ACTIONS OF HIGHER-RANK LATTICES ON FREE GROUPS

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Abstract. If G is a semisimple Lie group of real rank at least 2 and Γ is an irreducible lattice in G, then every homomorphism from Γ to the outer automorphism group of a finitely generated free group has finite image.

1. Introduction

In recent years, a powerful body of mathematics has emerged from efforts to extend rigidity phenomena from the context of irreducible lattices in semsimple Lie groups to a wider context that embraces mapping class groups and automorphism groups of finitely generated free groups (see [12] for references). An important focus of these efforts has been the conjecture that every map from an irreducible, higher-rank lattice Γ to a mapping class group or the automorphism group of a finitely generated free group must have finite image. This was proved in the case of mapping class groups by Farb, Kaimanovich and Mazur [18, 22]; subsequent proofs of their result have elucidated different aspects of the geometry of mapping class groups and their subgroups [8, 17, 16].

When Γ is non-uniform, one obtains a short proof in the mapping class group case by combining the Normal Subgroup Theorem ([31], Theorem 8.1.2) with the fact that all solvable subgroups of mapping class groups are virtually abelian [21, 9]. A similar argument, using [1] and [7] in place of [21] and [9], shows that any homomorphism from Γ to the outer automorphism group of a finitely generated free group must also factor through a finite group; see [11]. Our main objective in this note is the corresponding result for uniform lattices.

Theorem A. *Let Γ be a group. Suppose that no subgroup of finite index in Γ has a normal subgroup that maps surjectively to $\mathbb{Z}$. Then every homomorphism from Γ to the outer automorphism group of a finitely generated free group has finite image.*

In Theorem A and the variations on it in Section 3, we do not assume that Γ is finitely generated.

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We say that a group satisfying the hypothesis of Theorem A is $\mathbb{Z}$–*averse*. The Normal Subgroup Theorem of Kazhdan and Margulis [31] tells us that irreducible lattices in connected, higher-rank, semisimple Lie groups with finite centre have no infinite normal subgroups of infinite index. Since such lattices are not virtually cyclic, it follows that they are $\mathbb{Z}$–averse.

Corollary B. *If G is a connected, semisimple Lie group of real rank at least 2 that has finite centre, and Γ is an irreducible lattice in G, then every homomorphism from Γ to the outer automorphism group of a finitely generated free group has finite image.*

An additional argument allows one to avoid the hypothesis that G has finite centre (see Remark 3.2). Further examples of $\mathbb{Z}$–averse groups come from Bader and Shalom's recent work on the Normal Subgroup Theorem [4]. If a *hereditarily just infinite* group is not virtually cyclic then it is $\mathbb{Z}$–averse; examples are described in [30].

Our proof of Theorem A relies heavily on recent results of Bestvina and Feighn [5], Dahmani, Guirardel and Osin [16], and Handel and Mosher [20]. The work of Bestvina and Feighn was inspired in part by the desire to prove Corollary B, following the lines of the proof given in the case of mapping class groups by Bestvina and Fujiwara [8], which invokes Burger and Monod's theorem that irreducible lattices in higher-rank Lie groups have trivial bounded cohomology [13]. One can replace this use of bounded cohomology with an argument of Dahmani, Guirardel and Osin that applies small cancellation theory to the study of purely pseudo-Anosov subgroups; this is used in [16] to prove an analogue of Theorem A for homomorphisms to mapping class groups.

Whether one uses bounded cohomology or the alternative endgame from [16], the key step in the Bestvina–Feighn–Fujiwara approach is to get a finitely generated subgroup of $\mathrm{Out}(F_n)$ to act in a suitable way on a hyperbolic metric space. Bestvina and Feighn [5] construct such actions for subgroups of $\mathrm{Out}(F_n)$ that contain a fully irreducible automorphism, and hence deduce that a higher-rank lattice cannot map onto such a subgroup. If one could construct suitable actions for more general subgroups of $\mathrm{Out}(F_n)$, then Theorem A would follow. A significant step in this direction was taken recently by Handel and Mosher [20], who proved that if a subgroup $H < \mathrm{Out}(F_n)$ does not contain the class of a fully irreducible automorphism, then H has a subgroup of finite index that leaves the conjugacy class of a proper free factor of F invariant. Handel and Mosher also indicate that they hope to extend their work so as to prove Corollary B along the lines sketched above.

Arguments of a quite different sort allow one to see that a homomorphism from a uniform higher-rank lattice Γ to $\mathrm{Aut}(F_n)$ cannot contain any polynomially growing automorphisms of infinite order: on the one hand, Piggott [29] proves that the homomorphism $\mathrm{Aut}(F_n) \to \mathrm{GL}(m, \mathbb{Z})$ given by the action of $\mathrm{Aut}(F_n)$ on the first homology of some characteristic subgroup of finite index in F_n will map a power of such an automorphism to a non-trivial unipotent; on the other hand, Margulis super-rigidity implies that the image of any homomorphism $\Gamma \to \mathrm{GL}(m, \mathbb{Z})$ can contain only semisimple elements.

Our proof of Theorem A proceeds as follows. In Proposition 2.1 we shall use the results of Handel–Mosher and Dahmani–Guirardel–Osin to see that if Γ is $\mathbb{Z}$–averse, then the image of every homomorphism $\Gamma \to \mathrm{Out}(F_n)$ will have a subgroup of finite index that lies in the kernel $\overline{\mathrm{IA}}_n$ of the map $\mathrm{Out}(F_n) \to \mathrm{GL}(n, \mathbb{Z})$ given by the action of $\mathrm{Out}(F_n)$ on the first homology of F_n. In Section 2.2 we use Lie methods, à la Magnus, to prove that every non-trivial subgroup of $\overline{\mathrm{IA}}_n$ maps onto $\mathbb{Z}$ (Corollary 2.9). (This result also appears in Bass and Lubotzky's work on central series [3].) As no finite index subgroup of Γ maps onto $\mathbb{Z}$, this completes the proof of Theorem A.

We thank Laurent Bartholdi for a helpful conversation concerning central filtrations.

2. Proofs

We fix a $\mathbb{Z}$–averse group Γ. We do not assume that Γ is finitely generated.

2.1. Controlling the action of Γ on homology.

Proposition 2.1. *For every subgroup of finite index $\Lambda \subset \Gamma$ and every homomorphism $\phi : \Lambda \to \mathrm{Out}(F_n)$, the intersection $\phi(\Lambda) \cap \overline{\mathrm{IA}}_n$ has finite index in $\phi(\Lambda)$.*

Proof. The proof is by induction on n. The case $n = 1$ is trivial. $\mathrm{Out}(F_2)$ has a free subgroup of finite index and no subgroup of finite index in Γ can map onto a free group, so every homomorphism $\Lambda \to \mathrm{Out}(F_2)$ has finite image.

Suppose $n \geq 3$. Recall that $\psi \in \mathrm{Aut}(F_n)$ (and its image in $\mathrm{Out}(F_n)$) is said to be *fully irreducible* if no power of ψ sends a proper free factor of F_n to a conjugate of itself. Let $[\psi]$ denote the image of ψ in $\mathrm{Out}(F_n)$. Using the actions constructed in [5] and drawing on the approach to small cancellation theory developed in [15], Dahmani, Guirardel and Osin [16] prove that if ψ is fully irreducible then for some positive integer N, the normal closure of $[\psi]^N$ is a free group. It follows that any subgroup of $\mathrm{Out}(F_n)$ that contains a fully irreducible automorphism also contains an infinite normal subgroup that is free. In particular, $\phi(\Lambda)$ cannot contain a fully irreducible automorphism.

According to [20], if $\phi(\Lambda)$ does not contain a fully irreducible automorphism then a subgroup of finite index $H \subset \phi(\Lambda)$ leaves a free factor of F_n invariant up to conjugacy; say $F_n = L * L'$, where $\psi(L) = g_\psi^{-1} L g_\psi$ for all $[\psi] \in H$. Note that the image in $\mathrm{Out}(L)$ of $x \mapsto g_\psi \psi(x) g_\psi^{-1}$, which we denote $[\psi]_L$, depends only on the image of ψ in $\mathrm{Out}(F_n)$, and that $[\psi] \mapsto [\psi]_L$ defines a homomorphism from H to $\mathrm{Out}(L)$. Likewise, the action on the quotient $F_n/\langle\langle L \rangle\rangle$ induces a homomorphism $H \to \mathrm{Out}(L')$. By induction, we know that the induced action of H on the abelianization of both L and L' factors through a finite group. Thus the action of H on the abelianization of $F_n = L * L'$ lies in a block triangular subgroup (with respect to a basis that is the union of bases for L and L')

$$\begin{pmatrix} G & 0 \\ * & G' \end{pmatrix} \leq \mathrm{GL}(n, \mathbb{Z})$$

where G and G' are finite. This matrix group is finitely generated and virtually abelian, whereas Γ, and therefore H, does not have a subgroup of finite index that maps onto $\mathbb{Z}$. Thus the action of H on the homology of F_n factors through a finite group, and hence that of $\phi(\Lambda)$ does too, i.e. $\phi(\Lambda) \cap \overline{\mathrm{IA}}_n$ has finite index in $\phi(\Lambda)$. This completes the induction. □

2.2. **Central filtrations of $\mathrm{IA_n}$ and $\overline{\mathrm{IA}}_n$.** Let γ_c be the cth term in the lower central series of F_n; so $\gamma_1 = F_n$ and $\gamma_c = [\gamma_{c-1}, F_n]$. Let $\Gamma_c = F_n/\gamma_c$. As γ_c is characteristic, there is a natural map $\mathrm{Aut}(F_n) \to \mathrm{Aut}(\Gamma_c)$. Let G_c be the kernel of this map. Note that $G_1 = \mathrm{Aut}(F_n)$ and $G_2 = \mathrm{IA_n}$. Magnus [25] showed that $\cap_{i=0}^\infty \gamma_c = 0$; it follows that $\cap_{i=0}^\infty G_c = 0$. In fact, Andreadakis proved that $G_2, G_3, G_4, \ldots$ forms a central series for G_2 with each quotient a finitely generated free abelian group [2]. It now seems natural to regard his result in the context of *higher Johnson homomorphisms* [14, 28]. We include a proof for the convenience of the reader. Let $L_{c+1} = \gamma_c/\gamma_{c+1}$. Our commutator convention is $[x, y] := x^{-1}y^{-1}xy$.

Proposition 2.2. *For $c \geq 2$ there exists a homomorphism*

$$\tau_c : G_c \to \mathrm{Hom}(H_1(F_n), L_{c+1})$$

such that $\ker(\tau_c) = G_{c+1}$.

Proof. For all $\psi \in G_c$ and $x \in F_n$ there exists $w_x \in \gamma_c$ with $\psi(x) = xw_x$. Define $\tau_c(\psi)([x]) = w_x\gamma_{c+1}$. Note that if $x \in \gamma_2 = [F_n, F_n]$ then

$$\psi(x)\gamma_{c+1} = x\gamma_{c+1}. \tag{1}$$

Indeed the commutator relations $[x, yz] = [x, z][z, [y, x]][x, y]$ and $[xy, z] = [x, z][[x, z], y][y, z]$ imply

$$\begin{aligned}
\psi([x, y]) &= [\psi(x), \psi(y)] \\
&= [xw_x, yw_y] \\
&= [xw_x, w_y][w_y, [y, xw_x]][xw_x, y] \\
&= [xw_x, w_y][w_y, [y, xw_x]][x, y][[x, y], w_x][w_x, y]
\end{aligned}$$

and $[xw_x, w_y], [w_y, [y, xw_x]], [[x, y], w_x], [w_x, y]$ all lie in γ_{c+1}. It follows easily that τ_c is well-defined and a homomorphism. The automorphism ψ belongs to G_{c+1} if and only if w_x is contained in γ_{c+1} for all x in F_n, hence $\ker \tau_c = G_{c+1}$. □

Proposition 2.2 shows that G_c/G_{c+1} is isomorphic to a subgroup of the group $\mathrm{Hom}(H_1(F_n), L_{c+1})$. The abelian group L_{c+1} is a subgroup of the finitely generated free nilpotent group Γ_{c+1}; thus it is finitely generated and free abelian.

Corollary 2.3. *If $c \geq 2$ then G_c/G_{c+1} is a finitely generated free abelian group.*

If Γ is a non-trivial subgroup of $\mathrm{IA_n} = G_2$ then there exists $c \geq 2$ such that $\Gamma \leq G_c$ and $\Gamma \not\leq G_{c+1}$.

Corollary 2.4. *Every non-trivial subgroup of* $\mathrm{IA_n}$ *maps onto* $\mathbb{Z}$.

We would like to extend this analysis to subgroups of $\overline{\mathrm{IA}}_n$. Let H_c be the image of G_c under the projection $\pi : \mathrm{Aut}(F_n) \to \mathrm{Out}(F_n)$. Our goal in the remainder of this section is to prove the analogue of Corollary 2.3 for $H_2, H_3, \ldots$ (cf. [3]). We make use of a theorem of Magnus that $L = \oplus_{c=1}^{\infty} L_c$, along with the bracket operation induced by commutation in F_n, is a *free Lie $\mathbb{Z}$–algebra* generated by the images in $H_1(F_n) = L_1$ of a basis for the free group. This theorem and the required background on free Lie algebras is explained in Chapter 5 of [26]. Let p be a positive integer and let $(L)_p$ be the free Lie algebra obtained by taking the tensor product of $\mathbb{Z}/p\mathbb{Z}$ with L. Let L^c be the quotient algebra $L/\oplus_{i>c} L_i$. We will need the following fact, whose proof is sketched in exercise 3.3 of Chapter 2 in [10]. Throughout, $Z(A)$ denotes the centre of a Lie algebra A.

Proposition 2.5. *If $n \geq 2$, then $Z(L) = Z((L)_p) = 0$ and $Z(L^c)$ is the image of L_c under the quotient map $L \to L^c$.*

As $\overline{\mathrm{IA}}_1$ is trivial, we restrict ourselves to $n \geq 2$ for the remainder of this section. For each non-trivial $y \in F_n$ there exists a unique c such that $y \in \gamma_c$ and $y \notin \gamma_{c+1}$. We identify y with its (non-trivial) image in the submodule L_c of the Lie algebra L. Let $\mathrm{ad} : F_n \to \mathrm{Aut}(F_n)$ be the map induced by the action of F_n on itself by conjugation.

Corollary 2.6. *The kernel of the map $F_n \to \mathrm{Aut}(\Gamma_{c+1})$ induced by* ad *is γ_c. Hence $\mathrm{ad}(y)$ belongs to G_{c+1} if and only if y is in γ_c*

Proof. If $y \in \gamma_c$ then $x^{-1}y^{-1}xy$ lies in γ_{c+1} for each $x \in F_n$, therefore $\mathrm{ad}(y)$ lies in G_{c+1}. Conversely, if y is not in γ_c then by Proposition 2.5, its image in L^c is not central. As L^c is generated by the images of a basis $x_1, \ldots, x_n$ for F_n, this means that some $[x_i, y] \neq 0$ in L^c, hence $x_i^{-1}y^{-1}x_iy_i \notin \gamma_{c+1}$. □

Theorem 2.7. *For $c \geq 2$ the sequence*

$$0 \to \gamma_{c-1}/\gamma_c \to G_c/G_{c+1} \to H_c/H_{c+1} \to 0$$

is an exact sequence of free abelian groups, where the second and third maps are induced by ad *and π respectively.*

Proof. Surjectivity of the map $G_c/G_{c+1} \to H_c/H_{c+1}$ is trivial, and exactness of the remaining maps follows from Corollary 2.6. It remains to show that $H_c/H_{c+1} \cong (G_c/G_{c+1})/\mathrm{ad}(\gamma_{c-1})$ is torsion free. Suppose that $\psi \in G_c$ and there exists $y \in \gamma_{c-1}$ and $p \geq 1$ such that $\mathrm{ad}(y)G_{c+1} = \psi^p G_{c+1}$. For each x_i in a free generating set of F_n we have

$$\psi(x_i) = x_i w_i$$

for some w_i in γ_c. Equation (1) in Proposition 2.2 tells us that $\psi(w_i)\gamma_{c+1} = w_i\gamma_{c+1}$, therefore

$$\psi(x_i)^p\gamma_{c+1} = x_i w_i^p \gamma_{c+1}.$$

As $\mathrm{ad}(y)G_{c+1} = \psi^p G_{c+1}$, this rearranges to

$$x_i^{-1}y^{-1}x_i y\gamma_{c+1} = w_i^p\gamma_{c+1},$$

therefore $[x_i, y] = 0$ in the associated Lie algebra $(L)_p$. However, $(L)_p$ is generated by the images of $x_1, \ldots, x_n$, so the image of y lies in $Z((L)_p) = 0$. It follows that y lies in the kernel pL of the map $L \to (L)_p$. Hence there exists y_0 in γ_{c-1} such that $y\gamma_c = y_0^p\gamma_c$, and $\mathrm{ad}(y_0)^p G_{c+1} = \psi^p G_{c+1}$. But G_c/G_{c+1} is free abelian, so $\mathrm{ad}(y_0)G_{c+1} = \psi G_{c+1}$. □

The final fact we need concerning $\mathrm{Out}(F_n)$ is well known.

Lemma 2.8. *The intersection $\cap_{i=1}^{\infty} H_i$ is trivial in $\mathrm{Out}(F_n)$.*

Proof. By Proposition I.4.9 of [23], if $\psi(u)$ is conjugate to u in Γ_c for all c, then $\psi(u)$ is conjugate to u in F_n. And if ψ takes every element of F_n to a conjugate of itself, then ψ is inner ([19], Lemma 1). □

It follows that if Λ is a non-trivial subgroup of $\overline{\mathrm{IA}}_n = H_2$ then there exists $c \geq 2$ such that $\Lambda \leq H_c$ and $\Lambda \not\leq H_{c+1}$. In Theorem 2.7 we saw that for $c \geq 2$ the quotient H_c/H_{c+1} is a finitely generated free abelian group.

Corollary 2.9. *Every non-trivial subgroup of* $\overline{\mathrm{IA}}_n$ *maps onto* $\mathbb{Z}$*.*

As we explained at the end of the introduction, this completes the proof of Theorem A.

3. Alternative Hypotheses

It emerges from the proofs in the previous section that one can weaken the hypotheses of Theorem A as follows. Note that since we have not assumed Γ to be finitely generated, condition (1) is not equivalent to assuming that every finite index subgroup of Γ has finite abelianization.

Theorem 3.1. *Let* Γ *be a group. Suppose that each finite-index subgroup* $\Lambda \subset \Gamma$ *satisfies the following conditions:*

(1) Λ *does not map surjectively to* $\mathbb{Z}$*;*
(2) Λ *does not have a quotient containing a non-abelian, normal, free subgroup.*

Then every homomorphism $\Gamma \to \mathrm{Out}(F_n)$ *has finite image.*

Proof. The only additional argument that is needed concerns the normal closure I of $[\psi]^N$ in $\mathrm{Out}(F_n)$, as considered in the second paragraph of the proof of Proposition 2.1. We must exclude the possibility that the intersection of I with the image of $\phi : \Lambda \to \mathrm{Out}(F_n)$ is cyclic, generated by $[\psi]^m$ say. But if this were the case, $\langle [\psi]^m \rangle$ would be normal in $\phi(\Lambda)$. Since the normalizer in $\mathrm{Out}(F_n)$ of the subgroup generated by any fully irreducible element is virtually cyclic [6], it would follow that $\phi(\Lambda)$ itself was virtually cyclic, contradicting the fact that no finite-index subgroup of Λ maps onto $\mathbb{Z}$. □

Remark 3.2. The class of groups that satisfy the hypotheses of Theorem 3.1 is closed under certain extension operations. For example, if $1 \to A \to \hat{\Gamma} \to \Gamma \to 1$ is a short exact sequence and, in the notation of Theorem 3.1, we suppose that

- every finite-index subgroup of A satisfies (2),
- every finite-index subgroup of $\hat{\Gamma}$ satisfies (1), and

- every finite-index subgroup of Γ satisfies both (1) and (2),

then an elementary argument shows that every finite-index subgroup of $\hat{\Gamma}$ satisfies (2). (Hence every homomorphism $\hat{\Gamma} \to \mathrm{Out}(F_n)$ has finite image.)

In Theorem 3.1, condition (2) is used only to exclude the possibility that a homomorphic image of Γ in $\mathrm{Out}(F_n)$ might contain a fully irreducible element. An alternative way of ruling out such images is to use bounded cohomology, as in [5]. We briefly review some notation. A map $f : \Gamma \to \mathbb{R}$ is a *quasi-homomorphism* if the function

$$(g, h) \mapsto |f(g) + f(h) - f(gh)|$$

is bounded on $\Gamma \times \Gamma$. Let $V(\Gamma)$ be the vector space of all quasi-homomorphisms from Γ to $\mathbb{R}$. Two natural subspaces of $V(\Gamma)$ are $B(\Gamma)$, the vector space of bounded maps from Γ to $\mathbb{R}$, and $\mathrm{Hom}(\Gamma;\mathbb{R})$, the vector space of genuine homomorphisms. Define $\widetilde{\mathrm{QH}}(\Gamma) = V(\Gamma)/(B(\Gamma) + \mathrm{Hom}(\Gamma;\mathbb{R}))$.

Theorem 3.3. *Let Γ be a group. Suppose that for every finite index subgroup $\Lambda \subset \Gamma$ the second bounded cohomology of Λ is finite dimensional and* $\mathrm{Hom}(\Lambda, \mathbb{R}) = 0$*. Then every homomorphism $\phi : \Gamma \to$* $\mathrm{Out}(F_n)$ *has finite image.*

Proof. Bestvina and Feighn [5] show that if $H < \mathrm{Out}(F_n)$ contains a fully irreducible automorphism then either H is virtually cyclic or $\widetilde{\mathrm{QH}}(H)$ is infinite dimensional. If $\mathrm{Hom}(\Lambda;\mathbb{R}) = 0$ then a surjective map $\Lambda \to H$ induces an injection $\widetilde{\mathrm{QH}}(H) \to \widetilde{\mathrm{QH}}(\Lambda)$. The vector space $\widetilde{\mathrm{QH}}(\Lambda)$ injects into the second bounded cohomology of Λ (see [27]). Therefore, for all finite index subgroups $\Lambda \subset \Gamma$ and integers m the image of a homomorphism $\Lambda \to \mathrm{Out}(F_m)$ cannot contain a fully irreducible automorphism. It follows from Corollary 2.9 and the arguments in Proposition 2.1 that $\phi(\Gamma)$ is finite. □

In the light of [13], this alternative to Theorem A also implies Corollary B.

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