

# NORMALISERS IN LIMIT GROUPS

MARTIN R. BRIDSON AND JAMES HOWIE

**ABSTRACT.** Let  $\Gamma$  be a limit group,  $S \subset \Gamma$  a non-trivial subgroup, and  $N$  the normaliser of  $S$ . If  $H_1(S, \mathbb{Q})$  has finite  $\mathbb{Q}$ -dimension, then  $S$  is finitely generated and either  $N/S$  is finite or  $N$  is abelian. This result has applications to the study of subdirect products of limit groups.

Limit groups (or  $\omega$ -residually free groups) have received a good deal of attention in recent years, largely due to the work of Z. Sela ([18, 19] *et seq.*). See for example [1, 3, 7, 10, 16].

O. Kharlampovich, A. Myasnikov ([12, 13] *et seq.*) and others (see, for example, [9, 11, 14]) have studied limit groups extensively under the more traditional name of *fully residually free groups*, which appears to have been first introduced by B. Baumslag in [2]. This name reflects the fact that limit groups are precisely those finitely generated groups  $\Gamma$  such that for each finite subset  $T \subset \Gamma$  there exists a homomorphism from  $\Gamma$  to a free group that is injective on  $T$ . Limit groups are also known as  $\exists$ -free groups and  $\omega$ -residually free groups (see [17]).

Examples of limit groups include all finitely generated free or free abelian groups, and the fundamental groups of all closed surfaces of Euler characteristic at most  $-2$ . The free product of finitely many limit groups is again a limit group, which leads to further examples. More sophisticated examples are described in some of the articles cited above.

The purpose of this note is to contribute some results on the subgroup structure of limit groups. (Their relationship to work of others is discussed briefly in Section 3.)

**Theorem 1.** *If  $\Gamma$  is a limit group and  $H \subset \Gamma$  is a finitely generated, non-trivial subgroup, then either  $H$  has finite index in its normaliser or else the normaliser of  $H$  is abelian.*

We shall use the following result to circumvent the difficulty that *a priori* one does not know that normalisers in limit groups are finitely generated.

**Theorem 2.** *Let  $\Gamma$  be a limit group and  $S \subset \Gamma$  a subgroup. If  $H_1(S, \mathbb{Q})$  has finite  $\mathbb{Q}$ -dimension, then  $S$  is finitely generated (and hence is a limit group).*

Theorems 1 and 2 combine to give the result stated in the abstract. Theorem 1 plays an important role in our work on the subdirect products of limit groups [5, 6]. In the present note, we content ourselves with the following easy consequence of Theorem 1.

**Theorem 3.** *Suppose that  $\Gamma_1, \dots, \Gamma_n$  are limit groups and let*

$$S \subset \Gamma_1 \times \cdots \times \Gamma_n$$

*be an arbitrary subgroup. If  $L_i = \Gamma_i \cap S$  is non-trivial and finitely generated for  $i \leq r$ , then a subgroup  $S_0 \subset S$  of finite index splits as a direct product*

$$S_0 = L_1 \times \cdots \times L_r \times (S_0 \cap G_r),$$

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where  $G_r = \Gamma_{r+1} \times \cdots \times \Gamma_n$ .

Thus, up to finite index, the study of subdirect products of limit groups reduces to the case where the normal subgroups  $L_i \triangleleft S$  are all infinitely generated.

The remainder of this paper is structured as follows. In section 1 below we recall the definitions and record some elementary properties of limit groups and  $\omega$ -residually free towers. In section 2 we prove a result about normal subgroups of groups acting on trees. In section 3 we describe a large class of groups with the property that each finitely generated, non-trivial, normal subgroup is of finite index. This class includes all non-abelian limit groups. In sections 4, 5 and 6 we prove Theorems 2, 1 and 3 respectively.

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## 1. LIMIT GROUPS AND TOWERS

Our results rely on Sela's version [19] of the fundamental theorem that characterizes limit groups as the finitely generated subgroups of  $\omega$ -residually free tower groups. An equivalent theorem was proved by Kharlampovich and Myasnikov in [13], which appeared earlier. There,  $\omega$ -residually free tower groups were called NTQ-groups.

**Definition** An  $\omega$ -rft space of height  $h \in \mathbb{N}$  is defined by induction on  $h$ . An  $\omega$ -rft group is the fundamental group of an  $\omega$ -rft space.

A height 0 tower is the wedge (1-point union) of a finite collection of circles, closed hyperbolic surfaces and tori  $\mathbb{T}^n$  (of arbitrary dimension), except that the closed surface of Euler characteristic  $-1$  is excluded.

An  $\omega$ -rft space  $Y_n$  of height  $h$  is obtained from an  $\omega$ -rft space  $Y_{h-1}$  of height  $h-1$  by attaching a *block* of one of the following types:

*Abelian:*  $Y_h$  is obtained from  $Y_{h-1} \sqcup \mathbb{T}^m$  by identifying a coordinate circle in  $\mathbb{T}^m$  with any loop  $c$  in  $Y_{h-1}$  such that  $\langle c \rangle \cong \mathbb{Z}$  is a maximal abelian subgroup of  $\pi_1 Y_{h-1}$ .

*Quadratic:* One takes a connected, compact surface  $\Sigma$  that is either a punctured torus or has Euler characteristic at most  $-2$ , and obtains  $Y_h$  from  $Y_{h-1} \sqcup \Sigma$  by identifying each boundary component of  $\Sigma$  with a homotopically non-trivial loop in  $Y_{h-1}$ ; these identifications must be chosen so that there exists a retraction  $r : Y_h \rightarrow Y_{h-1}$ , and  $r_*(\pi_1 \Sigma) \subset \pi_1 Y_{h-1}$  must be non-abelian.

**Theorem 1.1.** *A given group is a limit group if and only if it is isomorphic to a finitely generated subgroup of an  $\omega$ -rft group.*

A useful sketch of the proof can be found in [1, Appendix].

This is a powerful theorem that allows one to prove many interesting facts about limit groups by induction on height.

The *height* of a limit group  $S$  is the minimal height of an  $\omega$ -rft group that has a subgroup isomorphic to  $S$ . Limit groups of height 0 are free products of finitely many free abelian groups and of surface groups of Euler characteristic at most  $-2$ .

The splitting described in the following lemma is that which the Seifert-van Kampen Theorem associates to the addition of the final block in the tower construction.

**Lemma 1.2.** *The fundamental group of an  $\omega$ -rft space of height  $h \geq 1$  splits as a 2-vertex graph of groups: one of the vertex groups  $A_1$  is the fundamental group of an  $\omega$ -rft space of height  $h-1$  and the other  $A_2$  is a free or free-abelian group of finite rank at least 2; the*

edge groups are maximal infinite cyclic subgroups of  $A_2$ . If  $A_2$  is abelian then there is only one edge group and this is a maximal abelian subgroup of  $A_1$ .

Since an arbitrary limit group (and hence an arbitrary subgroup of a limit group) is a subgroup of an  $\omega$ -rft group, one can apply Bass-Serre theory to deduce:

**Lemma 1.3.** *If  $S$  is a subgroup of a limit group of height  $h \geq 1$ , then  $S$  is the fundamental group of a bipartite graph of groups in which the edge groups are cyclic; the vertex groups fall into two types corresponding to the bipartite partition of vertices: type (i) vertex groups are isomorphic to subgroups of a limit group of height  $(h - 1)$ ; type (ii) vertex groups are all free or all free-abelian.*

Restricting attention to limit groups (rather than arbitrary subgroups) we will also use the following variant of the above decomposition.

**Lemma 1.4.** *If  $\Gamma$  is a freely indecomposable limit group of height  $h \geq 1$ , then it is the fundamental group of a finite graph of groups that has infinite cyclic edge groups and has a vertex group that is a non-abelian limit group of height  $\leq h - 1$ .*

*Proof.* Every limit group arises as a finitely generated subgroup of an  $\omega$ -rft group of the same height. Suppose  $L$  is an  $\omega$ -rft group of height  $h$  that contains  $\Gamma$  as a subgroup. Since  $\Gamma$  is finitely generated, the graph of groups structure given by Lemma 1.3 can be replaced by the finite core graph  $\mathcal{C}$  obtained by taking the quotient by  $\Gamma$  of a *minimal*  $\Gamma$ -invariant subtree of the Bass-Serre tree for  $L$ . The vertex groups of  $\mathcal{C}$  are finitely generated, and since  $\Gamma$  does not have height  $\leq h - 1$ , there is at least one vertex group of each type (in the terminology of Lemma 1.3). As  $\Gamma$  is freely indecomposable, the edge groups in  $\mathcal{C}$  are infinite cyclic.

If the block added at the top of the tower for  $L$  is quadratic, then each vertex group  $\Gamma_v$  of type (ii) in  $\mathcal{C}$  must be a non-abelian free group (and hence a non-abelian limit group of height  $0 \leq h - 1$ ). Indeed, distinct conjugates of the edge groups incident at the corresponding vertex group in  $L$  are maximal cyclic subgroups of the vertex group and generate a non-abelian free subgroup. Thus if  $\Gamma_v$  were cyclic, then there would be a single incident edge  $e$  at  $v$  and  $\Gamma_e \rightarrow \Gamma_v$  would be an isomorphism, contradicting the fact that  $\mathcal{C}$  was chosen to be minimal.

To complete the proof we shall argue that if the block added at the top of the tower for  $L$  is abelian, then each vertex of type (i) in  $\mathcal{C}$  is non-abelian.

We have  $L = A_1 *_{\langle \zeta \rangle} A_2$ , as in Lemma 1.2, where  $\langle \zeta \rangle \cong \mathbb{Z}$ . In  $\mathcal{C}$ , the edge groups incident at each vertex of type (i) (which are conjugates of subgroups of  $\langle \zeta \rangle$ ) intersect trivially and are maximal abelian. Thus if a vertex group  $\Gamma_v$  of type (i) were abelian, then it would be cyclic, there would be only one incident edge  $e$ , and the inclusion  $\Gamma_e \rightarrow \Gamma_v$  would be an isomorphism. As above, this would contradict the minimality of  $\mathcal{C}$ .  $\square$

## 2. NORMAL SUBGROUPS IN BASS-SERRE THEORY

As was evident in the previous section, we are assuming that the reader is familiar with the basic theory of groups acting on trees. We need the following fact from this theory.

**Proposition 2.1.** *If a finitely generated group  $\Gamma$  acts by isometries on a tree  $T$ , then either  $\Gamma$  fixes a point of  $T$  or else  $\Gamma$  contains a hyperbolic isometry.*

*In the latter case, the union of the axes of the hyperbolic elements is the unique minimal  $\Gamma$ -invariant subtree of  $T$ .*

The following dichotomy for normal subgroups in Bass-Serre theory ought to be well known but we are unaware of any appearance of it in the literature. (The results of [15] are similar in spirit but different in content.)

The first of the stated outcomes arises, for example, when  $\Gamma = G_0 *_A G_1$  and  $A$  is normal in both  $G_0$  and  $G_1$ . The second outcome arises, for example, when  $\Gamma$  is the amalgamation of  $N_0 \rtimes A$  and  $N_1 \rtimes A$ , and  $N = N_0 * N_1$ .

**Proposition 2.2.** *If the group  $\Gamma$  splits over the subgroup  $A \subset \Gamma$  and the normal subgroup  $N \triangleleft \Gamma$  is finitely generated, then either  $N \subset A$  or else  $AN$  has finite index in  $\Gamma$ .*

*Proof.* Consider the action of  $N$  on the Bass-Serre tree  $T$  associated to the splitting of  $\Gamma$  over  $A$ . If the fixed-point set of  $N$  is non-empty, then it is the whole of  $T$ , because it is  $\Gamma$ -invariant and the action is minimal. Thus  $N$  lies in the edge-stabiliser  $A$ .

If  $N$  does not have a fixed point, then by the preceding proposition it must contain hyperbolic isometries and the union of the axes of its hyperbolic elements is the minimal  $N$ -invariant subtree of  $T$ . Since  $N$  is normal this tree is also  $\Gamma$ -invariant, hence equal to  $T$ .

The edges of  $T$  are indexed by the cosets  $\Gamma/A$  and the action of  $g \in \Gamma$  is  $g.\gamma A = (g\gamma)A$ . Thus the edges of the graph  $X = N \backslash T$  are indexed by the double cosets  $N \backslash \Gamma / A$ . The quotient  $\Gamma/N$  acts on  $X$  and the stabiliser of the edge indexed  $NgA$  is the image in  $\Gamma/N$  of  $gAg^{-1}$ . It follows that some (hence every) edge in  $X$  has a finite orbit if and only the image of  $A$  in  $\Gamma/N$  is of finite index.

But  $X$  is a finite graph. Indeed, since  $N$  is finitely generated, there is a compact subgraph  $Y \subset X$  such that the inclusion of the corresponding graph of groups induces an isomorphism of fundamental groups, which implies that the action of  $N$  preserves the (connected) preimage of  $Y$  in  $T$ . Since the action of  $N$  on  $T$  is minimal, this preimage is the whole of  $T$ .  $\square$

### 3. A CLASS OF GROUPS WITH RESTRICTED NORMAL SUBGROUPS

In this section we describe a large class of groups that have the property that each of their non-trivial, finitely generated, normal subgroups is of finite index. This leads to the following special case of Theorem 1.

**Theorem 3.1.** *Let  $\Gamma$  be a non-abelian limit group and let  $N \neq 1$  be a normal subgroup. If  $N$  is finitely generated, then  $\Gamma/N$  is finite.*

Let  $\mathcal{J}$  denote the class of groups that have no proper infinite quotients<sup>1</sup>. (Thus  $\mathcal{J}$  is the union of the class of finite groups and the class of just-infinite groups.) Let  $\mathcal{G}$  be the smallest class of groups that contains  $\mathbb{Z}$  and all non-trivial free products of groups, and satisfies the following condition:

- \* If  $H$  is in  $\mathcal{G}$  and  $A$  is in  $\mathcal{J}$ , and if  $A$  has infinite index in  $H$ , then any amalgamated free product of the form  $H *_A B$  is in  $\mathcal{G}$ , and so is any HNN extension of the form  $H *_A$ .

**Theorem 3.2.** *If  $\Gamma$  is in  $\mathcal{G}$ , then every finitely generated, non-trivial, normal subgroup in  $\Gamma$  is of finite index.*

*Proof.* Given  $\Gamma$ , we argue by induction on its *level* in  $\mathcal{G}$ , namely the number of steps in which  $\Gamma$  is constructed by taking amalgamated free products and HNN extensions as in (\*).

<sup>1</sup>Richard Weidmann has pointed out that one could instead take  $\mathcal{J}$  to be the class of slender groups.

At level 0 we have  $\Gamma \cong \mathbb{Z}$  or  $\Gamma$  a non-trivial free product. In the former case the result is clear; in the latter it follows from Proposition 2.2 with  $A = \{1\}$ .

Assume that  $\Gamma = H *_A B$  or  $\Gamma = H *_A$ , where every finitely generated normal subgroup of  $H$  has finite index and  $A \in \mathcal{J}$  has infinite index in  $H$ . Let  $N \subset \Gamma$  be finitely generated and normal. According to Proposition 2.2, either  $N \subset A$  or else  $NA$  has finite index in  $\Gamma$ . The former case is impossible because it would imply  $N \subset H$  and by induction  $|H/N|$  would be finite, implying that  $|H/A|$  is finite, contrary to hypothesis. Thus the image of  $A$  has finite index in  $\Gamma/N$ . Since  $A$  has no proper infinite quotients, it follows that either  $N$  has finite index in  $\Gamma$  (as desired), or else  $A$  is infinite and  $N \cap A = \{1\}$ .

We claim that this last possibility is absurd. First note that since  $A$  has infinite index in  $H$  and projects to a subgroup of finite index in  $\Gamma/N$ , we know that  $H \cap N$  is non-trivial. Secondly, because  $A$  (and hence each of its conjugates in  $\Gamma$ ) intersects  $N$  trivially, the quotient of the Bass-Serre tree for  $\Gamma$  gives a graph of groups decomposition for  $N$  with trivial edge groups. One of the vertex groups in this decomposition is  $H \cap N$ . Thus  $H \cap N$  is a free factor of the finitely generated group  $N$ , and hence is finitely generated. By induction, it follows that  $|H/(H \cap N)|$  is finite. Hence  $|A/(A \cap N)|$  is finite, which is the contradiction we seek.  $\square$

**Proposition 3.3.** *All non-abelian limit groups lie in  $\mathcal{G}$ .*

*Proof.* This follows easily from the fact that an arbitrary limit group  $\Gamma$  is a subgroup of an  $\omega$ -rft group. A non-abelian limit group of height 0 is either a non-trivial free product or a surface group of Euler characteristic at most  $-2$  (which therefore splits as  $F *_\mathbb{Z}$  with  $F$  free). Such groups clearly lie in  $\mathcal{G}$ , so we may assume that the tower has height  $h \geq 1$  and that  $\Gamma$  is freely indecomposable.

As in Lemma 1.4, we decompose  $\Gamma$  as  $\pi_1 \mathcal{C}$ . That lemma singles out a non-abelian vertex group  $\Gamma_v$  that is a limit group of height  $\leq h - 1$ . By induction  $\Gamma_v$  lies in  $\mathcal{G}$ . By conflating the remaining vertices of  $\mathcal{C}$  to a single vertex, we can express  $\Gamma$  as the fundamental group of a 2-vertex graph of groups where one vertex group is  $\Gamma_v$  and the other is the fundamental group of the full subgraph of groups spanned by the remaining vertices of  $\mathcal{C}$ ; the edge groups are cyclic. A secondary induction on the number of edges in the decomposition completes the proof.  $\square$

If  $\Gamma$  is a hyperbolic limit group then it has cohomological dimension at most 2, so by a theorem of Bieri [4] its finitely presented normal subgroups are free or of finite index. Since any finitely generated subgroup of a limit group is a limit group, and limit groups are finitely presented, the novel content of Theorem 3.1 in the hyperbolic case is essentially contained in:

**Corollary 3.4.** *If  $F$  is a finitely generated, non-abelian free group, then no group of the form  $F \rtimes \mathbb{Z}$  is a limit group.*

The arguments of Ilya Kapovich [11] already cover Theorem 3.1 and Theorem 1 in the hyperbolic case. The referee of the current paper suggested an alternative proof for the general case based on Proposition 7 of [14].

In the version of this paper posted in May 2005 we noted that Theorem 1 was in keeping with the expectation that finitely generated subgroups of limit groups should be quasi-isometrically embedded (cf. [8]). Henry Wilton [21] has since proved a stronger result: if  $\Gamma$  is a limit group and  $H$  is a finitely generated subgroup, then there exists a finite-index subgroup  $\Gamma_0 \subset \Gamma$  that contains  $H$  as a retract.

If the normaliser  $N(H)$  intersects  $\Gamma_0$  in  $H$  then Theorem 1 follows immediately from Wilton's theorem. On the other hand, if  $t \in \Gamma_0 \setminus H$  normalizes  $H$  then the existence of a retraction  $\Gamma_0 \rightarrow H$  implies that for some  $h \in H$  the centraliser  $C(H)$  contains  $th^{-1} \neq 1$ . But limit groups are commutative-transitive, so this can only happen if  $H$  and  $C(H)$  are abelian and  $t \in C(H)$ . Thus  $\Gamma_0 \cap N(H)$  is abelian. And since limit groups are torsion-free, the commutative-transitive property implies that  $N(H)$  is also abelian.

#### 4. HOMOLOGICAL FINITENESS AND FINITE GENERATION

The purpose of this section is to prove Theorem 2, which is required in order to prove that the normaliser of any finitely generated subgroup of a limit group is finitely generated. Throughout this section, we work with rational homology. However, the proofs can easily be adapted to give analogous results for homology with coefficients in any field, or in  $\mathbb{Z}$ .

**Lemma 4.1.** *If  $\Gamma$  is a residually free group and  $S \subset \Gamma$  is a non-cyclic subgroup, then  $H_1(S, \mathbb{Q})$  has dimension at least 2 over  $\mathbb{Q}$ .*

*Proof.* The lemma is obvious if  $S$  is abelian, because  $\Gamma$  is torsion-free and contains no infinitely divisible elements. If  $S$  is non-abelian, then there exist  $s, t \in S$  that do not commute. Since  $\Gamma$  is residually free, there exists a homomorphism  $\phi : \Gamma \rightarrow F$  with  $F$  free and  $\phi([s, t]) \neq 1$ . Thus  $S$  maps onto a non-abelian free group, namely  $\phi(S)$ , and hence onto a free abelian group of rank at least 2.  $\square$

Recall the statement of Theorem 2.

**Theorem 2.** *Let  $\Gamma$  be a limit group and let  $S \subset \Gamma$  be a subgroup. If  $H_1(S, \mathbb{Q})$  has finite  $\mathbb{Q}$ -dimension, then  $S$  is finitely generated (and hence a limit group).*

*Proof.* Without loss of generality we may assume that  $\Gamma$  is an  $\omega$ -rft group, of height  $h$  say. If  $h = 0$  then  $S$  is a free product of free groups, surface groups and free abelian groups, and the result is clear. We may assume therefore that  $h > 1$ . We consider the action of  $S$  on the Bass-Serre tree of the tower, pass to a *minimal*  $S$ -invariant subtree and form the quotient graph of groups. Thus we obtain a graph of groups decomposition  $S = \pi_1(\mathcal{S}, X)$  of  $S$ , with  $X$  bipartite, in which the edge groups are cyclic and the vertex groups are of two kinds corresponding to the bipartite partition of vertices (cf. Lemma 1.3): type (i) vertex groups are the intersections of  $S$  with conjugates of a limit group  $B \subset \Gamma$  of height  $(h - 1)$ ; type (ii) are the intersections of  $S$  with the conjugates of the free or free-abelian group at the top of the tower for  $\Gamma$ .

It is enough to prove the theorem for freely indecomposable subgroups, so we may assume that all of the edge groups are infinite cyclic.

*Claim:* *If the graph  $X$  is finite, then  $S$  is finitely generated.*

To see this, note that  $H_1(S_e, \mathbb{Q}) \cong \mathbb{Q}$  for each of the finitely many edges  $e$  of  $X$ , so the Mayer-Vietoris sequence for  $S$  shows that  $H_1(S_v, \mathbb{Q})$  is finite-dimensional for each vertex  $v$  of  $X$ . By inductive hypothesis, each  $S_v$  is finitely generated, and hence so is  $S$ .

It remains to prove that  $X$  is finite. Certainly, since  $H_1(S, \mathbb{Q})$  is finite-dimensional, there is a finite connected subgraph  $Y$  of  $X$ , such that the inclusion-induced map

$$H_1(\pi_1(\mathcal{S}, Y), \mathbb{Q}) \rightarrow H_1(\pi_1(\mathcal{S}, X), \mathbb{Q}) = H_1(S, \mathbb{Q})$$

is surjective.

In particular,  $Y$  must contain all the simple closed paths in  $X$ , so all edges of  $X \setminus Y$  separate  $X$ . Suppose that  $e$  is such an edge. Let  $Z_0$  be the component of  $X \setminus \{e\}$  that

contains  $Y$ , and  $Z_1$  the other component. Then  $e$  gives rise to a splitting of  $S$  as an amalgamated free product

$$S = S_0 *_C S_1$$

with  $C = S_e$  infinite cyclic, and  $H_1(S_0, \mathbb{Q}) \rightarrow H_1(S, \mathbb{Q})$  surjective.

The Mayer-Vietoris sequence of this decomposition shows that  $H_1(C, \mathbb{Q}) \rightarrow H_1(S_1, \mathbb{Q})$  is surjective, so  $H_1(S_1, \mathbb{Q})$  has dimension at most 1. Since  $\Gamma$  is residually free, Lemma 4.1 tells us that  $S_1$  must be cyclic. It follows that all the vertex groups of  $Z_1$  are cyclic.

By the argument given in the proof of Lemma 1.4, the minimality of  $X$  means that vertex groups in  $X$  of at most one type can be cyclic (type (i) if the top of the tower for  $\Gamma$  is a quadratic block; type (ii) if it is an abelian block). Hence  $Z_1$  consists of a single vertex,  $u$  say. Moreover, if the top of the tower is an abelian block, then  $u$  is of type (ii), and  $S_e$  is a maximal cyclic subgroup of  $S_u$ , whence  $S_e = S_u$ , which contradicts the minimality of  $X$ .

We are reduced to the case where the top of the tower is a quadratic block, and  $X$  consists of the finite graph  $Y$ , together with a (possibly infinite) collection of type (i) vertices of valence 1, each attached to a type (ii) vertex of  $Y$  and each with infinite cyclic vertex group. It follows easily that  $H_1(\pi_1(\mathcal{S}, Y), \mathbb{Q}) \rightarrow H_1(S, \mathbb{Q})$  is an isomorphism, so  $H_1(\pi_1(\mathcal{S}, Y), \mathbb{Q})$  is finite-dimensional. Since  $Y$  is a finite graph and the edge groups are cyclic, it follows from the Mayer-Vietoris sequence for  $\pi_1(\mathcal{S}, Y)$  that  $H_1(S_v, \mathbb{Q})$  is finite-dimensional for each vertex  $v$  of  $Y$ .

To complete the proof that  $X$  is finite, we need only show that a type (ii) vertex  $v$  of  $Y$  can have only finitely many neighbours in  $X$ . To see this, note that  $S_v$  is a surface group, and the edge-groups of the edges incident at  $v$  form a subset of the nontrivial peripheral subgroups of  $S_v$ . Thus  $v$  is incident to at most  $1 + \dim_{\mathbb{Q}}(H_1(S_v, \mathbb{Q})) < \infty$  edges of  $X$ , as required.  $\square$

## 5. NORMALISERS ARE FINITELY GENERATED

In this section we prove Theorem 1:

**Theorem 1.** *If  $\Gamma$  is a limit group and  $H \subset \Gamma$  is a finitely generated, non-trivial subgroup, then either  $H$  has finite index in its normaliser or else the normaliser of  $H$  is abelian.*

Since subgroups of fully residually free groups are fully residually free, finitely generated subgroups of limit groups are limit groups. Thus Theorem 1 is an immediate consequence of Theorem 3.1 and the following result.

**Theorem 5.1.** *If  $\Gamma$  is a limit group and  $H \subset \Gamma$  is a finitely generated subgroup, then the normaliser in  $\Gamma$  of  $H$  is finitely generated.*

*Proof.* Let  $X$  be a finite generating set for  $H$  and let  $N$  be the normaliser of  $H$  in  $\Gamma$ . All abelian subgroups of limit groups are finitely generated [3], so we assume that  $N$  is not abelian.

Each finitely generated subgroup  $M \subset N$  is a limit group, so if  $X \subset M$ , then  $H$  has finite index in  $M$ , by Theorem 3.1. In particular, the dimension of  $H_1(M, \mathbb{Q})$  is bounded above by that of  $H_1(H, \mathbb{Q})$ . Since  $N$  is the union of its finitely generated subgroups and homology commutes with direct limits,  $H_1(N, \mathbb{Q})$  has finite dimension. Hence  $N$  is finitely generated, by Theorem 2.  $\square$

## 6. A SPLITTING THEOREM

The purpose of this section is to prove Theorem 3. In the statement we do not assume that  $S$  is finitely generated.

**Theorem 3.** *Suppose that  $\Gamma_1, \dots, \Gamma_n$  are limit groups and let*

$$S \subset \Gamma_1 \times \cdots \times \Gamma_n$$

*be an arbitrary subgroup. If  $L_i = \Gamma_i \cap S$  is non-trivial and finitely generated for  $i \leq r$ , then a subgroup  $S_0 \subset S$  of finite index splits as a direct product*

$$S_0 = L_1 \times \cdots \times L_r \times (S_0 \cap G_r),$$

*where  $G_r = \Gamma_{r+1} \times \cdots \times \Gamma_n$ .*

*Proof.* Let  $P_i$  be the projection of  $S$  to  $\Gamma_i$ . Since  $L_i$  is normal in  $S$ , it is normal in  $P_i$ . Finitely generated subgroups of limit groups are limit groups, so it follows from Theorem 3.1 that  $L_i$  has finite index in  $P_i$ .

We have shown that  $D = L_1 \times \cdots \times L_r$  has finite index in the projection of  $S$  to  $\Gamma_1 \times \cdots \times \Gamma_r$ . Thus the inverse image  $S_0 \subset S$  of  $D$  is of finite index, and by splitting the short exact sequence  $1 \rightarrow (S_0 \cap G_r) \rightarrow S_0 \rightarrow D \rightarrow 1$  of the projection we obtain the desired direct product decomposition of  $S_0$ .  $\square$

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**Authors’ addresses**

Martin R. Bridson  
Department of Mathematics  
Imperial College London  
London SW7 2AZ  
`m.bridson@imperial.ac.uk`

James Howie  
Department of Mathematics  
Heriot-Watt University  
Edinburgh EH14 4AS  
`J.Howie@hw.ac.uk`