

# THE DEHN FUNCTIONS OF $Out(F_n)$ AND $Aut(F_n)$

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ABSTRACT. For  $n \geq 3$ , the Dehn functions of  $Aut(F_n)$  and  $Out(F_n)$  are exponential.

Dehn functions provide upper bounds on the complexity of the word problem in finitely presented groups. They are examples of filling functions: if a group  $G$  acts properly and cocompactly on a simplicial complex  $X$ , then the Dehn function of  $G$  is asymptotically equivalent to the function that provides the optimal upper bound on the area of least-area discs in  $X$ , where the bound is expressed as a function of the length of the boundary of the disc. This article is concerned with the Dehn functions of automorphism groups of finitely-generated free groups.

Much of the contemporary study of  $Out(F_n)$  and  $Aut(F_n)$  is based on the deep analogy between these groups, mapping class groups, and lattices in semisimple Lie groups, particularly  $SL(n, \mathbb{Z})$ . The Dehn functions of mapping class groups are quadratic [9], as is the Dehn function of  $SL(n, \mathbb{Z})$  if  $n \geq 5$  (see [10]). In contrast, Epstein *et al.* [6] proved that the Dehn function of  $SL(3, \mathbb{Z})$  is exponential. Building on their result, we proved in [3] that  $Aut(F_3)$  and  $Out(F_3)$  also have exponential Dehn functions. Hatcher and Vogtmann [8] established an exponential upper bound on the Dehn function of  $Aut(F_n)$  and  $Out(F_n)$  for all  $n \geq 3$ . The comparison with  $SL(n, \mathbb{Z})$  might lead one to suspect that this last result is not optimal for large  $n$ , but in fact it is.

**Theorem 1.** *For  $n \geq 3$ , the Dehn functions of  $Aut(F_n)$  and  $Out(F_n)$  are exponential.*

This theorem answers Questions 35 and 37 of [4]. The lower bound needed to complete the proof is contained in a recent paper of Handel and Mosher [7]: they used their general results on quasi-retractions to reduce to the case  $n = 3$ . We learned of this work from Lee Mosher at Luminy in June 2010 and realized that one can also reduce Theorem 1 to the case  $n = 3$  using a simple observation about natural maps between different-rank Outer spaces and Auter spaces (Lemma 3). The purpose of this note is record this observation and the resulting proof of Theorem 1.

**0.1. Definitions.** Let  $A$  be a 1-connected simplicial complex. We consider simplicial loops  $\ell: S \rightarrow A^{(1)}$ , where  $S$  is a simplicial subdivision of the circle. A *simplicial filling* of  $\ell$  is a simplicial map  $L: D \rightarrow A^{(2)}$ , where  $D$  is a triangulation of the 2-disc and  $L|_{\partial D} = \ell$ . Such fillings always exist, by simplicial approximation. The filling area of  $\ell$ , denoted  $\text{Area}_A(\ell)$ , is the least number of triangles in the domain of any simplicial filling of  $\ell$ .

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The *Dehn function*<sup>1</sup> of  $A$  is the least function  $\delta_A : \mathbb{N} \rightarrow \mathbb{N}$  such that  $\text{Area}_A(\ell) \leq \delta_A(n)$  for all loops of length  $\leq n$  in  $A^{(1)}$ . The Dehn function of a finitely presented group  $G$  is the Dehn function of any 1-connected 2-complex on which it acts simplicially with finite stabilizers and compact quotient. This is well-defined up to  $\simeq$  equivalence and can be rephrased in terms of the complexity of the word problem for  $G$  — see [2]. (Two functions  $f, g : \mathbb{N} \rightarrow \mathbb{N}$  are  $\simeq$  equivalent if  $f \preceq g$  and  $g \preceq f$ , where  $f \preceq g$  means that there is a constant  $a > 1$  such that  $f(n) \leq a g(an + a) + an + a$ .)

**Lemma 1.** *If  $A$  and  $B$  are 1-connected simplicial complexes,  $F : A \rightarrow B$  is a simplicial map, and  $\ell$  is a loop in the 1-skeleton of  $A$ , then  $\text{Area}_A(\ell) \geq \text{Area}_B(F \circ \ell)$ .*

*Proof.* If  $L : D \rightarrow A$  is a simplicial filling of  $\ell$ , then  $F \circ L$  is a simplicial filling of  $F \circ \ell$ , with the same number of triangles in the domain  $D$ .  $\square$

**Corollary 1.** *Let  $A, B$  and  $C$  be 1-connected simplicial complexes with simplicial maps  $A \rightarrow B \rightarrow C$ . Let  $\ell_n$  be a sequence of simplicial loops in  $A$  whose length is bounded by a linear function of  $n$ , let  $\bar{\ell}_n$  be the image loops in  $C$  and let  $\alpha(n) = \text{Area}_C(\bar{\ell}_n)$ . Then the Dehn function of  $B$  satisfies  $\delta_B(n) \succeq \alpha(n)$ .*

**0.2. Simplicial complexes associated to  $\text{Aut}(F_n)$  and  $\text{Out}(F_n)$ .** Let  $K_n$  denote the spine of Outer space, as defined in [5], and  $L_n$  the spine of Auter space, as defined in [8]. (We allow marked graphs representing the points of these complexes to have separating edges.)

A vertex of  $K_n$  can be considered either as a marked graph  $(g, G)$  or as a free minimal action of  $F_n$  on a simplicial tree (namely the universal cover of  $G$ ). A vertex of  $L_n$  has the same descriptions except that there is a chosen basepoint in the marked graph (respected by the marking) or in the simplicial tree. Both  $K_n$  and  $L_n$  are flag complexes, so to define them it suffices to describe what it means for vertices to be adjacent. In the marked-graph description, vertices of  $K_n$  (or  $L_n$ ) are adjacent if one can be obtained from the other by a forest collapse (i.e. collapsing each component of a forest to a point).

**0.3. Three Natural Maps.** There is a *forgetful map*  $\phi_n : L_n \rightarrow K_n$  which simply forgets the basepoint; this map is simplicial.

Let  $m < n$ . We fix an ordered basis for  $F_n$ , identify  $F_m$  with the subgroup generated by the first  $m$  elements of the basis, and identify  $\text{Aut}(F_m)$  with the subgroup of  $\text{Aut}(F_n)$  that fixes the last  $n - m$  basis elements. We consider two maps associated to this choice of basis.

First, there is an equivariant *augmentation map*  $\iota : L_m \rightarrow L_n$  which attaches a bouquet of  $n - m$  circles to the basepoint of each marked graph and marks them with the last  $n - m$  basis elements of  $F_n$ . This map is simplicial, since a forest collapse has no effect on the bouquet of circles at the basepoint.

Secondly, there is a *restriction map*  $\rho : K_n \rightarrow K_m$  which is easiest to describe using trees. A point in  $K_n$  is a minimal free simplicial action of  $F_n$  on a tree  $T$  with no vertices of valence 2. We define  $\rho(T)$  to be the minimal invariant subtree for  $F_m < F_n$ ; more explicitly,  $\rho(T)$  is the union of the axes in  $T$  of all elements of  $F_m$ . (Vertices of  $T$  that have valence 2 in  $\rho(T)$  are no longer considered to be vertices.)

<sup>1</sup>The standard definition of area and Dehn function are phrased in terms of singular discs, but this version is  $\simeq$  equivalent.

One can also describe  $\rho$  in terms of marked graphs. The chosen embedding  $F_m < F_n$  corresponds to choosing an  $m$ -petal subrose  $R_m \subset R_n$ . A vertex in  $K_n$  is given by a graph  $G$  marked with a homotopy equivalence  $g : R_n \rightarrow G$  and the restriction of  $g$  to  $R_m$  lifts to a homotopy equivalence  $\widehat{g} : R_m \rightarrow \widehat{G}$ , where  $\widehat{G}$  is the covering space corresponding to  $g_*(F_m)$ . There is a canonical retraction  $r$  of  $\widehat{G}$  onto its compact core. Let  $\widehat{G}_0$  be the graph obtained by erasing all vertices of valence 2 from the compact core and define  $\rho(g, G) = (r \circ \widehat{g}, \widehat{G}_0)$ .

**Lemma 2.** *For  $m < n$ , the restriction map  $\rho : K_n \rightarrow K_m$  is simplicial.*

*Proof.* Any forest collapse in  $G$  is covered by a forest collapse in  $\widehat{G}$  that preserves the compact core, so  $\rho$  preserves adjacency.  $\square$

**Lemma 3.** *For  $m < n$ , the following diagram of simplicial maps commutes :*

$$\begin{array}{ccc} L_m & \xrightarrow{\iota} & L_n \\ \phi_m \downarrow & & \downarrow \phi_n \\ K_m & \xleftarrow{\rho} & K_n \end{array}$$

*Proof.* Given a marked graph with basepoint  $(g, G; v) \in L_n$ , the marked graph  $\iota(g, G; v)$  is obtained by attaching  $n - m$  loops at  $v$  labelled by the elements  $a_{m+1}, \dots, a_n$  of our fixed basis for  $F_n$ . Then  $(g_n, G_n) := \phi_n \circ \iota(g, G; v)$  is obtained by forgetting the basepoint, and the cover of  $(g_n, G_n)$  corresponding to  $F_m < F_n$  is obtained from a copy of  $(g, G)$  (with its labels) by attaching  $2(n - m)$  trees. (These trees are obtained from the Cayley graph of  $F_n$  as follows: one cuts at an edge labelled  $a_i^\varepsilon$ , with  $i \in \{m + 1, \dots, n\}$  and  $\varepsilon = \pm 1$ , takes one component of the result, and then attaches the hanging edge to the basepoint  $v$  of  $G$ .) The effect of  $\rho$  is to delete these trees.  $\square$

**0.4. Proof of the Theorem.** In the light of Corollary 1 and Lemma 3, it suffices to exhibit a sequence of loops  $\ell_i$  in the 1-skeleton of  $L_3$  whose lengths are bounded by a linear function of  $i$  and whose filling area when projected to  $K_3$  grows exponentially as a function of  $i$ . Such a sequence of loops is essentially described in [3]. What we actually described there were words in the generators of  $Aut(F_3)$  rather than loops in  $L_3$ , but standard quasi-isometric arguments show that this is equivalent. More explicitly, the words we considered were  $w_i = T^i A T^{-i} B T^i A^{-1} T^{-i} B^{-1}$  where

$$T: \begin{cases} x_1 \mapsto x_1^2 x_2 \\ x_2 \mapsto x_1 x_2 \\ x_3 \mapsto x_3 \end{cases} \quad A: \begin{cases} x_1 \mapsto x_1 \\ x_2 \mapsto x_2 \\ x_3 \mapsto x_1 x_3 \end{cases} \quad B: \begin{cases} x_1 \mapsto x_1 \\ x_2 \mapsto x_2 \\ x_3 \mapsto x_3 x_2 \end{cases}$$

To interpret these as loops in the 1-skeleton of  $L_3$  (and  $K_3$ ) we note that  $A = \lambda_{31}$  and  $B = \rho_{32}$  are elementary transvections and  $T$  is the composition of two elementary transvections:  $T = \lambda_{21} \circ \rho_{12}$ . Thus  $w_i$  is the product of  $8i + 4$  elementary transvections. There is a (connected) subcomplex of the 1-skeleton of  $L_3$  spanned by roses (graphs with a single vertex) and Nielsen graphs (which have  $(n - 2)$  loops at the base vertex and a further trivalent vertex). We say roses are adjacent if they have distance 2 in this graph.

Let  $I \in L_3$  be the rose marked by the identity map. Each elementary transvection  $\tau$  moves  $I$  to an adjacent rose  $\tau I$ , which is connected to  $I$  by a Nielsen graph  $N_\tau$ . A composition  $\tau_1 \dots \tau_k$  of elementary transvections gives a path through adjacent roses

$I, \tau_1 I, \tau_1 \tau_2 I, \dots, \tau_1 \tau_2 \dots \tau_k I$ ; the Nielsen graph connecting  $\sigma I$  to  $\sigma \tau I$  is  $\sigma N_\tau$ . Thus the word  $w_i$  corresponds to a loop  $\ell_i$  of length  $16i + 8$  in the 1-skeleton of  $L_3$ . Theorem A of [3] provides an exponential lower bound on the filling area of  $\phi \circ \ell_i$  in  $K_3$ .  $\square$

The square of maps in Lemma 3 ought to have many uses beyond the one in this note (cf. [7]). We mention just one, for illustrative purposes. This is a special case of the fact that every infinite cyclic subgroup of  $Out(F_n)$  is quasi-isometrically embedded [1].

**Proposition 1.** *The cyclic subgroup of  $Out(F_n)$  generated by any Nielsen transformation (elementary transvection) is quasi-isometrically embedded.*

*Proof.* Each Nielsen transformation is in the image of the map  $\Phi : Aut(F_2) \rightarrow Aut(F_n) \rightarrow Out(F_n)$  given by the inclusion of a free factor  $F_2 < F_n$ . Thus it suffices to prove that if a cyclic subgroup  $C = \langle c \rangle < Aut(F_2)$  has infinite image in  $Out(F_2)$ , then  $t \mapsto \Phi(c^t)$  is a quasi-geodesic. This is equivalent to the assertion that some (hence any)  $C$ -orbit in  $K_n$  is quasi-isometrically embedded, where  $C$  acts on  $K_n$  as  $\Phi(C)$  and  $K_n$  is given the piecewise Euclidean metric where all edges have length 1.

$K_2$  is a tree and  $C$  acts on  $K_2$  as a hyperbolic isometry, so the  $C$ -orbits in  $K_2$  are quasi-isometrically embedded. For each  $x \in L_2$ , the  $C$ -orbit of  $\phi_2(x)$  is the image of the quasi-geodesic  $t \mapsto c^t \cdot \phi_2(x) = \phi_2(c^t \cdot x)$ . We factor  $\phi_2$  as a composition of  $C$ -equivariant simplicial maps  $L_2 \xrightarrow{\iota} K_n \xrightarrow{\phi_n} K_2$ , as in Lemma 3, to deduce that the  $C$ -orbit of  $\phi_n \iota(x)$  in  $K_n$  is quasi-isometrically embedded.  $\square$

A slight variation on the above argument shows that if one lifts a free group of finite index  $\Lambda < Out(F_2)$  to  $Aut(F_2)$  and then maps it to  $Out(F_n)$  by choosing a free factor  $F_2 < F_n$ , then the inclusion  $\Lambda \hookrightarrow Out(F_n)$  will be a quasi-isometric embedding.

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