *is the generating function of the relative Chern classes given in* (2.2.20).

Let us briefly review the proof of this theorem. We began by interpreting the Thom polynomial as an equivariant Poincaré dual of a variety Θ_d in the space of map-jets in Proposition 2.2.11. Next, we constructed a birational model for Θ_d in Proposition 4.2.17, and then we applied a localization formula (3.3.3) to this model, which resulted in expression (4.3.24) for the Thom polynomial. Finally, by studying certain explicit relations and under the assumption that $d \ll n$, we uncovered a cancellation phenomenon, which lead to the simplified formula (4.4.25).

To finish the proof, note that the formulation of Theorem 4.4.16 is more general than to what we seem to be entitled: Proposition 4.4.1 includes the assumption $d \ll n$, while here we claim that our statement holds for any d and $n \le k$. The reason is Proposition 2.2.13, which, in particular, says that if an expression of a Thom polynomial in the relative Chern classes holds for large n, then the same expression works for any n.

It is not difficult to see that formula (4.4.25) manifestly satisfies all properties listed in Proposition 2.2.13. In particular, it only depends on the codimension $k-n$, and reducing the codimension by 1 leads to shifting the indices of the relative Chern classes down by 1. Another benefit of the result is that it shows: the *Thom series* introduced in [17] is governed by a finite object: $\widehat{Q}_d$.

Chapter 5

Calculation of $\widehat{Q}_d$

5.1 Calculation for small values of d. Explicit formulas for Thom polynomials

Theorem 4.4.16 reduces the computation of the Thom polynomials of the algebra A_d to that of the polynomial $\widehat{Q}_d$, which is the equivariant Poincaré dual of a B_d-orbit in a certain B_d-invariant subspace of 3-tensors in d dimensions. Note that the parameters n and k do not enter this picture; in particular, $\widehat{Q}_d$ only depends on d.

Clearly, in principle, the computation of $\widehat{Q}_d$ is a finite problem in commutative algebra, which, for each value of d, can be handled by a computer algebra package such as Macaulay. However, the number of variables and the degree of $\widehat{Q}_d$ grow rather quickly: they are of order d^3. More importantly, computer algebra programs have difficulties dealing with parametrized subvarieties already in very small examples.

At this point, we do not have an efficient method of computation for $\widehat{Q}_d$ in general. The purpose of this section is twofold: to show how to compute $\widehat{Q}_d$ for small degrees: $d = 2, 3, 4, 5, 6$, and we would like to describe the extra information we have uncovered about the orbit $B_d\hat{\varepsilon}_{\mathrm{ref}} \subset \widehat{\mathcal{N}}_d$ in Section 5.2.

5.1.1 The degree of $\widehat{Q}_d$

The degree of the polynomial $\widehat{Q}_d$ is the codimension of the orbit $B_d\varepsilon_{\mathrm{ref}}$, or that of its closure $\widehat{\mathcal{O}}_d$, in $\widehat{\mathcal{N}}_d$.

Recall that $\widehat{\mathcal{N}}_d$ has a basis indexed by the set of indices $\{m + r \leq l \leq d\}$. An elementary computation shows that $\dim \widehat{\mathcal{N}}_d$ is given by a cubic quasi-polynomial in d with leading term $d^3/24$

On the other hand, we have

$$\dim(B_d\hat{\varepsilon}_{\mathrm{ref}}) = \dim(B_d) - \dim(H_d) = \binom{d+1}{2} - d = \binom{d}{2}.$$

Next, denote by $\widehat{\mathcal{N}}_d^0$ the *minimal* or defect-zero part of $\widehat{\mathcal{N}}_d$ spanned by the vectors $\{q^l_{mr};\ m+r=l\le d\}$, and let $\mathrm{pr}_0 : \widehat{\mathcal{N}}_d \to \widehat{\mathcal{N}}_d^0$ be the natural projection; note that $\hat{\varepsilon}_{\mathrm{ref}} \in \widehat{\mathcal{N}}_d^0$. Recall that $B_d = T_d U_d$, where $U_d \subset B_d$ is the subgroup of unipotent matrices, which have 1s on the diagonal. It is easy to check that U_d acts trivially on $\widehat{\mathcal{N}}_d^0$, and its action commutes with the projection pr_0. This motivates the introduction of the toric orbit $T_d\hat{\varepsilon}_{\mathrm{ref}} \subset \widehat{\mathcal{N}}_d^0$ and its closure $\widehat{\mathcal{T}} \subset \widehat{\mathcal{N}}_d^0$.

Lemma 5.1.1. *The projection* pr_0 *restricted to the orbit* $B_d\hat{\varepsilon}_{\mathrm{ref}}$ *establishes a fibration over the toric orbit* $T_d\hat{\varepsilon}_{\mathrm{ref}}$*. This map extends to a map between the closures* $\widehat{\mathcal{O}} \to \widehat{\mathcal{T}}$*, where* $\widehat{\mathcal{T}} = \overline{T_d\hat{\varepsilon}_{\mathrm{ref}}}$*.*

Proof. The first part is obvious. For the second statement, we need to show that $\mathrm{pr}_0(\widehat{\mathcal{O}}_d) = \widehat{\mathcal{T}}_d$ holds. The $\subset$ part follows from the fact that $\mathrm{pr}_0(B_d\hat{\varepsilon}_{\mathrm{ref}}) = T_d\hat{\varepsilon}_{\mathrm{ref}}$. For the $\supset$ part notice that $T_d \subset B_d$, so $\mathcal{T}_d\hat{\varepsilon}_{\mathrm{ref}} \subset B_d\hat{\varepsilon}_{\mathrm{ref}}$. □

One can show that the polynomial $\mathrm{eP}[\widehat{\mathcal{T}}, \widehat{\mathcal{N}}_d^0]$ gives a certain "piece" of $\widehat{Q}_d = \mathrm{eP}[\widehat{\mathcal{O}}, \widehat{\mathcal{N}}_d]$. This is relevant because there are standard algorithms to compute the equivariant Poincaré dual of a toric orbit – we presented some of these in the example of the toric orbit in §2.2.2 – but no such algorithm is known for Borel orbits. In section 5.2 we describe a simple relation between $\widehat{Q}_d^0 = \mathrm{eP}[\widehat{\mathcal{T}}, \widehat{\mathcal{N}}_d^0]$ and $\widehat{Q}_d = \mathrm{eP}[\widehat{\mathcal{O}}, \widehat{\mathcal{N}}_d]$.

Recall, that $\widehat{Q}_d$ is a homogeneous polynomial in d variables, which correspond to a basis of the d-dimensional weight lattice. After a proper linear coordinate change we will distinguish the first coordinate, and view $\mathrm{eP}[\widehat{\mathcal{O}}_d, \widehat{\mathcal{N}}_d]$ as a degree $\alpha(d)$ polynomial in this coordinate, whose coefficients are homogeneous polynomials in the remaining $d-1$ variables. It turns out that the leading coefficient, i.e the coefficient of $w_1^{\alpha(d)}$ is $\mathrm{eP}[\widehat{\mathcal{T}}_d, \widehat{\mathcal{N}}_d^0]$, and this is the main result of 5.2.

Lemma 5.1.1 implies, in particular, that the codimension of $B_d\hat{\varepsilon}_{\mathrm{ref}}$ is the sum of the codimensions of $\widehat{\mathcal{T}}$ in $\widehat{\mathcal{N}}_d^0$ and the codimension in the fiberwise directions. We collect the appropriate numeric values in the following table:

d	$\dim\widehat{\mathcal{O}} = \binom{d}{2}$	$\dim\widehat{\mathcal{N}}_d$	$\deg\widehat{Q}_d = \mathrm{codim}(\widehat{\mathcal{O}})$	$\dim(\widehat{\mathcal{T}}) = d-1$	$\dim\widehat{\mathcal{N}}_d^0$	$\mathrm{codim}(\widehat{\mathcal{T}})$
1	0	0	0	0	0	0
2	1	1	0	1	1	0
3	3	3	0	2	2	0
4	6	7	1	3	4	1
5	10	13	3	4	6	2
6	15	22	7	5	9	4

The first 3 columns list the codimension of the closure of the Borel orbit $\widehat{\mathcal{O}}$ in $\widehat{\mathcal{N}}_d$, while the last three - the codimension of the closure of the toric orbit $\widehat{\mathcal{T}}$ in $\widehat{\mathcal{N}}_d^0$.

Now we are ready for the computations.

5.1.2 The cases d=1,2,3

In these cases $\deg \widehat{Q}_d = 0$ and thus $\widehat{Q}_d = 1$; geometrically, this means that $\mathcal{O}_d = \widetilde{\mathcal{E}}_d$, and thus $\widehat{\mathcal{O}}_d = \widehat{\mathcal{N}}_d$. The case of $d = 1$ was described in §3.2.

For $d = 2$ we obtain

$$\mathrm{eP}[\Theta_2] = \operatorname*{Res}_{z_1=\infty} \operatorname*{Res}_{z_2=\infty} \frac{z_1 - z_2}{2z_1 - z_2} \mathrm{RC}\left(\frac{1}{z_1}\right) \mathrm{RC}\left(\frac{1}{z_2}\right) z_1^{k-n} z_2^{k-n} \, dz_1 dz_2. \tag{5.1.1}$$

Expanding the iterated residue, one immediately recovers Ronga's formula [42]:

$$\mathrm{eP}[\Theta_2] = c_{k-n+1}^2 + \sum_{i=1}^{k-n+1} 2^{i-1} c_{k-n+1-i} c_{k-n+1+i}. \tag{5.1.2}$$

For $d = 3$, the formula is

$$\mathrm{eP}[\Theta_3] = (-1) \operatorname*{Res}_{z_1=\infty} \operatorname*{Res}_{z_2=\infty} \operatorname*{Res}_{z_3=\infty} \frac{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)}{(2z_1 - z_2)(z_1 + z_2 - z_3)(2z_1 - z_3)}$$
$$\mathrm{RC}\left(\frac{1}{z_1}\right) \mathrm{RC}\left(\frac{1}{z_2}\right) \mathrm{RC}\left(\frac{1}{z_3}\right) z_1^{k-n} z_2^{k-n} z_3^{k-n} \, dz_1 dz_2 dz_3. \tag{5.1.3}$$

This is a much more compact and conceptual formula for $\mathrm{eP}[\Theta_3]$ than the one given in [3].

5.1.3 The basic equations in general

As our table in §5.1.1 shows, the polynomial $\widehat{Q}_d$ is not trivial when $d > 3$. As a step towards its computation, we describe a set of equations satisfied by $\widehat{\mathcal{O}} \subset \widehat{\mathcal{N}}_d$ and $\widehat{\mathcal{T}} \subset \widehat{\mathcal{N}}_d^0$. We will call these equations "basic".

The equations will be written in terms of the local variables $\hat{u}_\tau^l$ introduced in §4.4.1, where now we assume that $|\tau| = 2$. Clearly, these variables form a dual basis to the basis $\{q_{mr}^l\}$ of $\widehat{\mathcal{N}}_d$. We will streamline our notation by writing $\hat{u}_{mr}^l$ instead of $\hat{u}_{[m,r]}^l$; naturally, we have $\hat{u}_{mr}^l = \hat{u}_{rm}^l$, and $r + m \le l$.

The construction is as follows. If $i + j + m \le l$, then the sequence

$$\boldsymbol{\pi}(i, j, m; l) = ([1], [2], \dots, [l-1], [i, j, m], [l+1], \dots, [d-1], [d])$$

is admissible but not complete, hence $\mathbf{u}^{\boldsymbol{\pi}(i,j,m)}$ will appear as a term of some of the relations $\mathrm{Rel}(\boldsymbol{\rho}, \tau)$ introduced in Proposition 4.4.8. In fact, it appears in three different relations:

$$\text{for } \tau = [i],\ \rho_l = [j, m], \text{ for } \tau = [j],\ \rho_l = [i, m], \text{ and for } \tau = [m],\ \rho_l = [i, j];$$

in all cases $\rho_r = [r]$ for $r \ne l$. Next, we reduce the relation $\mathrm{Rel}(\boldsymbol{\rho}, \tau)$ according to the prescription of Lemma 5.2.25. After the reduction, only the terms corresponding to the identity permutation and those corresponding to the transpositions of the form (s, l) survive; for example, in the case $\tau = [m]$, we obtain the "localized" relation

$$\hat{u}^l_{ijm} = \sum_{s=j+m}^{l-i} \hat{u}^s_{jm} \hat{u}^l_{is}. \tag{5.1.4}$$

Note that the number of terms on the right hand side is $l - (i + j + m) + 1$, which is the defect of $\hat{u}^l_{ijm}$ plus 1.

We obtain two other expressions for $\hat{u}^l_{ijm}$ when we choose τ to be $[j]$ or $[k]$, and the resulting equalities provide us with quadratic relations among our variables $\hat{u}^l_{mr}$, $m + r \leq l \leq d$.

Proposition 5.1.2. *Let $(i, j, m; l)$ be a quadruple of nonnegative integers satisfying $i + j + m \leq l \leq d$. Then the ideal of the variety $\widehat{\mathcal{O}} \subset \widehat{\mathcal{N}}_d$ contains the relations*

$$R(i, j, m; l): \quad \sum_{s=j+m}^{l-i} \hat{u}^s_{jm} \hat{u}^l_{is} = \sum_{s=i+m}^{l-j} \hat{u}^s_{im} \hat{u}^l_{js} = \sum_{s=i+j}^{l-m} \hat{u}^s_{ij} \hat{u}^l_{ms}. \tag{5.1.5}$$

Remark 5.1.3.

- In general, the quadruple $(i, j, m; l)$ gives us 2 relations. If $i = j \neq m$, then the number of relations reduces to 1, and if $i = j = m$, then (5.1.5) is vacuous.
- The equalities $R(i, j, m; l)$ with $i + j + m = l$ are relations of the toric orbit closure $\widehat{\mathcal{T}} \subset \widehat{\mathcal{N}}^0_d$. We will call these equations *toric.*

5.1.4 Basic equations and commuting matrices

The basic equations (5.1.5) have a particularly simple, compact form introducing the following d matrices. For $i = 1, 2, \dots, d$ let

$$(M_k)_{i,j} = \begin{cases} \hat{u}^j_{ik}, \text{ if } i + k \leq j \leq d, \\ 0 \text{ otherwise.} \end{cases} \tag{5.1.6}$$

Note that $M_d = 0$.

Example 5.1.4. $d = 4$. The matrices are

$$M_1 = \begin{pmatrix} 0 & \hat{u}^2_{11} & \hat{u}^3_{11} & \hat{u}^4_{11} \\ 0 & 0 & \hat{u}^3_{12} & \hat{u}^4_{12} \\ 0 & 0 & 0 & \hat{u}^4_{13} \\ 0 & 0 & 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0 & \hat{u}^3_{12} & \hat{u}^4_{12} \\ 0 & 0 & 0 & \hat{u}^4_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 0 & 0 & 0 & \hat{u}^4_{13} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, M_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Remark 5.1.5. • All these matrices are upper-triangular, moreover, the main diagonal and the neighbouring $i-1$ diagonals of M_i is 0.

• Since $\hat{u}^j_{ik} = \hat{u}^j_{ki}$, for $i \neq k$ the ith row of M_k is equal to the kth row of M_i.

One can easily check the following

Observation 5.1.6. The basic equations can be written as $M_iM_j = M_jM_i$ for any $1 \leq i \leq j \leq d$.

So $M_1, \ldots, M_d$ are commuting matrices. The non-basic equations, however do not have such a simple description.

5.1.5 d=4,5,6

This is the first nontrivial case: here $\deg \widehat{Q}_4 = 1$, i.e. $\widehat{\mathcal{O}}_4 = \overline{B_4\hat{\varepsilon}_{\mathrm{ref}}}$ is a hypersurface in $\widehat{\mathcal{N}}_4$. Checking the table at the end of § 5.1.1, we see that in this case the codimension of the toric piece $\widehat{\mathcal{T}}_4$ in $\widehat{\mathcal{N}}^0_4$ is the same as the codimension of $\widehat{\mathcal{O}}_4$ in $\widehat{\mathcal{N}}_4$. This means that $\widehat{Q}_4 = \mathrm{eP}[\widehat{\mathcal{T}}_4, \widehat{\mathcal{N}}^0_4]$.

It is not surprising then to find that the only basic equation is a toric one, corresponding to the quadruple $(1,1,2,4)$:

$$R(1,1,2;4): \quad \hat{u}^2_{11}\hat{u}^4_{22} = \hat{u}^3_{12}\hat{u}^4_{13}. \tag{5.1.7}$$

We note that this toric hypersurface is essentially our basic example introduced in §2.2.2. The variety defined by (5.1.7) in $\widehat{\mathcal{N}}_4$ is irreducible, and has the same dimension as $\widehat{\mathcal{O}}_4$, therefore it coincides with $\widehat{\mathcal{O}}_4$. We have already determined the equivariant Poincaré dual in this case in a number of ways: it is the sum of the weights of any of the monomials in the equation. Applying this here, we arrive at the formula

$$\widehat{Q}_4(z_1,z_2,z_3,z_4) = (2z_1 - z_2) + (2z_2 - z_4) = 2z_1 + z_2 - z_4. \tag{5.1.8}$$

As a result we obtain

$$\mathrm{eP}[\Theta_4] = \operatorname*{Res}_{z_1=\infty} \operatorname*{Res}_{z_2=\infty} \operatorname*{Res}_{z_3=\infty} \operatorname*{Res}_{z_4=\infty} \prod_{l=1}^{4} \mathrm{RC}\left(\frac{1}{z_l}\right) z_l^{k-n}\, dz_l$$
$$\frac{(z_1-z_2)(z_1-z_3)(z_1-z_4)(z_2-z_3)(z_2-z_4)(z_3-z_4)(2z_1+z_2-z_4)}{(2z_1-z_2)(z_1+z_2-z_3)(2z_1-z_3)(z_1+z_3-z_4)(2z_2-z_4)(z_1+z_2-z_4)(2z_1-z_4)}.$$

d=5: Again, we consult our table. We have $\dim \widehat{\mathcal{N}}_5 = 13$ and $\operatorname{codim} \widehat{\mathcal{O}}_5 = 3$, while $\dim \widehat{\mathcal{N}}^0_5 = 6$ and $\operatorname{codim} \widehat{\mathcal{T}}_5 = 2$.

Let us list our variables.

$$\begin{array}{rc} 6 \text{ toric}: & \hat{u}^5_{14}, \hat{u}^5_{23}, \hat{u}^4_{13}, \hat{u}^4_{22}, \hat{u}^3_{12}, \hat{u}^2_{11} \\ 4 \text{ defect-1}: & \hat{u}^5_{13}, \hat{u}^5_{22}, \hat{u}^4_{12}, \hat{u}^3_{11}, \\ 2 \text{ defect-2}: & \hat{u}^5_{12}, \hat{u}^4_{11}, \text{and} \\ 1 \text{ defect-3}: & \hat{u}^5_{11}. \end{array}$$

There are 3 toric equations, which necessarily involve the toric variables only:

$$\begin{array}{rl} R(1,1,2;4): & \hat{u}^3_{12}\hat{u}^4_{13} = \hat{u}^2_{11}\hat{u}^4_{22} \\ R(1,1,3;5): & \hat{u}^5_{14}\hat{u}^4_{13} = \hat{u}^5_{23}\hat{u}^2_{11} \\ R(1,2,2;5): & \hat{u}^5_{14}\hat{u}^4_{22} = \hat{u}^5_{23}\hat{u}^3_{12} \end{array} \tag{5.1.9}$$

and one defect-1 equation:

$$R(1,1,2;5): \quad \hat{u}^5_{13}\hat{u}^3_{12} + \hat{u}^5_{14}\hat{u}^4_{12} = \hat{u}^2_{11}\hat{u}^5_{22} + \hat{u}^5_{23}\hat{u}^3_{11} \tag{5.1.10}$$

We observe that the toric equations (5.1.9) describe the vanishing of the 3 maximal minors of a 2×3 matrix. This is an irreducible toric variety, thus we can again argue that it coincides with $\widehat{\mathcal{T}}_5$. Fortunately, this variety is a special case of the A_1-singularity, this time with $n = 2$ and $k = 3$. Substituting the appropriate weights into (3.2.2), we obtain:

$$\begin{aligned} &\mathrm{eP}[\widehat{\mathcal{T}}_5, \widehat{\mathcal{N}}^0_d] = \\ &= \frac{(z_1+z_2-z_3)(2z_1-z_2)(z_1+z_4-z_5)-(2z_2-z_4)(z_1+z_3-z_4)(z_2+z_3-z_5)}{z_1+z_4-z_2-z_3} = \\ &= 2z_1^2+3z_1z_2-2z_1z_5+2z_2z_3-z_2z_4-z_2z_5-z_3z_4+z_4z_5. \end{aligned} \tag{5.1.11}$$

Let M_5 denote the variety determined by the basic equations. Notice, that for fixed

$$\hat{u}^2_{11}, \hat{u}^3_{12}, \hat{u}^5_{14}, \hat{u}^5_{23}$$

(5.1.10) is linear in the remaining variables, so the map $\mathrm{pr}_0 : M_d \to \widehat{\mathcal{T}}_5$ is the projection of a vector bundle onto its base; the fibers of this vector bundle are hyperplanes in the 7-dimensional complement of $\widehat{\mathcal{N}}^0_5$ in $\widehat{\mathcal{N}}_5$. Since $\widehat{\mathcal{T}}_5$ is irreducible, so is M_5, and therefore $M_5 = \widehat{\mathcal{O}}_5$.

Using the computer algebra program Macaulay, one can check that $\mathrm{eP}[\widehat{\mathcal{O}}_5, \widehat{\mathcal{N}}_5]$ is the product of $\mathrm{eP}[\widehat{\mathcal{T}}_5, \widehat{\mathcal{N}}^0_5]$ and the weight of the relation $R(1,1,2;5)$. The latter equals $2z_1 + z_2 - z_5$, hence the final result is

$$\widehat{Q}_5(z_1,z_2,z_3,z_4,z_5) = (2z_1+z_2-z_5)(2z_1^2+3z_1z_2-2z_1z_5+2z_2z_3-z_2z_4-z_2z_5-z_3z_4+z_4z_5).$$

d=6

Now $\widehat{Q}_6$ is a degree-7 polynomial in 6 variables, and one needs the help of a computer algebra program to do the calculations. Here we summarize our computations with Macaulay.

Let M_6 denote, again, the variety defined by the basic equations. It turns out, that the codimension of M_6 in $\widehat{\mathcal{N}}_6$ is equal to the codimension of $\widehat{\mathcal{O}}_6$, however, M_6 contains two maximal dimensional components, namely,

$$M_6^1 = \langle \hat{u}_{11}^2, \hat{u}_{12}^3, \hat{u}_{11}^3, \hat{u}_{14}^5, \hat{u}_{14}^6, \hat{u}_{15}^6, \hat{u}_{24}^6 \rangle$$

and

$$M_6^2 = \langle \text{basic equations}, R \rangle,$$

where the extra relation (which is not basic equation) is

$$R = \hat{u}_{12}^4\hat{u}_{12}^4\hat{u}_{23}^5\hat{u}_{33}^6 + \hat{u}_{22}^4\hat{u}_{13}^4\hat{u}_{12}^5\hat{u}_{33}^6 + \hat{u}_{13}^4\hat{u}_{13}^4\hat{u}_{22}^5\hat{u}_{23}^6 + \hat{u}_{22}^4\hat{u}_{13}^4\hat{u}_{23}^5\hat{u}_{13}^6$$
$$- \hat{u}_{22}^4\hat{u}_{11}^4\hat{u}_{23}^5\hat{u}_{33}^6 - \hat{u}_{13}^4\hat{u}_{12}^4\hat{u}_{22}^5\hat{u}_{33}^6 - \hat{u}_{22}^4\hat{u}_{13}^4\hat{u}_{13}^5\hat{u}_{23}^6 - \hat{u}_{13}^4\hat{u}_{13}^4\hat{u}_{23}^5\hat{u}_{22}^6 = 0$$

The weight of R is $2z_1 + 3z_2 + 3z_3 - 2z_4 - z_5 - z_6$. Since $\widehat{\mathcal{O}}_6$ is irreducible, we have $\widehat{\mathcal{O}}_6 = M_6^2$. The other component, M_6^1, is a linear subspace, and we obtain $\widehat{Q}_6$ as

$$\widehat{Q}_6 = \mathrm{eP}[M_6] - \mathrm{eP}[M_6^1].$$

Using Macaulay, it turns out that $\mathrm{Spec}(\mathcal{IN}_6)$ has 59 components, some of them with multiplicity bigger than 1, and the final form of $\widehat{Q}_6$ is too long to present.

5.2 Computing $\widehat{Q}_d$ in general

In this section we describe a simple relation between $\mathrm{eP}[\widehat{\mathcal{T}}_d, \widehat{\mathcal{N}}_d^0]$ and $\mathrm{eP}[\widehat{\mathcal{O}}_d, \widehat{\mathcal{N}}_d]$. Recall, that $\mathrm{eP}[\widehat{\mathcal{T}}_d, \widehat{\mathcal{N}}_d^0]$ is the multidegree of a toric variety; computation of this is a well-known and investigated problem in commutative algebraic geometry. The multidegree can be read off the polytope of the toric variety several different ways. (see [37] for details) However, our main formula (4.4.25) contains $\mathrm{eP}[\widehat{\mathcal{O}}, \widehat{\mathcal{N}}_d]$, which is the multidegree of a Borel orbit. Multidegrees of Borel orbits is a harder task, and almost nothing is known in general.

Since $\widehat{\mathcal{T}}_d = \overline{T_d \hat{\varepsilon}_{\text{ref}}}$ and $\widehat{\mathcal{O}}_d = \overline{B_d \hat{\varepsilon}_{\text{ref}}}$, we may think of $\widehat{\mathcal{T}}_d$ as the toric 'head' of $\widehat{\mathcal{O}}_d$: a toric subvariety of $\widehat{\mathcal{O}}_d$.

$\widehat{Q}_d$ is a homogeneous polynomial in d variables, which correspond to a basis of the d-dimensional weight lattice. After a proper linear coordinate change

$$(z_1, \dots, z_d) \rightsquigarrow (w_1, \dots, w_d)$$

w_1 will play a distinguished role, and $\mathrm{eP}[\widehat{\mathcal{T}}_d, \widehat{\mathcal{N}}_d^0]$ is a homogeneous polynomial in $w_2, \dots, w_d$. We view $\mathrm{eP}[\widehat{\mathcal{O}}_d, \widehat{\mathcal{N}}_d]$ as a degree $\alpha(d)$ polynomial in this coordinate, whose coefficients are homogeneous polynomials in the remaining $d-1$ coordinates. It turns out that the leading coefficient, i.e the coefficient of $w_1^{\alpha(d)}$ is $\mathrm{eP}[\widehat{\mathcal{T}}_d, \widehat{\mathcal{N}}_d^0]$, and this is the main result of this section.

5.2.1 The degree of $\widehat{Q}_d$ in general

We use the same terms and notations as in the previous section. Recall the structure of the Borel orbit $B_d\hat{\varepsilon}_{\mathrm{ref}}$ and its closure, $\widehat{\mathcal{O}}_d = \overline{B_d\hat{\varepsilon}_{\mathrm{ref}}}$ in $\widehat{\mathcal{N}}_d$. $\widehat{\mathcal{N}}_d$ has a basis q^l_{mr}, indexed by the set of indices $\{m+r \leq l \leq d\}$.

We denoted by $\widehat{\mathcal{N}}^0_d$ the *minimal* or defect-zero part of $\widehat{\mathcal{N}}_d$ spanned by the vectors $\{q^l_{mr};\ m+r = l \leq d\}$, and $\mathrm{pr}_0 : \widehat{\mathcal{N}}_d \to \widehat{\mathcal{N}}^0_d$ is the natural projection; note that $\hat{\varepsilon}_{\mathrm{ref}} \in \widehat{\mathcal{N}}^0_d$. Consequently, we denote by $\widehat{\mathcal{N}}^{>}_d$ the positive-defect part of $\widehat{\mathcal{N}}_d$ spanned by the vectors $\{q^l_{mr};\ m+r < l \leq d\}$. Recall that $B_d = T_dU_d$, where $U_d \subset B_d$ is the subgroup of unipotent matrices, and $\widehat{\mathcal{T}}_d = \overline{T_d\hat{\varepsilon}_{\mathrm{ref}}} \subset \widehat{\mathcal{N}}^0_d$. The projection pr_0 restricted to the orbit $B_d\hat{\varepsilon}_{\mathrm{ref}}$ establishes a fibration over the toric orbit $T_d\hat{\varepsilon}_{\mathrm{ref}}$. This map extends to a map between the closures $\widehat{\mathcal{O}}_d \to \widehat{\mathcal{T}}_d$.

Lemma 5.1.1 implies, in particular, that the codimension of $B_d\hat{\varepsilon}_{\mathrm{ref}}$ is the sum of the codimensions of $\widehat{\mathcal{T}}$ in $\widehat{\mathcal{N}}^0_d$ and the codimension in the fiberwise directions. We use the shortened notation $\widehat{Q}^0_d = \mathrm{eP}[\widehat{\mathcal{T}}_d, \widehat{\mathcal{N}}^0_d]$ and recall that $\widehat{Q}_d = \mathrm{eP}[\widehat{\mathcal{O}}_d, \widehat{\mathcal{N}}_d]$. We colleccted the appropriate numeric values in the table in section 5.1.1

Let us compute the degree of the homogeneous polynomial $\widehat{Q}_d$: this is the codimension of the orbit $B_d\varepsilon_{\mathrm{ref}}$, or that of its closure $\widehat{\mathcal{O}}_d$, in $\widehat{\mathcal{N}}_d$.

Let t_l denote the number of defect-zero vectors q^l_{mr} with $m+r = l \leq d$, i.e $t_{2k} = t_{2k+1} = k$.

Lemma 5.2.1. *1.* $\dim \widehat{\mathcal{N}}^0_d = t_2 + t_3 + \ldots + t_d$

2. $\dim \widehat{\mathcal{N}}_d = t_d + 2t_{d-1} + 3t_{d-2} + \ldots + (d-1)t_2$

3. $\dim(\widehat{\mathcal{T}}_d) = d-1$

4. $\dim(\widehat{\mathcal{O}}_d) = \dim(B_d) - \dim(H_d) = \binom{d+1}{2} - d = \binom{d}{2}$.

Proof. The first part is obvious. The basis of $\widehat{\mathcal{N}}_d$ described above is the union of the basis of $\widehat{\mathcal{N}}^0_d, \widehat{\mathcal{N}}^0_{d-1}, \ldots, \widehat{\mathcal{N}}^0_2$. This implies the second equation. The last two are straightforward. $\square$

Corollary 5.2.2. *1.* $\dim \widehat{\mathcal{N}}_d = \begin{cases} \frac{(4k-1)k(k+1)}{6} & \text{if } d = 2k \\ \frac{(4k+5)k(k+1)}{6} & \text{if } d = 2k+1 \end{cases}$

2. $\deg \widehat{Q}_d = \dim Y_2 - \binom{d}{2} = \begin{cases} \frac{4k^3-9k^2+5k}{6} & \text{if } d = 2k \\ \frac{4k^3-3k^2-k}{6} & \text{if } d = 2k+1. \end{cases}$

3. $\deg \widehat{Q}^0_d = \dim \widehat{\mathcal{N}}^0_d - (d-1) = \begin{cases} (k-1)^2 & \text{if } d = 2k \\ k(k-1) & \text{if } d = 2k+1 \end{cases}$

As a consequence we have the following relations

Proposition 5.2.3. *1.*

$$\deg \widehat{Q}_d - \deg \widehat{Q}_d^0 = \deg \widehat{Q}_{d-1} \qquad (5.2.1)$$

2.

$$\deg \widehat{Q}_d - \deg \widehat{Q}_d^0 = \dim \widehat{\mathcal{N}}_{d-3} = \dim \widehat{\mathcal{N}}_{d-2}^{>} \qquad (5.2.2)$$

Now, (5.2.1) says that the codimension of the generic fiber in Lemma 5.1.1 is equal to the codimension of $\widehat{\mathcal{O}}_{d-1}$ in $\widehat{\mathcal{N}}_{d-1}$. This can give the false impression that $\widehat{Q}_d = \widehat{Q}_d^0 \cdot \widehat{Q}_{d-1}$, but it is true only for $d \leq 5$.

The RHS of 5.2.2 is the number of vectors q_{mr}^l with $m + r < l \leq d - 2$. So this is the number of vectors with positive defect on level $l \leq d - 2$.

As before, $\hat{u}_{mr}^l$ denotes the dual coordinate of q_{mr}^l.

Notation 5.2.4. We have $\widehat{\mathcal{O}}_d = \mathrm{Spec}(\mathbb{C}[\hat{u}_{mr}^l; m + r \leq l \leq d]/\mathcal{I}_d)$ for some ideal $\mathcal{I}_d \lhd \mathbb{C}[\hat{u}_{mr}^l; m+r \leq l \leq d]$, and $\widehat{\mathcal{T}}_d = \mathrm{Spec}(\mathbb{C}[\hat{u}_{mr}^l; m+r = l \leq d]/\mathcal{J}_d)$ for some ideal $\mathcal{J}_d \lhd \mathbb{C}[\hat{u}_{mr}^l; m+r = l \leq d]$. Note that $\mathcal{J}_d = \mathcal{I}_d \cap \mathbb{C}[\hat{u}_{mr}^l; m+r = l \leq d]$.

5.2.2 Initial ideals, Irredundant Irreducible Decomposition

This subsection is a concise summary of the material we need in commutative algebra related to initial ideals and irredundant irreducible decomposition of monomial ideals. For monomial orders and initial ideals we refer [14] Chapter 15, for the irredundant irreducible decomposition [37], Chapter 3 and 5.

Definition 5.2.5. A monomial order on the polynomial ring $\mathbb{C}[x_1, \ldots, x_s]$ is a total order $\succ$ on the monomials, such that if M_1, M_2 and $1 \neq N$ are monomials then $M_1 > M_2$ implies $NM_1 > NM_2 > M_2$.

The most common example for monomial order is the lexicographic order defined as follows.

Definition 5.2.6. Let $>$ be an order of the variables $x_1, \ldots, x_s$, $x_1 > x_2 > \ldots > x_s$. The generated lexicographic monomial order on $\mathbb{C}[x_1, \ldots, x_s]$ is defined as follows: $x_1^{a_1} \ldots x_s^{a_s} > x_1^{b_1} \ldots x_s^{b_s}$ iff $a_i > b_i$ for the first index with $a_i \neq b_i$.

Proposition 5.2.7. *([14], $15.2) The lexicographic order of monomials associated to any order of the variables is a monomial order.*

Let $\succ$ be a monomial order on $\mathbb{C}[x_1, \ldots, x_s]$, and $I \lhd \mathbb{C}[x_1, \ldots, x_s]$ an ideal. For any $p \in I$ denote $in_\succ(p)$ the greatest term of p with respect ot the order $\succ$.

Definition 5.2.8. The ideal $in_\succ(I)$ of $\mathbb{C}[x_1, \ldots, x_s]$ generated by the initial terms $in_\succ(p)$ for all $p \in I$ is called the initial monomial ideal of I w.r.t $\succ$.

The scheme $A = \mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_s]/I)$ has a flat deformation, whose zero-fiber is $\mathrm{In}(A) = \mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_s]/in_{\succ}(I))$. By the axioms of section 2.2.1, $\mathrm{In}(A)$ has the same multidegree as A.

We call a monomial ideal $P \lhd \mathbb{C}[x_1, \ldots, x_s]$ *irreducible*, if it is generated by powers of the coordinates, i.e has the form $P = \mathbf{x}^{\mathbf{a}} = \langle x_i^{a_i} | a_i \geq 1 \rangle$. Any irreducible monomial ideal corresponds to a vector $\mathbf{a} \in \mathbb{Z}^s_{\geq 0}$ in this way.

Proposition 5.2.9. *([37], Theorem 5.27) Every monomial ideal $M \lhd \mathbb{C}[x_1, \ldots, x_s]$ can be decomposed into the intersection of irreducible ideals,*

$$M = \bigcap_{\mathbf{a} \in A} \mathbf{x}^{\mathbf{a}}.$$

Moreover, there is a unique decomposition with smallest number of components; this is called the irredundant irreducible decomposition.

The ideals of the irredundant irreducible decomposition are primary ideals, and intersecting the irreducible ideals with the same radical prime ideal give a primary decomposition of M. More on primary decompositions see [14], Chapter 15.

Example 5.2.10. The irredundant irreducible decomposition of the monomial ideal $\langle xy, y^2 \rangle \lhd \mathbb{C}[x, y]$ is

$$\langle xy, y^2 \rangle = \langle y \rangle \cap \langle x, y^2 \rangle.$$

The terms correspond to the vectors $(0, 1), (1, 2) \in \mathbb{Z}^2_{\geq 0}$.

The degree of $\mathrm{mdeg}[M, \mathbb{C}[\mathbf{x}]]$ is equal to the codimension of the maximal dimensional components of $\mathrm{Spec}(M)$. In this example the maximal dimensional component is $\{y = 0\}$, and in general:

Proposition 5.2.11. *Let $M = \bigcap_{\mathbf{a} \in A} \mathbf{x}^{\mathbf{a}}$ be the irredundant irreducible decomposition of the monomial ideal M. Then for any $\mathbf{a} \in A$:*

$$\sharp \{i : a_i \geq 1\} \geq \deg(\mathrm{mdeg}[M, \mathbb{C}[x_1, \ldots, x_s]]).$$

We use the following two observation later on to prove the main result of this chapter:

Proposition 5.2.12. *Let $M \lhd \mathbb{C}[x_1, \ldots, x_s]$ be a monomial ideal with irredundant irreducible decompoisiton*

$$M = \bigcap_{\mathbf{a} \in A} \langle x_1^{a_1}, \ldots, x_s^{a_s} \rangle$$

Define the ideal $\hat{M} \lhd \mathbb{C}[x_1, \ldots, x_s, y_1, \ldots, y_t]$ by

$$\hat{M} = \bigcap_{\mathbf{a} \in A} \langle x_1^{a_1}, \ldots, x_s^{a_s}, y_1, \ldots, y_t \rangle$$

Suppose we have a diagonal torus action on $\mathbb{C}^{s+t}$ with weights $\omega_1, \ldots, \omega_s, \zeta_1, \ldots, \zeta_t$. Then

$$\mathrm{mdeg}[\hat{M}, \mathbb{C}[x_1, \ldots, x_s, y_1, \ldots, y_t]] = \zeta_1 \zeta_2 \cdots \zeta_t \cdot \mathrm{mdeg}[M, \mathbb{C}[x_1, \ldots, x_s]].$$

Proof. Straightforward from Lemma 2.2.2. □

The following proposition follows from the uniqueness of the irredundant irreducible decomposition for monomial ideals. For two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^s_{\geq 0}$ we write $\mathbf{a} < \mathbf{b}$ if $a_i \leq b_i$ for all i and $\mathbf{a} \neq \mathbf{b}$.

Proposition 5.2.13. *Let $M \subset \mathbb{C}[x_1, \ldots, x_s]$ be a monomial ideal, and suppose* $\mathrm{Spec}(M)$ *(i.e its maximal dimensional components) has codimension c. Let $\mathbf{a} \in \mathbb{Z}^s_{\geq 0}$ be a vector such that $\sharp\{i : a_i \geq 1\} = c$. Suppose that any $p \in M$ is divisible by one of $\{x_i^{a_i} | a_i \geq 1\}$, but this is not true for $\mathbf{b} \in \mathbb{Z}^s_{\geq 0}$, $\mathbf{b} > \mathbf{a}$, $\sharp\{i : b_i \geq 1\} = c$. Then $\langle x_i^{a_i} | a_i \geq 1 \rangle$ is a maximal irreducible component in the irredundant irreducible decomposition of M.*

Proof. By the divisibility property,

$$M \subset \langle x_i^{a_i} | a_i \geq 1 \rangle, \tag{5.2.3}$$

The codimension of the underlying subspace is equal to the codimension of M, so the irreducible irredundant decomposition must contain a term

$$\langle x_i^{b_i} | b_i \geq a_i; b_i = 0 \text{ iff } a_i = 0 \rangle$$

But by the assumption of the Proposition, none of a_i can be increased such that (5.2.3) still holds, so $b_i = a_i$.

□

5.2.3 The main theorem

We will use the lexicographic product of 4 partial orders on the coordinates $\hat{u}^l_{mr}$, which defines a total order on these coordinates, and the lexicographic monomial order associated to this order to define our initial ideals. Here are the definitions.

Definition 5.2.14. Let $>_1, >_2, \ldots$ be partial orders of the variables $x_1, \ldots, x_s$. The lexicographic product of $>_1, >_2, \ldots$ is a partial order $\succ$ of the the variables, such that $x_i \succ x_j$ if $x_i >_k x_j$ for the first k such that x_i and x_j are compatible.

Definition 5.2.15. We call $\hat{u}^l_{mr}$ and the corresponding u^l_{mr} *good coordinate*, if $m + r < l \leq d - 2$.

The partial orders $>_1$,$>_2$,$>_3$ and $>_4$ are defined in the following way

Definition 5.2.16. 1.

$$\hat{u}^l_{mr} >_1 \hat{u}^{\bar{l}}_{\overline{mr}} \iff \hat{u}^l_{mr} \text{ is good, but } \hat{u}^{\bar{l}}_{\overline{mr}} \text{ is not good coordinate}$$

2.

$$\hat{u}^l_{mr} >_2 \hat{u}^{\bar{l}}_{\overline{mr}} \iff l - (m + r) > \bar{l} - (\overline{m} + \overline{r})$$

3.
$$\hat{u}^l_{mr} >_3 \hat{u}^{\bar{l}}_{\overline{mr}} \Longleftrightarrow l < \bar{l}$$

4.
$$\hat{u}^l_{mr} >_4 \hat{u}^{\bar{l}}_{\overline{mr}} \Longleftrightarrow m < \overline{m}$$

Note that by definition $m \leq r$ and $\overline{m} \leq \overline{r}$.

5. Let $\succ$ denote the lexicographic product of $>_1$,$>_2$, $>_3$ and $>_4$.

Proposition 5.2.17. *$\succ$ is a total order of the coordinates $\hat{u}^\bullet$, i.e any two coordinates are compatible.*

Proof. First, the good coordinates are all larger than the non-good ones w.r.t $\succ$. Then the order of the good coordinates is determined by $>_2, >_3, >_4$, and the same holds for the non-good coordinates. The bigger the defect, the larger the coordinate (this is $>_2$), and if two coordinates have the same defect, then their upper indices decide their order (this is $>_3$.) Finally, two coordiantes with the same defect and upper index must differ in their lower indeces, determining a total order of the coordinates.

□

Example 5.2.18. For $d = 5$ there is 1 good coordinate: $\hat{u}^3_{11}$, and the order is

$$\hat{u}^3_{11} \succ \hat{u}^5_{11} \succ \hat{u}^4_{11} \succ \hat{u}^5_{12} \succ \hat{u}^4_{12} \succ \hat{u}^5_{13} \succ \hat{u}^5_{22} \succ \hat{u}^3_{12} \succ \hat{u}^4_{13} \succ \hat{u}^4_{22} \succ^5_{14} \succ \hat{u}^5_{23}$$

We denote with the same $\succ$ the generated lexicographic monomial order on $\mathbb{C}[\hat{u}^\bullet]$.

Notation 5.2.19. Let $\mathcal{IN}_d$ denote the initial monomial ideal of the ideal $\mathcal{I}_d$ and $\mathcal{JN}_d$ the initial monomial ideal of $\mathcal{J}_d$, with respect to $\succ$, see definition 5.2.4. Note that $\mathcal{JN}_d = \mathcal{IN}_d \cap \mathbb{C}[u^l_{mr}; m + r = l \leq d]$.

By the deformation invariance of the multidegrees,

$$\mathrm{mdeg}[\mathcal{IN}_d, \mathbb{C}[\hat{u}^l_{mr}; m + r \leq l \leq d]] = \mathrm{mdeg}[\mathcal{I}_d, \mathbb{C}[\hat{u}^l_{mr}; m + r \leq l \leq d]],$$

and

$$\mathrm{mdeg}[\mathcal{JN}_d, \mathbb{C}[\hat{u}^l_{mr}; m + r = l \leq d]] = \mathrm{mdeg}[\mathcal{J}_d, \mathbb{C}[\hat{u}^l_{mr}; m + r = l \leq d]],$$

To state the main result of this section we need some extra notation.

Let $\mathcal{IN}_d = \bigcap_{\mathbf{a} \in A} \mathbf{P_a}$ be the irredundant irreducible decomposition. Here $\mathbf{P_a}$ is irreducible monomial ideal of the form

$$\mathbf{P_a} = \langle \mathbf{u}_1^{a_1}, \ldots \mathbf{u}_\alpha^{a_\alpha}, \mathbf{u}_{\alpha+1}^{a_{\alpha+1}}, \ldots \mathbf{u}_{\alpha+\beta}^{a_{\alpha+\beta}} \rangle \tag{5.2.4}$$

where $\mathbf{u}_i = \hat{u}^{l_i}_{m_i r_i}$, and $a_i \in \mathbb{Z}^+$ for $i = 1, \ldots, \alpha + \beta$. The first α coordinates have postive defect and the last β coordinates are the zero-defect coordinates, i.e

$$m_1 + r_1 - l_1 > 0, m_2 + r_2 - l_2 > 0, \ldots, m_\alpha + r_\alpha - l_\alpha > 0 \tag{5.2.5}$$

$$m_{\alpha+1} + r_{\alpha+1} - l_{\alpha+1} = \ldots = m_{\alpha+\beta} + r_{\alpha+\beta} - l_{\alpha+\beta} = 0 \tag{5.2.6}$$

By Proposition 5.2.11, $\alpha + \beta \geq \deg \widehat{Q}_d$, with equality for the maximal dimensional components of $\widehat{\mathcal{O}}_d$. The lower dimensional components are redundant when computing multidegree, so we restrict our attention to the $\alpha + \beta = \deg \widehat{Q}_d$ case. By Proposition 5.2.11, $\beta \geq \deg \widehat{Q}^0_d$, so (5.2.2) implies $\alpha \leq \dim \widehat{\mathcal{N}}^{>}_{d-2}$.

Definition 5.2.20. We call $\mathbf{P_a}$ *minimal*, if $\alpha + \beta = \deg \widehat{Q}_d$, and it has mininal number of zero-defect generators, that is $\beta = \deg \widehat{Q}^0_d = \mathrm{codim}(\widehat{\mathcal{T}}_d \subset \widehat{\mathcal{N}}^0_d)$ and $\alpha = \dim \widehat{\mathcal{N}}^{>}_{d-2}$.

From now on, we investigate only the minimal elements of the irredundant irreducible decomposition. So we throw away the non-minimal elements, and A will denote the set of the minimals, i.e $\mathbf{P}_a$ is minimal for all $a \in A$. Moreover, if it is necessary, we emphasize the dependence of α, β on d, and we write $\alpha = \alpha(d) = \dim \widehat{\mathcal{N}}^{>}_{d-2}, \beta = \beta(d) = \mathrm{codim}(\widehat{\mathcal{T}}_d \subset \widehat{\mathcal{N}}^0_d)$.

Theorem 5.2.21. *1. For any* $\mathbf{a} \in A$

$$\langle \mathbf{u}^{a_{\alpha+1}}_{\alpha+1}, \ldots, \mathbf{u}^{a_{\alpha+\beta}}_{\alpha+\beta} \rangle \lhd \mathbb{C}[\hat{u}^l_{mr}; m + r = l \leq d]$$

is an element of the irredundant irreducible decomposition of $\mathcal{JN}_d$.

2. *For any* $\mathbf{a} \in A$ *the set* $\{\mathbf{u}^{a_1}_1, \ldots, \mathbf{u}^{a_\alpha}_\alpha\}$ *equals to the set* $\{\hat{u}^l_{mr}; m + r < l \leq d - 2\}$.

3. *The reverse is also true. Let* $\mathbf{P} = \langle \mathbf{u}_1, \ldots, \mathbf{u}_\beta \rangle$ *be a term in the irredundant irreducible decomposition of* $\mathcal{JN}_d$*, then*

$$\langle \mathbf{u}_1, \ldots, \mathbf{u}_\beta, \hat{u}^l_{mr} : m + r < l \leq d - 2 \rangle \lhd \mathbb{C}[\hat{u}^l_{mr}; m + r \leq l \leq d]$$

is a term of the irredundant irreducible decomposition of $\mathcal{IN}_d$.

Recall that the weights $z_1, \ldots, z_d$ form a basis of the d-dimesional weight space, and the weight of $\hat{u}^l_{mr}$ under the torus action is $z_m + z_r - z_l$. The defect is $l - (m + r)$. Let we choose another basis measuring the defect in a more appropriate way. Introduce

$$w_1 = z_1 \ , w_2 = 2z_1 - z_2 \ , \ w_3 = z_1 + z_2 - z_3 \ldots , \ w_d = z_1 + z_{d-1} - z_d \tag{5.2.7}$$

The inverse transformation is

$$z_1 = w_1 \ , \ z_2 = 2w_1 - w_2 \ , \ \ldots , \ z_d = dw_1 - w_2 - w_3 - \ldots - w_d \tag{5.2.8}$$

The weight of $\hat{u}^l_{mr}$ in the new coordinates is

$$(m+r-l)w_1 - w_2 - \ldots - w_m + w_{r+1} + \ldots w_l. \tag{5.2.9}$$

Note that the coefficient of w_1 is the opposite of the defect.

There are exactly $\alpha(d)$ positive-defect coordinates among the generators of a minimal term in the irredundant irreducible decomposition, so the degree of $\widehat{Q}(w_1, \ldots, w_d)$ in w_1 is α. The top coefficient is a degree-β homogeneous polynomial in $w_2, \ldots, w_d$. The second and third part of Theorem 5.2.21 say that the conditions of Proposition 5.2.12 hold with $M = \mathcal{JN}_d$ and $\hat{M} = \mathcal{IN}_d$, so we have

Corollary 5.2.22.

$$\mathrm{coeff}_{w_1^{\alpha(d)}}(\widehat{Q}_d(w_1, \ldots, w_d)) = C_d \cdot \widehat{Q}^0_d$$

with the constant

$$C_d = (-1)^{t_{d-3}+t_{d-4}+\ldots+t_2}(-2)^{t_{d-4}+\ldots+t_2} \cdot \ldots \cdot (-d+4)^{t_2} = \begin{cases} (-1)^{(k-2)(k-1)}(-2)^{(k-2)(k-2)} \cdot \ldots \cdot (-2k+4)^1 & \text{if } d = 2k \\ (-1)^{(k-1)(k-1)}(-2)^{(k-2)(k-1)} \cdot \ldots \cdot (-2k+3)^1 & \text{if } d = 2k+1 \end{cases} \tag{5.2.10}$$

The theorem implies something stronger. By the additivity axiom, $\widehat{Q}_d$ is the weighted sum of the multidegrees of the maximal dimensional components of $\mathrm{Spec}(\mathcal{IN}_d)$, weighted by the multiplicities. The maximal dimensional components correspond to irreducible ideals in the irredundant irreducible decomposition, some of these are minimal, and we call the corresponding maximal dimensional component of $\mathrm{Spec}(M)$ minimal. Let $\widehat{Q}^{top}_d$ denote the sum of the multidegrees of the minimal components with multiplicity. Then

Corollary 5.2.23.

$$\widehat{Q}^{top}_d = \widehat{Q}^0_d \cdot \prod_{m+r<l\leq d-2} (z_m + z_r - z_l) \tag{5.2.11}$$

The rest of this section is devoted to the proof of Theorem 5.2.21.

5.2.4 An interlude: the case of $d = 6$.

We illustrate the statements of Theorem 5.2.21 in the $d = 6$ case. We consult our table in Section §5.1.1 again. QQ_6 is a homogeneous degree 7 polynomial in 6 variables:$w_1, \ldots, w_6$, and $\widehat{Q}^0_6$ is a homogeneous degree 4 polynomial in $w_2, \ldots, w_6$. So $\alpha(6) = 4$, and Corollary 5.2.22 states, that the coefficient of w_1^4 in $\widehat{Q}_6$ is $\widehat{Q}^0_6$. For checking Theorem 5.2.21 for this case, we struggle for the irredundant irreducible decomposition of $\mathcal{IN}_6$.

We have $\dim \widehat{\mathcal{N}}_6 = 22$ with basis

$$\left\{ \hat{u}^l_{mr} | m + r \leq l \leq 6 \right\},$$

and $\dim \widehat{\mathcal{N}}^0_6 = 9$, with basis

$$\left\{ \hat{u}^l_{mr} | m + r = l \leq 6 \right\}.$$

For simplicity, we denote the coordinates with upper index i with the ith letter of the alphabet. For instance, the basis of $\widehat{\mathcal{N}}^0_6$ is

$$b_{11}, c_{12}, d_{13}, d_{22}, e_{14}, e_{23}, f_{15}, f_{24}, f_{33}.$$

The ideal $\mathcal{I}_6$ is generated by the basic relations described in Section 5.1.3, and $\mathcal{JN}_d$ is generated by the toric basic relations. Using Macaulay for the computations, we get the following irredundant irreducible decompositions:

$$\begin{aligned}
\mathcal{IN}_6 = {} & \langle c_{11}, d_{12}, d_{11}, b_{11}, c_{12}, d_{22}, d_{13} \rangle \cap \langle c_{11}, d_{12}, d_{11}, b_{11}, c_{12}, d_{13}, e_{14} \rangle \\
& \cap \langle c_{11}, d_{12}, d_{11}, b_{11}, c_{12}, d_{13}, f_{33} \rangle \cap \langle c_{11}, d_{12}, d_{11}, b_{11}, d_{13}, e_{23}, f_{33} \rangle \\
& \cap \langle c_{11}, d_{12}, d_{11}, b_{11}, c_{12}, f_{33}, f_{24} \rangle \cap \langle c_{11}, d_{12}, d_{11}, b_{11}, e_{23}, f_{33}, f_{24} \rangle \\
& \cap \langle c_{11}, d_{12}, d_{11}, b_{11}, c_{12}, d_{22}, f_{24} \rangle \cap \langle c_{11}, d_{12}, d_{11}, c_{12}, d_{22}, e_{23}, f_{24} \rangle \\
& \cap \langle c_{11}, d_{12}, d_{11}, b_{11}, c_{12}, e_{14}, f_{24} \rangle \cap \langle c_{11}, d_{12}, d_{11}, d_{22}, e_{23}, f_{33}, f_{24} \rangle \\
& \cap \langle b_{11}, d_{22}, d_{13}, d_{12}, d_{11}, e_{23}, f_{33} \rangle \cap \langle b_{11}, d_{13}, d_{12}, d_{11}, e_{23}, e_{14}, f_{33} \rangle \\
& \cap \langle b_{11}, c_{12}, c_{11}, d_{11}, e_{23}, e_{14}, f_{24} \rangle \cap \langle c_{12}, c_{11}, d_{22}, d_{11}, e_{23}, e_{14}, f_{24} \rangle \cap \\
& \cdots \\
& \cap \langle d_{22}, e_{23}, e_{14}, f_{33}, f_{24}, f_{15}, f_{23} \rangle.
\end{aligned}$$

The number of terms is 59. Since $\alpha(6) = 4$, $\beta(6) = 3$, the first 10 terms of the decomposition are minimal. All the minimal terms contain c_{11}, d_{12}, d_{11}, this is the second statement of Theorem 5.2.21. Each minimal element has 4 toric coordinates, and putting together these we get the irredundant irreducible representation of $\mathcal{JN}_6$; this is the third statement of Theorem 5.2.21.:

$$\begin{aligned}
\mathcal{JN}_6 = {} & \langle b_{11}, c_{12}, d_{22}, d_{13} \rangle \cap \langle b_{11}, c_{12}, d_{13}, e_{14} \rangle \\
& \cap \langle b_{11}, c_{12}, d_{13}, f_{33} \rangle \cap \langle b_{11}, d_{13}, e_{23}, f_{33} \rangle \\
& \cap \langle b_{11}, c_{12}, f_{33}, f_{24} \rangle \cap \langle b_{11}, e_{23}, f_{33}, f_{24} \rangle \\
& \cap \langle b_{11}, c_{12}, d_{22}, f_{24} \rangle \cap \langle c_{12}, d_{22}, e_{23}, f_{24} \rangle \\
& \cap \langle b_{11}, c_{12}, e_{14}, f_{24} \rangle \cap \langle d_{22}, e_{23}, f_{33}, f_{24} \rangle.
\end{aligned}$$

Let $I_1, \ldots, I_{10}$ denote the minimal components of the irredundant irreducible decomposition. These correspond to maximal dimensional components of $\mathrm{Spec}(\mathcal{IN}_6)$, with multiplicity 1, so by the additivity and normalization axioms

$$\mathrm{mdeg}[\mathcal{IN}_d] = \widehat{Q}_6^{top} + \text{other terms} = \mathrm{mdeg}[I_1] + \ldots + \mathrm{mdeg}[I_{10}] + \text{other terms}, \tag{5.2.12}$$

where

$$\mathrm{mdeg}[I_1] = (2z_1 - z_3)(z_1 + z_2 - z_4)(2z_1 - z_4)(2z_1 - z_2)(z_1 + z_2 - z_3)(2z_2 - z_4)(z_1 + z_3 - z_4),$$

$$\ldots$$

$$\mathrm{mdeg}[I_{10}] = (2z_1 - z_3)(z_1 + z_2 - z_4)(2z_1 - z_4)(2z_2 - z_4)(z_2 + z_3 - z_5)(2z_3 - z_6)(z_2 + z_4 - z_6).$$

The first 3 factors are the same, so

$$\mathrm{mdeg}[\mathcal{IN}_d] = (2z_1 - z_3)(z_1 + z_2 - z_4)(2z_1 - z_4) \cdot [(2z_1 - z_2)(z_1 + z_2 - z_3)(2z_2 - z_4)(z_1 + z_3 - z_4) + \ldots$$

$$\ldots + (2z_2 - z_4)(z_2 + z_3 - z_5)(2z_3 - z_6)(z_2 + z_4 - z_6)] + \text{other terms}$$

and the sum in brackets is equal $\widehat{Q}_6^0$, so

$$\widehat{Q}_6^{top} = (2z_1 - z_3)(z_1 + z_2 - z_4)(2z_1 - z_4)\widehat{Q}_6^0,$$

this is Proposition 5.2.23. Using (5.2.7)

$$\mathrm{mdeg}[\mathcal{IN}_6] = (2z_1 - z_3)(z_1 + z_2 - z_4)(2z_1 - z_4) \cdot$$

$$[(2z_1 - z_2)(z_1 + z_2 - z_3)(2z_2 - z_4)(z_1 + z_3 - z_4) + \ldots + (2z_2 - z_4)(z_2 + z_3 - z_5)(2z_3 - z_6)(z_2 + z_4 - z_6)] +$$

$$+ \text{other terms} =$$

$$= (-w_1 + w_2 + w_3)(-w_1 + w_3 + w_4)(-2w_1 + w_2 + w_3 + w_4) \cdot [w_2 w_3(-w_2 + w_3 + w_4)w_4 + \ldots$$

$$\ldots + (-2w_2 - w_3)(-w_2 + w_4 + w_5)(-w_3 + w_4 + w_5 + w_6)(-w_2 + w_4 + w_5 + w_6)] +$$

$$+ \text{lower degree terms in } w_1$$

Hence

$$\mathrm{coeff}_{w_1^3} \widehat{Q}_6 = -2\widehat{Q}_6^0.$$

This is Corollary 5.2.22.

5.2.5 Proof of Theorem 5.2.21

Recall that the elements of $\mathcal{I}_d$ are local equivalents of the relations on the orbit closure $\mathcal{O}$ in $\widetilde{\mathcal{E}}$ near the distinguished fixed point $\tilde{\varepsilon}_{\text{dist}}$. At this point the coordinates u^l_τ of $\widetilde{\mathcal{E}} = \mathcal{E}/B_R$ can be eliminated for $|\tau| > 2$, in other words the Zariski tangent space to $\mathcal{O}$ at $\tilde{\varepsilon}_{\text{dist}}$ is contained in the linear subspace of $T_{\tilde{\varepsilon}_{\text{dist}}}\widetilde{\mathcal{E}}$ spanned by the tangent vectors indexed by the length-2 partitions, which was denoted by $\widehat{\mathcal{N}}_d$. The projection of $T_{\tilde{\varepsilon}_{\text{dist}}}\mathcal{O}$ onto $\widehat{\mathcal{N}}_d$ is $\widehat{\mathcal{O}}_d$, and $\widehat{Q}_d$ is the multidegree of $\widehat{\mathcal{O}}_d$ in $\widehat{\mathcal{N}}_d$.

Recall from Section 4.4.3, that the space $\mathbb{C}[u^\bullet]$ of polynomials on $\mathcal{E}$ forms a left-right representation of the group $B_L \times B_R$; the variables are u^l_τ, $1 \le l \le d$, $\sum(\tau) \le l$. In particular, it has two multigradings inherited from the T_R and T_L actions: $\deg_R(u^l_\tau)$ is the lth basis vector in $\mathbb{Z}^d$, while $\deg_L(u^l_\tau)$ is the vector of multiplicities $(\text{mult}(i,\tau),\ i = 1, \dots, d)$. A combination of these gradings will be particularly important for us: $\text{defect}(u^l_\tau) = l - \text{sum}(\pi)$; this induces a $\mathbb{Z}^{\ge 0}$-grading on $\mathbb{C}[u^\bullet]$. Denote the nilpotent Lie algebras of strictly upper-triangular matrices corresponding to B_R and B_L by $\mathfrak{n}_R$ and $\mathfrak{n}_L$, respectively. These Lie algebras are generated by the simple root vectors

$$\Delta_L = \{E^L_{l,l+1};\ l = 1, \dots, d-1\}, \quad \Delta_R = \{E^R_{l,l+1};\ l = 1, \dots, d-1\}.$$

The action of these root vectors on the coordinates of $\mathcal{E}$ is the following:

$$\begin{cases} \mathfrak{n}_R u^l_\tau = u^l_\tau \mathfrak{n}_L = 0, \text{ if } \text{sum}(\tau) = l, \\ E^R_{m,m+1} u^l_\tau = \delta_{l,m+1} u^{l-1}_\tau,\ u^l_\tau E^L_{m,m+1} = \text{mult}(m,\tau)\, u^l_{\tau - m \cup m+1}, \text{ if } \text{sum}(\tau) < l. \end{cases} \tag{5.2.13}$$

where $\delta_{a,b}$ is the Kronecker delta.

A relation Z for $\mathcal{O}$ in $\mathcal{E}$ is an element of the ideal $I_\mathcal{O} < \mathbb{C}[u^\bullet]$ characterized by the following properties, see (4.4.15).

Proposition 5.2.24. *$Z \in I_\mathcal{O}$ iff it satisfies*

1. $\mathfrak{n}_R Z = 0$
2. $[Z\mathfrak{n}^N_L](\varepsilon_{\text{ref}}) = 0$ *for all* $N = 1, \dots$

As we mentioned in the proof of Proposition ..., if $Z \in io$, then so are all of its $T_R \times T_L$-homogeneous components. This means that without loss of generality we can assume in what follows, that

$$Z \text{ is } T_R \times T_L\text{−homogeneous relation.}$$

By Lemma 4.4.5, near $\tilde{\varepsilon}_{\text{dist}}$, Z has a particular simple form.

Lemma 5.2.25. *Let $Z \in I_\mathcal{O}$ be a global relation in $\mathbb{C}[u^\bullet]$. Then the corresponding local relation $\widetilde{Z}$ near the fixed point $\tilde{\varepsilon}_{\text{dist}}$ in terms of the coordinates $\hat{u}^l_{mr}$ ($m + r \le l$) and $\hat{u}^l_m$ ($m \le l$) is obtained by*

- *setting* $u_{l,l}$ *to* 1*, for* $l = 1, \ldots, d$*,*
- *setting* $u_{m,l}$ *to* 0*, for* $1 \leq m < l \leq d$*,*
- *replacing the remaining variables* u^l_τ *by* $\hat{u}^l_\tau$*.*

The elements of $\mathcal{I}_d$ are the local forms of elements in $I_{\mathcal{O}}$ near $\tilde{\varepsilon}_{\mathrm{dist}}$. So each $\widetilde{Z} \in \mathcal{I}_d$ comes from a global form $Z \in I_{\mathcal{O}}$, which is a polynomial in $\mathbb{C}[u^l_{mr} : m + r \leq l \leq d, u^l_m : m \leq l \leq d]$ satisfying the properties of Prop. 5.2.24.

Finally before starting the proof, we introduce the following

Definition 5.2.26.
- The *defect* of Z is the defect of its monomials. Since Z is a pure $T_R \times T_L$ weight, this is well-defined. The defect of a monomial is the sum of the defects of its terms. We call the zero-defect relations *toric* relations.
- We say that the index m (and r) has defect k in the variable u^l_{mr} (resp. u^l_m) if $l - (m + r) = k$ (resp. $l - m = k$). The notation is $\mathrm{def}(m, \pi_3)$ ($\mathrm{def}(m, \pi_2)$).

The first part of Theorem 5.2.21 follows from the fact that $\mathcal{JN}_d$ is formed by the toric elements of $\mathcal{IN}_d$. Consequently, any $M \in \mathcal{JN}_d$ is a toric element of $\mathcal{IN}_d$, so it is divisible by $\mathbf{u}_i^{a_i}$ for some $i > \alpha$, and this is not true for bigger vector $\mathbf{b} > \mathbf{a}$.

5.2.6 Proof of the second part

Now we prove the second part of Theorem 5.2.21. Recall that we call the elements of

$$\left\{\hat{u}^l_{mr} : m + r < l \leq d - 2\right\}$$

good coordinates. The theorem comes from the following two lemmas

Lemma 5.2.27. *Let* $\hat{u}^l_{mr}$ *be a good coordinate. Then* $\mathcal{IN}_d$ *contains a monomial* $\widetilde{M}(m, r, l) = \mathbf{u}_1 \mathbf{u}_2 \cdots \mathbf{u}_s$ *whith the following properties:*

1. $\mathbf{u}_1 = \hat{u}^l_{mr}$
2. $\mathbf{u}_2, \ldots, \mathbf{u}_s$ *are toric coordinates, and* $\mathbf{u}_i \neq \mathbf{u}_j$ *for* $i \neq j$*.*
3. *There is* $\widetilde{Z} \in \mathcal{I}_d$ *such that* $\widetilde{M}$ *is the initial monomial of* $\widetilde{Z}$ *and* $\hat{u}^l_{mr}$ *does not occur in other monomials of* $\widetilde{Z}$*.*

Lemma 5.2.28. *Suppose* $\widetilde{M} \in \mathcal{IN}_d$ *is a monomial satisfying Lemma 5.2.27 (1),(2),(3). Then* $\hat{u}^l_{mr}$ *is a generator of* $\mathbf{P_a}$ *for every* $\mathbf{a} \in A$*, i.e a generator of every minimal element of the irredundant irreducible decomposition of* $\mathcal{IN}_d$*.*

Notation 5.2.29. To simplify our life, if $\pi_2 = \{m, l\}$ is a pair with $m \leq l \leq d$ and $\pi_3 = \{m, r, l\}$ is a triple with $m + r \leq l \leq d$ then we also use the notations $\hat{u}_{\pi_2} = \hat{u}^l_m$ and $\hat{u}_{\pi_3} = \hat{u}^l_{mr}$ for the coordinates on $\widetilde{\mathcal{E}}$, and $u_{\pi_2} = u^l_m$ and $u_{\pi_3} = u^l_{mr}$ for the coordinates on $\mathcal{E}$. We refer a coordinate u_{π_2} (resp. u_{π_3}) as two-index (resp. three-index) coordinate.

Proof of Lemma (5.2.28) Suppose

$$\widetilde{M} = \hat{u}^l_{mr} \prod_{\pi_3 \in \Pi_3} \hat{u}_{\pi_3} \in \mathcal{IN}_d \tag{5.2.14}$$

where $l - (m + r) > 0$ but the other terms are toric coordinates. Note that there are no two-index coordinates by Lemma 5.2.25.

Suppose there is $\mathbf{a} \in A$ such that $\hat{u}^l_{mr} \notin \mathbf{P_a}$. Let $\widetilde{Z} \in \mathcal{I}_d$ be an element with initial monomial $\widetilde{M}$ satisfying (3) of Lemma 5.2.27 and $Z \in I_{\mathcal{O}}$ a global form of $\widetilde{Z}$. Note that $\widetilde{M}$ comes from a monomial M of Z applying Lemma 5.2.25.

$$M = u^l_{mr} \prod_{\pi_3 \in \Pi_3} u_{\pi_3} \prod_{\pi_2 \in \Pi_2} u_{\pi_2}, \tag{5.2.15}$$

where Π_3 is the same set as in (5.2.14), and Π_2 contains only toric two-indices.

Since $\widetilde{M} \in \mathcal{IN}_d$ but $\hat{u}^l_{mr} \notin \mathbf{P_a}$, there is a $\tau_3 \in \Pi_3$ such that $\hat{u}_{\tau_3} \in \mathbf{P_a}$, that is, the set

$$\Pi_{\mathbf{P}} = \Pi_3 \cap \mathbf{P_a} \tag{5.2.16}$$

is nonempty. The generators of $\mathcal{J}_d$ have the form $\tilde{M}_1 - \tilde{M}_2$ (every toric ideal is generated by polynomials of this form) Define the following sets of monomials for $\tau \in \Pi_{\mathbf{P}}$

$$\mathbf{Mon}^1_\tau = \left\{ \widetilde{M_1} : \ \widetilde{M_1} - \widetilde{M_2} \in \mathcal{J}_d, \ \hat{u}_\tau \in \widetilde{M_1} \right\} \tag{5.2.17}$$

$$\mathbf{Mon}^2_\tau = \left\{ \widetilde{M_2} : \ \widetilde{M_1} - \widetilde{M_2} \in \mathcal{J}_d, \ \hat{u}_\tau \in \widetilde{M_1} \right\} \tag{5.2.18}$$

If both $\widetilde{M_1}$ and $\widetilde{M_2}$ are divisible by $\hat{u}_\tau$, we can simplify the toric relation $\widetilde{M_1} - \widetilde{M_2}$, so we suppose that $\mathbf{Mon}^1_\tau \cap \mathbf{Mon}^2_\tau = 0$.

Choose $\widetilde{M^2_\tau} \in \mathbf{Mon}^2_\tau$ for all $\tau \in \Pi_{\mathbf{P}}$. Let $\widetilde{M^1_\tau} \in \mathbf{Mon}^1_\tau$ the pair of $\widetilde{M^2_\tau}$, i.e $\widetilde{M^1_\tau} - \widetilde{M^2_\tau} \in \mathcal{J}_d$. The following small observation is the crucial step in the proof.

Observation 5.2.30.

$$\check{M} = \hat{u}^l_{mr} \prod_{\pi_3 \in \Pi_3 \setminus \Pi_{\mathbf{P}}} \hat{u}_{\pi_3} \prod_{\tau \in \Pi_{\mathbf{P}}} \widetilde{M^2_\tau} \tag{5.2.19}$$

is an element of $\mathcal{IN}_d$.

Proof of Observation (5.2.30)

The initial monomial of

$$\check{Z} = \widetilde{Z} \prod_{\tau \in \Pi_{\mathbf{P}}} \frac{\widetilde{M^1_\tau}}{\hat{u}_\tau} \tag{5.2.20}$$

is

$$In(\check{Z}) = \widetilde{M} \prod_{\tau \in \Pi_{\mathbf{P}}} \frac{\tilde{M}^1_\tau}{\hat{u}_\tau} = \hat{u}^l_{mr} \prod_{\pi_3 \in \Pi_3 \setminus \Pi_{\mathbf{P}}} \hat{u}_{\pi_3} \prod_{\tau \in \Pi_{\mathbf{P}}} \widetilde{M^1_\tau} \tag{5.2.21}$$

Let $\check{Z}^{\mathbf{repl}} \in I_{\mathcal{O}}$ be the modified polynomial arising from $\check{Z}$ by replacing $In(\check{Z})$ by $\check{M}$. (Since $In(\check{Z}) - \check{M} = 0$ a toric relation, $\check{Z}^{\mathbf{repl}}$ is also relation.) We claim that the initial monomial of $\check{Z}^{\mathbf{repl}}$ is $\check{M}$. This is guaranteed by the term $\hat{u}^l_{mr}$ in $\check{M}$, namely $\check{Z}^{\mathbf{repl}}$ has the following properties:

1. $\hat{u}^l_{mr}$ is a term in $\check{M}$
2. $\hat{u}^l_{mr}$ is not a term in any other monomial of $\check{Z}^{\mathbf{repl}}$
3. $\hat{u}^l_{mr}$ is the greatest element w.r.t $\succ$ among all the coordinates of $\check{Z}^{\mathbf{repl}}$.

Only (2) and (3) need some explanation. But the same is true for $\widetilde{Z}$ by the prescribed properties of Lemma 5.2.27, and multiplying by toric coordinates and substituting $\mathbf{Mon}^1$ by $\mathbf{Mon}^2$ does not bother the positive-definite coordinates, so the same is true for $\check{Z}$ and $\check{Z}^{\mathbf{repl}}$. So Observation (5.2.30) is proved.

Recall that $\mathbf{P_a}$ have the form

$$\mathbf{P_a} = \langle \mathbf{u}_1^{a_1}, \dots \mathbf{u}_\alpha^{a_\alpha}, \mathbf{u}_{\alpha+1}^{a_{\alpha+1}}, \dots \mathbf{u}_{\alpha+\beta}^{a_{\alpha+\beta}} \rangle \tag{5.2.22}$$

where $\mathbf{u}_i = \hat{u}^{l_i}_{m_i r_i}$, and $a_i \in \mathbb{Z}^+$ for $i = 1, \dots, \alpha+\beta$. As in 5.2.4, the first α coordinates have positive defect and the last β coordinates are the zero-defect coordinates.

Take a look at $\check{M} \in \mathcal{IN}_d$. Since $\hat{u}^l_{mr} \notin \mathbf{P_a}$, $\prod_{\pi_3 \in \Pi_3 \setminus \Pi_{\mathbf{P}}} \hat{u}_{\pi_3} \notin \mathbf{P_a}$, there must be $\tau \in \Pi_{\mathbf{P}}$ and $\alpha + 1 \le i \le \alpha + \beta$ such that $\mathbf{u}_i | \widetilde{M}^2_\tau$.

This implies

Observation 5.2.31. There must be a $\tau_0 \in \Pi_{\mathbf{P}}$, such that for any $\widetilde{M}^2_{\tau_0} \in \mathbf{Mon}_{\tau_0}$ there is $\alpha + 1 \le i \le \alpha + \beta$ with $\mathbf{u}_i | \widetilde{M}^2_{\tau_0}$.

For simplicity, suppose $\mathbf{u}_{\tau_0} = \mathbf{u}_{\alpha+1}$. Introduce

$$\mathcal{JN}_{\tau_0} = \left\{ \widetilde{M} \in \mathcal{JN}_d : \mathbf{u}_{\tau_0} \nmid M \right\} \tag{5.2.23}$$

Observation 5.2.31 implies

Observation 5.2.32.

$$I_{\tau_0} = \langle \mathbf{Mon}^2_{\tau_0}, \mathcal{JN}_{\tau_0} \rangle \subset \langle \mathbf{u}_{\alpha+2}, \dots, \mathbf{u}_{\alpha+\beta} \rangle \tag{5.2.24}$$

We define a new order $\succ_0$ on the toric coordinates, namely

$$\hat{u}^l_{mr} \succ_0 \hat{u}^{\bar{l}}_{\overline{mr}}$$

if

$$\hat{u}^l_{mr} = \mathbf{u}_{\alpha+1} \neq \hat{u}^{\bar{l}}_{\overline{mr}}$$

or neither of them is $\mathbf{u}_{\alpha+1}$, but $\hat{u}^l_{mr} \succ \hat{u}^{\bar{l}}_{\overline{mr}}$

Let $\succ_{wt}$ be the lexicographic monomial order corresponding to $\succ_0$ on the coordinates. Note that $\succ_{wt}$ is a weighted lexicographic monomial order, where $\mathbf{u}_{\alpha+1}$ has weight ε, the other coordinates have weight 1 with $\varepsilon \ll 1$.

The following observation is straightforward from the definition:

Observation 5.2.33. Let $\mathcal{JN}_{wt}$ be the initial monomial ideal of $\mathcal{J}_d$ with respect to $\succ_{wt}$. Then $\mathcal{JN}_{wt} = I_{\tau_0} = \langle \mathbf{Mon}^2_{\tau_0} \cup \mathcal{JN}_{\tau_0} \rangle$

By Observations 5.2.32 and 5.2.33, Spec($\mathcal{JN}_{wt}$) has a component of codimension $\beta - 1$, but the maximal dimensional components of Spec($\mathcal{JN}_d$) have codimension β by the minimality of $\mathbf{P_a}$, which is impossible, since both $\mathcal{JN}_d$ and $\mathcal{JN}_w t$ are initial ideals of $\mathcal{J}_d$ with respect to some monomial orders. Lemma 5.2.28 is proved.

Proof of Lemma 5.2.27

Briefly, we need to find relations in $\mathcal{I}_d \lhd \mathbb{C}[\hat{u}^\bullet]$ with initial monomial ideal satisfying the requirements of the lemma. It is interesting, that we can find such relations in the subideal $\mathcal{R}_d \subset \mathcal{I}_d$ generated by the basic equations! Observe that in this way we use only $\mathcal{R}_d$ and the toric elements of $\mathcal{I}_d$ to prove the first half of Theorem 5.2.21.

The good coordinates $\hat{u}^l_{1r}$ $(r+1 < l \le d-2)$ are easy to handle: the initial term of the basic equation $R(1,r,2;l+2)$ satisfies (1) and (2), and $R(1,r,2;l+2)$ itself satisfies (3) of Lemma 5.2.27. Namely, the initial monomial of

$$R(1,r,2;l+2) = (\hat{u}^l_{1,r}\hat{u}^{l+2}_{2,l} + \hat{u}^{l-1}_{1,r}\hat{u}^{l+2}_{2,l-1} + \ldots + \hat{u}^{r+1}_{1,r}\hat{u}^{l+2}_{2,r+1}) - \\ - (\hat{u}^{l+1}_{2,r}\hat{u}^{l+2}_{1,l+1} + \hat{u}^{l}_{2,r}\hat{u}^{l+2}_{1,l} + \ldots + \hat{u}^{r+2}_{2,r}\hat{u}^{l+2}_{1,r+2}) \quad (5.2.25)$$

is

$$\hat{u}^l_{1,r}\hat{u}^{l+2}_{2,l} \quad (5.2.26)$$

since $\hat{u}^l_{1,r}$ and $\hat{u}^{l+1}_{2,r}$ have the same maximal defect among the coordinates of this basic relation (hence they are maximal with respect to $>_2$) but

$$\hat{u}^l_{1,r} >_3 \hat{u}^{l+1}_{2,r}. \quad (5.2.27)$$

The other properties of Lemma 5.2.27 are straightforward.

The remaining good coordinates are the initial monomials of a linear combination of 2 basic relations. For the good coordinate $\hat{u}^l_{m,r}$, $m > 1, m+r < l \le d-2$ these basic equations are $R(m,r,1;l+1)$ and $R(m+1,r,1;l+2)$:

$$R(m,r,1;l+1) = (\hat{u}^l_{m,r}\hat{u}^{l+1}_{1,l} + \hat{u}^{l-1}_{m,r}\hat{u}^{l+1}_{1,l-1} + \ldots + \hat{u}^{m+r}_{m,r}\hat{u}^{l+1}_{1,m+r}) - \\ - (\hat{u}^{l-m+1}_{1,r}\hat{u}^{l+1}_{m,l-m+1} + \hat{u}^{l-m}_{1,r}\hat{u}^{l+1}_{m,l-m} + \ldots + \hat{u}^{r+1}_{1,r}\hat{u}^{l+1}_{m,r+1}) = 0 \quad (5.2.28)$$

and

$$R(m+1,r,1;l+2) = (\hat{u}^{l+1}_{m+1,r}\hat{u}^{l+2}_{1,l+1} + \hat{u}^{l}_{m+1,r}\hat{u}^{l+2}_{1,l} + \ldots + \hat{u}^{m+r+1}_{m+1,r}\hat{u}^{l+2}_{1,m+r+1}) - \\ - (\hat{u}^{l-m+1}_{1,r}\hat{u}^{l+2}_{m+1,l-m+1} + \hat{u}^{l-m}_{1,r}\hat{u}^{l+2}_{m+1,l-m} + \ldots + \hat{u}^{r+1}_{1,r}\hat{u}^{l+2}_{m+1,r+1}) \quad (5.2.29)$$

The maximum of the occurring defects in (5.2.28) and (5.2.29) is $l-m-r$, and only

$$\hat{u}^l_{m,r}, \hat{u}^{l+1}_{1,m+r}, \hat{u}^{l-m+1}_{1,r}, \hat{u}^{l+1}_{m,r+1}$$

in (5.2.28) and

$$\hat{u}^{l+1}_{m+1,r}, \hat{u}^{l+2}_{1,m+r+1}, \hat{u}^{l-m+1}_{1,r}, \hat{u}^{l+2}_{m+1,r+1}$$

in (5.2.29) have this maximal defect. The good coordinates among these are $\hat{u}^l_{m,r}$, $\hat{u}^{l-m+1}_{1,r}$ if $l = d-2$ and if $l < d-2$ there are even more.

For a partial order $>$ we use $a \geq b$ when $a > b$ or a it not comparable with b. Let $a >_{1,2,3} b$ stand for $a \geq_1 b, a \geq_2 b, a >_3 b$. If $m < r+1$ the coordinates above have the following order:

$$\hat{u}^{l-m+1}_{1,r} \geq_{1,2,3} \hat{u}^l_{m,r} \geq_{1,2,3} \hat{u}^{l+1}_{1,m+r} \geq_{1,2,3,4} \hat{u}^{l+1}_{m,r+1} \geq \\ \geq_{1,2,3,4} \hat{u}^{l+1}_{m+1,r} \geq_{1,2,3} \hat{u}^{l+2}_{1,m+r+1} \geq_{1,2,3,4} \hat{u}^{l+2}_{m+1,r+1} \quad (5.2.30)$$

(If $m = r+1$ $\hat{u}^{l+1}_{m,r+1}$ and $\hat{u}^{l+1}_{m+1,r}$ change order.) But we can get rid of $\hat{u}^{l-m+1}_{1,r}$ easily: multiplying (5.2.28) and (5.2.29) by the toric $\hat{u}^{l+2}_{m+1,l-m+1}$,$\hat{u}^{l+1}_{m,l-m+1}$, respectively, in the difference $v^{l-m+1}_{1,r}$ cancels, and we get

$$\hat{u}^l_{m,r}\hat{u}^{l+1}_{1,l}\hat{u}^{l+2}_{m+1,l-m+1} + \ldots = 0$$

which is a relation in $\mathcal{R}$ which satisfies (1),(2),(3) of Lemma 5.2.27.

5.2.7 Proof of the third part

Now we turn to the third part of Theorem 5.2.21. Recall that $\mathcal{JN}_d$ is the initial ideal of $\mathcal{J}_d$ with respect to the monomial order defined in Definition 5.2.16, where $\mathcal{J}_d \lhd \mathbb{C}[\hat{u}^l_{m,r} : m+r = l \leq d]$ is the ideal of $\mathcal{T}_d$, and $\mathcal{IN}_d$ is the initial ideal of $\mathcal{I}_d$ w.r.t the same monomial order, where $\mathcal{I}_d \lhd \mathbb{C}[\hat{u}^l_{m,r} : m+r \leq l \leq d]$ is the ideal of $\mathcal{O}_d$. By definition, $\mathcal{JN}_d \lhd \mathbb{C}[\hat{u}^l_{m,r} : m+r = l \leq d]$, and $\mathcal{IN}_d \lhd \mathbb{C}[\hat{u}^l_{m,r} : m+r \leq l \leq d]$, and let

$$\mathcal{JN}^{big}_d \lhd \mathbb{C}[\hat{u}^l_{m,r} : m+r \leq l \leq d]$$

denote the ideal generated by $\mathcal{JN}_d$. If

$$\langle \mathbf{u}_1, \ldots, \mathbf{u}_\beta \rangle$$

is a term of the irredundant irreducible decomposition of $\mathcal{JN}_d$, then

$$\mathcal{JN}^{big}_d \subset \langle \mathbf{u}_1, \ldots, \mathbf{u}_\beta \rangle \lhd \mathbb{C}[\hat{u}^l_{m,r} : m+r \leq l \leq d$$

Let $\mathcal{IN}^{>}_d = \mathcal{IN}_d \setminus \mathcal{JN}^{big}_d$ denote the elements of $\mathcal{IN}_d$ which are not in $\mathcal{JN}^{big}_d$. These have positive defect, and by Proposition 5.2.13 for the third part of Theorem 5.2.21 is sufficient to prove

Lemma 5.2.34.

$$\mathcal{IN}^{>}_d \subset \langle \hat{u}^l_{mr} : m+r < l \leq d-2 \rangle. \quad (5.2.31)$$

Indeed, let $\mathbf{P} = \langle \mathbf{u}_1, \ldots, \mathbf{u}_\beta \rangle$ be an element of the irredundant irreducible decomposition of $\mathcal{JN}_d$. Then (5.2.31) induces that

$$\langle \mathbf{u}_1, \ldots, \mathbf{u}_\beta, (\hat{u}^l_{mr})^{a(m,r,l)} : m + r < l \leq d - 2 \rangle$$

is an element of the irredundant irreducible decomposition of $\mathcal{IN}_d$ with some $a(m, r, l) \geq 1$, but Lemma 5.2.28 with Lemma 5.2.27 says that $a(m, r, l) = 1$ must hold.

Proof of Lemma 5.2.34

Let $\widetilde{M} \in \mathcal{IN}^{>}_d$, and $\widetilde{Z} \in \mathcal{I}_d$ be a relation, whose initial monomial w.r.t $\succ$ is $\widetilde{M}$. Let $Z \in I_{\mathcal{O}}$ be a global form of $\widetilde{Z}$.

Let M_κ ($\kappa \in K(Z)$) be the monomials of Z, i.e

$$Z = \sum_{\kappa \in K(Z)} C_\kappa M_\kappa = \sum_{\kappa \in K(Z)} C_\kappa \prod_{\pi_2 \in \Pi^\kappa_2} \prod_{\pi_3 \in \Pi^\kappa_3} u_{\pi_2} u_{\pi_3} \tag{5.2.32}$$

When M is a monomial term of Z we also use Π^M_2 and Π^M_3 for the two- and three-index coordinates of M. To avoid the exponents, we think of these as multisets, i.e elements of Π^M_2 can be equal, and the same for Π^M_3.

Since Z is $T_L \times T_R$-homogeneous, Π^κ_2 has the same number of elements for all $\kappa \in K(Z)$, and the same holds for Π^κ_3. This is obvious after comparing the T_L weights and T_R weights of two monomials.

By Lemma 5.2.25 $\widetilde{Z}$ is the sum of those $C_\kappa M_\kappa$'s where Π^κ_2 does not contain a positive-defect coordinate, i.e (m, l) with $m < l$. In particular, if $\widetilde{Z} \neq 0$ then there is no positive-defect two-index coordinate in some M_κ. Using the definition of $\prec$ Lemma 5.2.34 follows from

Observation 5.2.35. Let Z be a global form of $\widetilde{Z}$, and the initial monomial of $\widetilde{Z}$ is $\widetilde{M} \in \mathcal{IN}^{>}_d$. Then there is $\kappa \in K(Z)$ such that M_κ contains at least one good coordinate term, but it does not have positive-defect two-index term, i.e $u^l_m \notin \Pi^\kappa_2$ if $m < l$.

We prove by induction on the defect of Z. The $\mathrm{def}(Z) = 1$ case is proved separately in the end of this section below. Now suppose that $\mathrm{def}(Z) \geq 2$, and Observation 5.2.35 is true for relations with smaller defect.

Definition 5.2.36. We say that $Z \in I_{\mathcal{O}}$ is a *primitive* element of the ideal if there is no $\check{Z} \in I_{\mathcal{O}}$ and $p \in \mathbb{C}[u^\bullet]$ such that

$$Z = p\check{Z}$$

Clearly, it is enough to prove Prop. 5.2.35 for primitive elements, and we suppose that Z is such an element of $I_{\mathcal{O}}$.

Definition 5.2.37.

$$\zeta(Z) = \min\{l : \exists \kappa \in K(Z),\ \mathrm{and(m,l)} \in \Pi^\kappa_2 \text{ or } \mathrm{(m,r,l)} \in \Pi^\kappa_3 \text{ s.t. m} < \mathrm{l \text{ or } m + r < l}\}$$

$$\xi(Z) = \min\{m : \exists \kappa \in K(Z), 1 \leq m \leq r \leq d \text{ such that } \mathrm{m + r < l, (m,r,l)} \in \Pi^\kappa_3\}$$

In words, $\zeta(Z)$ is the minimal l such that there is positive-defect variable on level l, and $\xi(Z)$ is the minimal m, such that there is positive-defect three-index variable in Z with lower index m.

Note that the definition of $\xi(Z)$ is meaningless if the defined set is empty. However, if there is no positive-defect coordinate in Π_3^κ for all $\kappa \in K(Z)$, then Π_2^κ contains at least one positive-defect coordinate for all κ, and therefore $\widetilde{Z} = 0$ by Lemma 5.2.24.

Suppose that Observation 5.2.35 is false, and take a $Z \in I_{\mathcal{O}}$ with the following properties:

- Z is a counterexample.
- Z is primitive with defect ≥ 2.
- $\zeta(Z)$ is maximal among the counterexamples.

We use $\xi = \xi(Z)$. Take $Z^* = ZE^L_{\xi,\xi+1}$. Since $\operatorname{def} Z^* = \operatorname{def} Z - 1$, by the induction hypothesis Z^* has a monomial term M^* without positive-defect two-index coordinate u_{π_2}:

$$M^* = u^{l_1}_{m_1 r_1} u^{l_2}_{m_2 r_2} \dots u^{l_s}_{m_s r_s} u^{l_{s+1}}_{l_{s+1}} \dots u^{l_t}_{l_t} \tag{5.2.33}$$

where one of the first s terms is a good coordinate, say $u^{l_1}_{m_1 r_1}$.

This monomial comes from a monomial M of Z, by changing the lower index ξ to $\xi + 1$ in some term of M, where ξ has positive defect. We call this *shifting* coordinate.

Case 1) *The shifting coordinate is one of the first s.*
Then either $u^{l_1}_{m_1 r_1}$, or the shifting coordinate is a good coordinate in M and the two-index coordinates have defect 0, since they coincide with the two-index coordinates of M^*. So Z satisfies Observation 5.2.35.
Case 2) *The shifting coordinate is among the two-index coordinates,*
say the $s + 1$th. That is, $\xi + 1 = l_{s+1}$ and M contains $u^{\xi+1}_{\xi}$, but all the other two-index coordinates have zero defect, and $\Pi_3^M = \Pi_3^{M^*}$.

We prove the following

Observation 5.2.38. *Z must have a monomial N, which has a coordinate u^l_{mr} with $m \geq 1, r \geq 1, m + r = \xi, l = \xi + 1$.*

Since m and r have defect 1 in this coordinate and one of them is less than ξ, this contradicts to the definition of $\xi(Z)$.

Proof of Observation 5.2.38

Suppose that $u^{\xi+1}_{m(\xi-m)}$ does not occur in Z with $m \geq 1$ and $\xi - m \geq 1$. Then by the minimality of ξ the possible positive-defect coordinate on the $\xi+1$th level is $u^{\xi+1}_{\xi}$. Take $Z^{\text{shift}} = E^R_{\xi,\xi+1} Z$. Then $Z^{\text{shift}} \in I_{\mathcal{O}}$ by Prop. 5.2.24. Since $E^R_{\xi,\xi+1} u^{\xi+1}_{\xi} = u^{\xi}_{\xi}$, any monomial M^{shift} of Z^{shift} must contain the coordinate u_ξ^ξ which comes from $u_\xi^{\xi+1}$ in the corresponding monomial of Z. So

$$Z^{\text{shift}} = u_\xi^\xi \cdot \check{Z}. \tag{5.2.34}$$

Since $Z^{\text{shift}} \in I_{\mathcal{O}}$, and u_ξ^ξ is a zero-defect coordinate, $\check{Z}$ must also satisfy the properties in Lemma 5.2.24, so $\check{Z} \in I_{\mathcal{O}}$. That is

$$Z^{\text{rem}} = Z - u_\xi^{\xi+1} \cdot \check{Z} \in I_{\mathcal{O}}. \tag{5.2.35}$$

If $Z^{\text{rem}} = 0$ then Z is not primitive; contradiction. Otherwise, since $\zeta(Z^{\text{rem}}) > \zeta(Z)$, Z^{rem} is not a counterexample, so it contains a monomial with a good coordinate but no double index with positive defect, and this is also a monomial of Z. Prop. 5.2.38 and Lemma 5.2.35 is proved.

Note that we proved the following more general

Proposition 5.2.39. *If the primitive Z contains a monomial with a term u_{l-1}^l but u_m^l and u_{mr}^l do not occur when $m < l-1, m+r < l-1$ then some u_{mr}^l with $m+r = l-1$ must occur in Z*

To finish the proof of the second half of Theorem 5.2.21 only the $\operatorname{def}(Z) = 1$ case is a left:

Proof of Observation 5.2.35 for $\operatorname{def}(Z) = 1$

We can suppose that Z is primitive. Each monomial of Z has exactly one coordinate of defect 1. So we have to prove that one of them is a good coordinate. We prove a slightly weaker fact, namely: Z contains one of

$$\left\{u_{mr}^l : m+r \leq d-2, l-(m+r) = 1; u_m^l : m \leq d-2, l-m = 1\right\}.$$

If Z contains a 1-defect two-index coordinate, then it also contains a 1-defect three-index coordinate on the same level by Prop. 5.2.39, so this weaker result is sufficient.

From now on $d = 2k$. The odd case can be handled similarly. Suppose that there is no 1-defect coordinate on level $\leq d-2$ in Z, i.e the occurring 1-defect coordinates for $d = 2k$ are

$$u_{2k-2}^{2k-1}, u_{1(2k-3)}^{2k-1}, u_{2(2k-4)}^{2k-1}, \dots u_{(k-1)(k-1)}^{2k-1} \tag{5.2.36}$$

on level $2k-1$ and

$$u_{2k-1}^{2k}, u_{1(2k-2)}^{2k}, u_{2(2k-3)}^{2k}, \dots u_{(k-1)(k)}^{2k} \tag{5.2.37}$$

on level $2k$. We denote by $A_{m,r}$ (resp. $B_{m,r}$) the sum of the coefficients of the monomials which have their unique 1-defect coordinate on the $2k$th (resp. $2k-1$th) level. We use A_m and B_m when the unique 1-defect coordinate is a two-index one.

By Prop. 5.2.24 $ZE^L_{i,i+1} \in I_{\mathcal{O}}$ is a toric relation ($i = 1, 2, \ldots, d-1$), and $ZE^L_{i,i+1}(\varepsilon_{\mathrm{ref}}) = 0$. This can be rewritten using $A_{m,r}$ and $B_{m,r}$ as following:

$$\begin{aligned} B_{i,2k-2-i} + A_{i,2k-1-i} = 0 \text{ for i} = 1, \ldots, \text{k} - 2, \text{k}, \ldots 2\text{k} - 3 \\ 2B_{k-1,k-1} + A_{k-1,k} = 0 \text{ for i} = \text{k} - 1 \\ B_{2k-2} + A_{1,2k-2} = 0 \text{ for i} = 2\text{k} - 2 \\ A_{2k-1} = 0 \text{ for i} = 2\text{k} - 1 \end{aligned}$$

This gives us the following:

$$\begin{aligned} &B_{2k-2} = -A_{1,2k-2} = B_{1,2k-3} = -A_{2,2k-3} = \\ &\qquad = B_{2,2k-4} = -A_{3,2k-4} = \ldots = B_{k-2,k} = -A_{k-1,k} = 2B_{k-1,k-1} \end{aligned} \tag{5.2.38}$$

On the other hand $E^R_{d-2,d-1}Z(\varepsilon_{\mathrm{ref}}) = 0$, in coordinates

$$B_{2k-2} + B_{1,2k-3} + B_{2,2k-4} + \ldots + B_{k-1,k-1} = 0 \tag{5.2.39}$$

and $E^R_{d-1,d}Z(\varepsilon_{\mathrm{ref}}) = 0$, i.e

$$A_{2k-1} + A_{1,2k-2} + A_{2,2k-3} + \ldots + A_{k-1,k} = 0 \tag{5.2.40}$$

Adding equations of (5.2.38) we get

$$B_{2k-2} + A_{1,2k-2} + B_{1,2k-3} + A_{2,2k-3} + \ldots + B_{k-2,k} + A_{k-1,k} = 0 \tag{5.2.41}$$

The left hand side of (5.2.39)+(5.2.40) is almost the same, the difference is $0 = A_{2k-1} + B_{k-1,k-1} = B_{k-1,k-1}$. From (5.2.38) then all A's and B's are 0, which means that Z has the form

$$Z = \sum_{m+r=l-1} u^l_{m,r} R(m,r,l) + \sum_{m=l-1} u^l_m R(m,l)$$

where $R(m,r,l), R(m,r)$ are toric relations in $\mathcal{J}_d$. So Z is an element of the ideal in $\mathbb{C}[u^l_{m,r} : m + r \leq l \leq d]$, generated by $\mathcal{J}_d$, therefore the initial monomial is in $\mathcal{JN}^{big}_d$, which is a contradiction, Observation 5.2.35 is proved.

Chapter 6

List of Notations and Bibliography

- -$\mathcal{J}(n)$: algebra of power series in n variables, without constant term; see §2.1.1.
  –$\mathcal{J}_d(n)$: space of d-jets of holomorphic functions on $\mathbb{C}^n$ near the origin; see §2.1.1.
  -$\mathcal{J}_d(n,k)$: *d-jets* of maps $(\mathbb{C}^n,0) \to (\mathbb{C}^k,0)$, see §2.1.1.

- Lin: linear part of a germ or jet, see §2.1.1.

- $\mathrm{Diff}_d(n)$: the group consisting of d-jets of diffeomorphisms of $\mathbb{C}^n$ at the origin; see §2.1.1.

- -A_Ψ: the nilpotent algebra of the map germ Ψ; see §2.1.1.
  -A_d: the nilpotent algebra $t\mathbb{C}[t]/t^{d+1}$; see §2.1.2.

- -$\Theta_A, \Theta_A^{n\to k}$: set of jets with nilpotent algebra A; see (2.1.2).
  -$\Theta_d, \Theta_d^{n\to k}$ short notation for Θ_{A_d}; see §2.1.2.

- $\mathcal{K}, \mathcal{K}_d(n,k)$: the contact group; see (2.1.3).

- $\mathrm{eP}[\Sigma, W]$: equivariant Poincaré dual class of $\Sigma \subset W$; see the axiomatic definition in §2.2.1.

- -$\mathrm{Euler}^T(W)$: the equivariant Euler class of a vector space W with respect to the action of the torus T; see §2.2.1.
  -$\mathrm{emult}_p[M,Z]$: equivariant multiplicity of M in Z at $p \in M$; see (2.2.12) in §2.2.5.
  -$\mathrm{Thom}(W)$: equivariant Thom class of W; see §2.2.5.

- $\mathrm{RC}(q)$: the generating function of the relative Chern classes; see (2.2.20).

- -$\mathrm{Tp}_A^{n\to k}, \mathrm{Tp}_A^{n\to k}(\boldsymbol{\lambda}, \boldsymbol{\theta})$: the Thom polynomial of Θ_A for a nilpotent algebra A, see (2.2.12). -$\mathrm{Tp}_d^{n\to k}$: the Thom polynomial in case $A = A_d$.
  -TD_d^j: the Thom-Damon series defined in Proposition 2.2.13.

- -$|\pi|, \mathrm{sum}(\pi)$, $\max(\pi)$, $\mathrm{perm}(\pi)$: the length, the sum, the maximal element and the number of different permutations of the partition π, respectively, see Notation 4.1.2.
  -$\pi[m]$: the set of all partitions of m, see (4.1.5).

- $\mathbf{v}_\pi = \prod_{j=1}^{l} v_{i_j}$, $\Psi(\mathbf{v}_\pi) = \Psi(v_{i_1}, \ldots, v_{i_l})$, see (4.1.6).

- -$\mathcal{J}_d^{\mathrm{reg}}(1,n)$: set of regular curves, i.e with nonvanishing linear part, see (4.1.1).
  -$\Psi = (\Psi^1, \ldots, \Psi^d) = (A, B, C, \ldots)$: element of $\mathcal{J}_d(n,k)$, see (4.1.4).

- $\gamma \in \mathcal{J}_d(1,n)$: γ always stands for a test curve, see (4.1.3).

- -$\mathrm{Sol}_{\varepsilon(\gamma)} \subset \mathcal{J}_d(n,k)$: the linear subspace of solutions in $\mathcal{J}_d(n,k)$ corresponding to the curve γ, see (4.1.9).
  -Sol_ε: the linear subspace of solutions in $\mathcal{J}_d(n,k)$ corresponding $\varepsilon \in \mathcal{F}_d(n)$, see the Definition 4.2.4.
  -$\mathrm{Sol}_{\tilde{\varepsilon}}$: the linear subspace of solutions in $\mathcal{J}_d(n,k)$ corresponding to $\tilde{\varepsilon} \in \widetilde{\mathcal{F}}_d(n)$, see Definition 4.2.4.
  -$\mathrm{Sol}_{\widetilde{\mathcal{F}}}$: the vector bundle with fibers $\mathrm{Sol}_{\tilde{\varepsilon}}$ and base $\widetilde{\mathcal{F}}_d(n)$, see Definition 4.2.4.

- $\mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Sym}^{\bullet}_d\mathbb{C}^n)$: the filtration preserving maps with respect to the filtrations (4.2.5) and (4.2.4), see (4.2.6).

- ψ: the fundamental map $\mathrm{Hom}(\mathbb{C}^d_L, \mathbb{C}^n) \longrightarrow \mathrm{Hom}(\mathbb{C}^d_R, \mathrm{Sym}^{\bullet}_d\mathbb{C}^n)$ defined in (4.2.3).

- -$\mathcal{F}_d(n)$: the subspace of nondegenerate systems in $\mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Sym}^{\bullet}_d\mathbb{C}^n)$, see (4.2.7).
  -$\mathcal{F}_d^{\mathrm{reg}}(n)$: the regular part of $\mathcal{F}_d(n)$, see (4.2.8).
  -$\widetilde{\mathcal{F}}_d(n)$: this is $\mathcal{F}_d(n)/B_R$: see Lemma 4.2.2.

- $\varepsilon, \tilde{\varepsilon}$: ε is an element of $\mathcal{F}_d(n)$, its image under the projection is $\tilde{\varepsilon} \in \widetilde{\mathcal{F}}_d(n)$, see Definition 4.2.4.

- V : The bundle over $\widetilde{\mathcal{F}}_d(n)$ associated to the standard representation of B_R on $\mathbb{C}^d$, see Lemma 4.2.5.

- $\mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^d_L, \mathbb{C}^n)$: the regular part of $\mathrm{Hom}(\mathbb{C}^d, \mathbb{C}^n)$, see (4.2.24).

- $\mathrm{Flag}_d(\mathbb{C}^n) = \mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^d_L, \mathbb{C}^n)/B_d$ the smooth variety of full flags of d-dimensional subspaces of $\mathbb{C}^n$, see Lemma 4.2.12.

- $\gamma_{\mathrm{ref}}, \mathbf{f}_{\mathrm{ref}}, \pi_{\mathrm{Fl}}$ denotes the reference sequence in $\mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^d_L, \mathbb{C}^n)$, the corresponding flag in $\mathrm{Flag}_d(\mathbb{C}^n)$, and the projection from $\mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^d_L, \mathbb{C}^n)$ to $\mathrm{Flag}_d(\mathbb{C}^n)$, respectively, see Definition 4.2.13.

- $\mathrm{Ind}(X)$: X is a space with a left B_L-action, $\mathrm{Ind}(X) = \mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^d_L, \mathbb{C}^n) \times_{B_L} X$, see Definition 4.2.13

- $\mathrm{Ym}^\bullet\mathbb{C}^d_L$: the filtered subspace of $\mathrm{Sym}^\bullet_d\mathbb{C}^n$, introduced in (4.2.31).

- $\mathrm{Hom}^\triangle(\mathbb{C}^d_R, \mathrm{Ym}^\bullet\mathbb{C}^d_L)$: the space of filtration preserving maps, with filtrations introduced in (4.2.31) and (4.2.5), this is a subspace of $\mathrm{Hom}^\triangle(\mathbb{C}^d_R, \mathrm{Sym}^\bullet_d\mathbb{C}^n)$.

- -$\mathcal{E}$: the nondegenerate part of $\mathrm{Hom}^\triangle(\mathbb{C}^d_R, \mathrm{Ym}^\bullet\mathbb{C}^d_L)$, see (4.2.33)
  -$\widetilde{\mathcal{E}}$: the quotient $\mathcal{E}/B_R$, see Proposition; 4.2.15.

- -$\varepsilon_{\mathrm{ref}}$: the reference system $\psi(\gamma_{\mathrm{ref}})$ in $\mathcal{E}$, see (4.2.26).
  -$\tilde{\varepsilon}_{\mathrm{ref}}$: the corresponding point in $\widetilde{\mathcal{E}}$, i.e $\tilde{\varepsilon}_{\mathrm{ref}} = \widetilde{\mathrm{pr}}(\varepsilon_{\mathrm{ref}})$ where $\widetilde{\mathrm{pr}} : \mathcal{E} \to \widetilde{\mathcal{E}}$ is the projection, see Definition 4.2.4.

- -$\mathbf{\Pi}_d$: set of admissible sequences, see Definition 4.3.6.
  -$\mathbf{\Pi}_{\mathcal{O}}$: the set of admissible sequences corresponding to fixed points in $\mathcal{O}$, see (4.4.4).

- $\varepsilon_{\boldsymbol{\pi}}, \tilde{\varepsilon}_{\boldsymbol{\pi}}$: the point of $\mathcal{E}$ corresponding to the admissible sequence $\boldsymbol{\pi}$, and the image $\widetilde{\mathrm{pr}}(\varepsilon_{\boldsymbol{\pi}}) \in \widetilde{\mathcal{E}}$, respectively; see (4.3.12)

- $\mathcal{O} = \overline{B_d\tilde{\varepsilon}_{\mathrm{ref}}} \subset \widetilde{\mathcal{E}}$, the closure of the Borel orbit, see Diagram (4.3.1).

- -$\mathcal{N}_{\boldsymbol{\pi}}$: the linear subspace of $\mathcal{E}$ introduced in Definition 4.3.9.
  -$\mathcal{O}_{\boldsymbol{\pi}}$: the piece of the orbit closure $\mathcal{O}$ in the chart $\mathcal{N}_{\boldsymbol{\pi}}$, see (4.3.18).

- -$u^l_\pi, \hat{u}^l_{\tau|\boldsymbol{\pi}}$: the coordinates on $\mathcal{E}$ and $\mathcal{N}_\pi$ defined in (4.2.30) and (4.3.14), respectively.
  -$\hat{u}^m_\tau$: shortened notation for $\hat{u}^m_{\tau|\boldsymbol{\pi}_{\mathrm{dist}}}$ for the coordinates on the space $\mathcal{N}_{\mathrm{dist}}$.

- -$\boldsymbol{\pi}_{\mathrm{dist}}$: distinguished sequence of partitions, see (4.4.1).
  -$\tilde{\varepsilon}_{\mathrm{dist}}$: the corresponding distinguished fixed point in $\widetilde{\mathcal{E}}$, shortened notation for $\tilde{\varepsilon}_{\boldsymbol{\pi}_{\mathrm{dist}}}$.
  -$\mathcal{N}_{\mathrm{dist}}$: shortened notation for $\mathcal{N}_{\boldsymbol{\pi}_{\mathrm{dist}}}$, the chart corresponding to the distinguished sequence of partitions.
  -$\mathcal{O}_{\mathrm{dist}}$: shortened notation for $\mathcal{O}_{\boldsymbol{\pi}_{\mathrm{dist}}}$.

- -$\widehat{\mathcal{N}}_d \subset \mathcal{N}_{\mathrm{dist}}$ is the linear subspace generated by partitions not longer than 2, see (4.4.22).
  -$\widehat{\mathcal{O}}_d = \widehat{\mathrm{pr}}(\mathcal{O}_{\mathrm{dist}}) \subset \widehat{\mathcal{N}}_d$, see (4.4.23)

- -Q_{Fl}: the equivariant Poincaré dual of the fiber over the reference flag $\mathbf{f}_{\mathrm{ref}}$, see (4.3.4).
  -$Q_{\boldsymbol{\pi}}$: the equivariant Poincaré dual of $\mathcal{O}_{\boldsymbol{\pi}}$ in $\mathcal{N}_{\boldsymbol{\pi}}$, see (4.3.22).
  -Q_{dist}: shortened notation for $Q_{\boldsymbol{\pi}_{\mathrm{dist}}}$; the equivariant Poincaré dual of $\mathcal{O}_{\mathrm{dist}}$ in $\mathcal{N}_{\mathrm{dist}}$.
  -$\widehat{Q}_d$: The equivariant Poincaré dual of $\widehat{\mathcal{O}}_d$ in $\widehat{\mathcal{N}}_d$, see (4.4.23).

- -$I_{\mathcal{O}}$: the ideal of the subvariety $\mathcal{O} \subset \widetilde{\mathcal{E}}$, see Definition 4.4.15.

- deg$(p(\mathbf{z}); S)$,coeff(L, z_l),lead$(q(\mathbf{z}); m)$ denotes the degree, some coefficient and leading coefficient of some polynomials defined in (4.4.2).

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