UNDECIDABILITY AND THE DEVELOPABILITY OF PERMUTOIDS AND RIGID PSEUDOGROUPS

MARTIN R. BRIDSON AND HENRY WILTON

Abstract. A *permutoid* is a set of partial permutations that contains the identity and is such that partial compositions, when defined, have at most one extension in the set. In 2004 Peter Cameron conjectured that there can exist no algorithm that determines whether or not a permutoid based on a finite set can be completed to a finite permutation group. In this note we prove Cameron's conjecture by relating it to our recent work on the profinite triviality problem for finitely presented groups. We also prove that the existence problem for finite developments of rigid pseudogroups is unsolvable.

1. Introduction

Across many contexts in mathematics one encounters extension problems of the following sort: given a set S of partially-defined automorphisms of an object X, one seeks an object $Y \supset X$ and a set of automorphisms $\hat{S}$ of Y such that each $s \in S$ has an extension $\hat{s} \in \hat{S}$. In the category of finite sets, this problem is trivial because any partial permutation of a set can be extended to a permutation of that set. Less trivially, Hrushovski [7] showed that extensions always exist in the category of finite graphs. But if one requires extensions to respect (partially defined) compositions in S, such existence problems become more subtle. In 2004 Peter Cameron [4] conjectured that there does not exist an algorithm that can solve the following extension problem.

Problem 1.1. Given *partial permutations* $p_1, ..., p_m$ *of a finite set* X *(that is, bijections between subsets of* X*) such that*

(1) $p_1 = \mathrm{id}_X$*, and*
(2) *for all* i, j *with* $\mathrm{dom}(p_i) \cap \mathrm{ran}(P_j) \neq \varnothing$*, there is at most one* k *such that* p_k *extends* $p_i \cdot p_j$

Date: 27 June 2014.

2010 *Mathematics Subject Classification.* 20F10, 05C60, (20M18, 08A50).

Both authors thank the EPSRC for its financial support. Bridson's work is also supported by a Wolfson Research Merit Award from the Royal Society; he thanks them and he thanks the Erwin Schrödinger International Institute for Mathematical Physics in Vienna for its hospitality during the writing of this article.

decide *whether or not there exists a finite set Y containing X, and permutations f_i of Y extending p_i for $i = 1, ..., m$, such that if p_k extends $p_i \cdot p_j$ then $f_i \circ f_j = f_k$.*

We shall prove that this problem is indeed algorithmically unsolvable by relating it to our recent work on the triviality problem for finitely presented profinite groups [3]. In order to achieve this, we develop some formalism: a collection of partial permutations as in Problem 1.1 is called a *permutoid*; in Section 2 we define morphisms, quotients, and developments of permutoids. In this terminology, Cameron's conjecture is that there does not exist an algorithm that can decide whether or not a finite permutoid is developable. Cameron [4] associated a permutoid to a finite group presentation (cf. Proposition 3.6) and observed that if the group has no finite quotients then the permutoid is not developable. If the converse were to hold, Cameron's conjecture would follow easily from the constructions in [3], but unfortunately it does not (see Remark 4.4). It is to obviate this difficulty that we introduce quotient permutoids.

In the final section of this paper we shall explain how our main construction also can be adapted to prove a similar undecidability result for rigid pseudogroups.

2. Partial Permutations and Permutoids

A *partial permutation* of a set X is a bijection between two non-empty subsets of X. We denote the domain and range of a partial permutation p by $\mathrm{dom}(p)$ and $\mathrm{ran}(p)$ respectively. By definition, q *extends* p if $\mathrm{dom}(p) \subset \mathrm{dom}(q)$ and $q(x) = p(x)$ for all $x \in \mathrm{dom}(p)$. The composition $p \cdot q$ of two partial permutations p, q on X is defined if $\mathrm{ran}(p) \cap \mathrm{dom}(q)$ is non-empty: $p \cdot q(x) = p(q(x))$ for $x \in p^{-1}(\mathrm{ran}(p) \cap \mathrm{dom}(q))$.

Definition 2.1. A *permutoid* $(\Pi; X)$ is a set Π of partial permutations of a set X such that

(1) Π contains 1_X, the identity map of X;
(2) for all $p, q \in \Pi$ there exists at most one $r \in \Pi$ such that r extends $p \cdot q$ (if the composition exists).

The permutoid is *finite* if X is finite, and *trivial* if $\Pi = \{1_X\}$.

A *morphism* of permutoids $(\Pi; X) \overset{(\phi,\Phi)}{\to} (\Pi'; X')$ is a pair of maps $\phi : \Pi \to \Pi'$ and $\Phi : X \to X'$ so that:

(1) $\phi(1_X) = 1_{X'}$;
(2) $\Phi(\mathrm{dom}(p)) \subseteq \mathrm{dom}\phi(p)$ and $\phi(p)(\Phi(x)) = \Phi(p(x))$ for all $p \in \Pi$ and $x \in \mathrm{dom}(p)$;
(3) if r extends $p \cdot q$, with $p, q, r \in \Pi$, then $\phi(r)$ extends $\phi(p) \cdot \phi(q)$.

The morphism (ϕ, Φ) is an *isomorphism* if Φ and ϕ are bijections and $\phi(p) = \Phi \circ p \circ \Phi^{-1}$ for all $p \in \Pi$.

The morphism (ϕ, Φ) is a *quotient map* if Φ and ϕ are surjections.

The morphism (ϕ, Φ) is an *extension* if Φ is injective.

An extension $(\Pi; X) \overset{(\phi,\Phi)}{\to} (\Pi'; X')$ is *complete* if $\Pi' \subset \mathrm{Perm}(X')$; in other words $\mathrm{dom}(p') = \mathrm{ran}(p') = X'$ for all $p' \in \Pi'$.

If a finite permutoid $(\Pi; X)$ admits a finite complete extension, then $(\Pi; X)$ is said to be *developable* and $(\Pi'; X')$ is called a development.

Remarks 2.2. (1) Cameron's Conjecture asserts that there is no algorithm that can determine the developability of a finite permutoid.

(2) If $(\Pi; X) \overset{(\phi,\Phi)}{\to} (\Pi'; X')$ is an extension, then ϕ will fail to be injective precisely when Π contains two distinct restrictions of some $p \in \Pi'$. For example, if Φ is the identity map on X and p is a permutation with at least two points x_1, x_2 in its support, then we obtain an extension $(\Pi; X) \overset{(\phi,\mathrm{id})}{\to} (\Pi'; X)$ by defining $p_i = p|_{x_i}$, $\Pi = \{\mathrm{id}, p_1, p_2\}$, $\Pi' = \{\mathrm{id}, p\}$ and $\phi(p_i) = p$.

Definition 2.3. The *universal group* of a permutoid $(\Pi; X)$ is

$$\Gamma(\Pi; X) := \langle \Pi \mid pq = r \text{ if } r \text{ extends } p \cdot q \,\rangle.$$

Lemma 2.4. (1) *If* $(\Pi; X) \overset{(\phi,\Phi)}{\to} (\Pi'; X')$ *is a morphism, then* $p \mapsto \phi(p)$ *defines a homomorphism of groups*

$$\phi_* : \Gamma(\Pi; X) \to \Gamma(\Pi'; X'),$$

and if (ϕ, Φ) *is a quotient morphism then* ϕ_* *is surjective.*

(2) *If* $\Pi \subseteq \mathrm{Perm}(X)$ *then there is an epimorphism* $\Gamma(\Pi; X) \to \langle \Pi \rangle \leqslant \mathrm{Perm}(X)$.

(3) *If a non-trivial finite permutoid* $(\Pi; X)$ *is developable, then* $\Gamma(\Pi; X)$ *has a non-trivial finite quotient.*

Proof. Parts (1) and (2) are immediate from the definitions. For (3), let $(\Pi; X) \overset{(\phi,\Phi)}{\to} (\Pi'; X')$ be a finite complete extension. Note that if $p \in \Pi$ is not id_X then $p(x) \neq x$ for some $x \in X$, and hence $\phi(p) \neq \mathrm{id}_{X'}$. It follows that the image of ϕ_* is non-trivial, and so (3) follows from (2). □

3. Cameron permutoids

A *marked group* is a pair (G, A) where G is a group and A is a generating set. Let ρ be a positive integer. Let $B_\rho \subset G$ be the set of elements that can be expressed as a word of length at most ρ in the generators and their inverses, and define $B_{2\rho}$ similarly. Define p_1 to be the identity map on $B_{2\rho}$, and for each $b \in B_\rho \smallsetminus \{1\}$ define $p_b : B_\rho \to B_{2\rho}$ to be the restriction of left multiplication by b; that is, $p_b(x) = bx$. Let $\Pi_\rho = \{p_b \mid b \in B_\rho\}$.

Lemma 3.1. *The pair* $(\Pi_\rho; B_{2\rho})$ *is a permutoid. If* A *is finite then this permutoid is finite.*

Proof. Each element $g \in G$ is uniquely determined by its action by left multiplication on any point $x \in G$. Thus, for all $b, b' \in B_\rho$, if bb' lies in B_ρ then $p_{bb'}$ is the unique element of Π_ρ extending $p_b \cdot p_{b'}$, and if not then no element of Π_ρ extends $p_b \cdot p_{b'}$. □

Definition 3.2. Given a marked group (G, A) and a positive integer ρ, $(\Pi_\rho; B_{2\rho})$ is called[1] a *Cameron permutoid.* If $\mathcal{P} \equiv \langle A \mid R\rangle$ is a finite presentation for a group G, then we write $\mathcal{B}_\rho(\mathcal{P})$ to denote the Cameron permutoid $(\Pi_\rho; B_{2\rho})$.

Remark 3.3. It is important to note that, in order to construct $\mathcal{B}_\rho(\mathcal{P})$ from a finite presentation $\mathcal{P}$, one needs to be able to calculate which words of length at most ρ in the generators represent equal elements of the group, and for each pair of such elements b, x, one needs to calculate bx. This can be achieved in an algorithmic manner *provided* that one has a solution to the word problem in $|\mathcal{P}|$. And in order to achieve the construction for all $\rho > 0$ and all presentations in a class $\mathfrak{P}$, one needs a *uniform solution* to the word problem for the groups in $\mathfrak{P}$.

We have arranged the definitions so that the following lemma is obvious.

Lemma 3.4. *For all presentations $\mathcal{P} \equiv \langle A \mid R\rangle$ and $\mathcal{P}' \equiv \langle A \mid R'\rangle$ with $R \subset R'$, the natural epimorphism $|\mathcal{P}| \xrightarrow{\pi} |\mathcal{P}'|$ induces a quotient map of permutoids $\mathcal{B}_\rho(\mathcal{P}) \xrightarrow{(\phi,\Phi)} \mathcal{B}_\rho(\mathcal{P}')$, where Φ is the restriction of π to $B_{2\rho}$ and $\phi(p_b) = p_{\pi(b)}$.*

If ρ is large enough then there is a natural isomorphism $\Gamma(\mathcal{B}_\rho(\mathcal{P})) \cong |\mathcal{P}|$. In order to see this, we need the following well-known triangulation procedure.

Lemma 3.5. *Let $\mathcal{P} \equiv \langle A \mid R\rangle$ be a finite presentation, let m be an integer greater than half the length of the longest relation in R, let B be the set of elements of $G = |\mathcal{P}|$ that can be expressed as words of length at most m in the free group $F(A)$, let T be the set of words $w \in F(B)$ of length three that equal the identity in G, and let $\mathcal{T} \equiv \langle B \mid T\rangle$. Then, the natural map $A \to B \subset G$ induces an isomorphism $|\mathcal{P}| \to |\mathcal{T}|$.*

Proposition 3.6. *If $\rho \geqslant 1$, then there is a natural epimorphism of groups $\Gamma(\mathcal{B}_\rho(\mathcal{P})) \to |\mathcal{P}|$, and if ρ is greater than half the length of the longest relator in R, this is an isomorphism.*

Proof. By definition,

$$\Gamma(\mathcal{B}_\rho(\mathcal{P})) = \langle p_b \ (b \in B_\rho) \mid p_{b_1}p_{b_2} = p_{b_3} \text{ if } b_1b_2 = b_3 \text{ in } |\mathcal{P}|\rangle,$$

where B_ρ is the ball of radius ρ about the identity in $|\mathcal{P}|$ (with word metric d_A). The homomorphism $\Gamma(\mathcal{B}_\rho(\mathcal{P})) \to |\mathcal{P}|$ defined by $p_b \mapsto b$ is onto (since $\rho \geqslant 1$ and the image of B_1 generates $|\mathcal{P}|$). And if ρ is

[1] in recognition of the fact that Peter Cameron considered these objects in [4]

greater than half the length of the longest relator in R, then modulo an obvious change of notation this is the isomorphism of Lemma 3.5. □

We need one final fact.

Lemma 3.7. *For all marked groups (G, A) and all positive integers $\rho > \rho' > 0$, there is an extension of permutoids $(\Pi_{\rho'}; B_{2\rho'}) \to (\Pi_\rho; B_{2\rho})$ given by the inclusion $B_{2\rho'} \hookrightarrow B_{2\rho}$ and the map $\Pi_{\rho'} \to \Pi_\rho$ that extends left-multiplication from $B_{\rho'}$ to B_ρ.*

Corollary 3.8. *If $\mathcal{P}$ is a finite presentation of a finite group G then, for all positive integers ρ, the permutoid $\mathcal{B}_\rho(\mathcal{P})$ is developable.*

Proof. If ρ is sufficiently large then $B_\rho = B_{2\rho} = G$ and $\Pi_\rho < \mathrm{Perm}(G)$ is the subgroup consisting of left multiplications. □

Remark 3.9. A permutoid defines a *pree* in an obvious manner. By definition, a *pree* is a non-empty set P with a partially defined binary operation, i.e. a subset $D \subseteq P \times P$ and a map $m : D \to P$. This terminology is due to Stallings [10] (also [9]); Baer [1] had earlier used the term *add* to describe such objects. Both Baer and Stallings established criteria that guarantee a pree will embed in the associated group

$$G(P, m) := \langle P \mid pq = m(p, q) \text{ for all } (p, q) \in D\rangle.$$

4. Finite completions and finite quotients

In the language of permutoids, Cameron's Conjecture (Problem 1.1) is that *developability is an undecidable property.*

Theorem 4.1. *There does not exist an algorithm that, given a finite permutoid $(\Pi; X)$, can determine whether or not $(\Pi; X)$ is developable.*

Remark 4.2. It is clear that the isomorphism classes of finite permutoids form a recursive set, and a naive search will eventually find a complete extension of a finite permutoid if such exists. The content of the above theorem, then, is that there is no algorithm that can enumerate the isomorphism classes of finite permutoids that do not have a complete finite extension.

In [3, Theorem B] we constructed a recursive set of finite presentations for biautomatic groups such that there is no algorithm that can determine which of these groups has a non-trivial finite quotient. The class of (bi)automatic groups admits a uniform solution to the word problem [5, pp. 32, 112]. Theorem 4.1 therefore follows immediately from [3, Theorem B] and the next proposition.

Proposition 4.3. *Let $\mathfrak{P}$ be a class of finite presentations for groups drawn from a class in which there is a uniform solution to the word problem. If there were an algorithm that could determine whether or not a finite permutoid was developable, then there would be an algorithm* *that, given any presentation $\mathcal{P} \in \mathfrak{P}$, could determine whether or not the group $|\mathcal{P}|$ had a non-trivial finite quotient.*

Proof. Given $\mathcal{P} \in \mathfrak{P}$, take ρ to be at least half the length of the longest relator and use the solution to the word problem to construct $\mathcal{B}_\rho(\mathcal{P})$ (cf. Remark 3.3). Then list representatives P_i for the finitely many isomorphism classes of the non-trivial quotient permutoids. The proposition now follows from the claim that $|\mathcal{P}|$ has a non-trivial finite quotient if and only if one of the P_i is developable.

On the one hand, if one of the P_i is developable then $\Gamma(P_i)$ has a finite quotient, by Lemma 2.4(3), and hence, by Lemma 2.4(1), so does $\Gamma(\mathcal{B}_\rho(\mathcal{P}))$, which, by Proposition 3.6, is isomorphic to $|\mathcal{P}|$. Conversely, if $|\mathcal{P}|$ has a non-trivial finite quotient, with presentation $\mathcal{P}' = \langle A \mid R' \rangle$ say, where $R \subset R'$, then $\mathcal{B}_\rho(\mathcal{P}')$ will be a quotient permutoid of $\mathcal{B}_\rho(\mathcal{P})$, and the P_i isomorphic to it will be developable, by Corollary 3.8. □

Remark 4.4. The key observation that if $\mathcal{B}_\rho(\mathcal{P})$ has a complete finite extension then $|\mathcal{P}|$ has a non-trivial finite quotient is due to Peter Cameron [4]. Note, however, that the converse to this observation does not hold: in general $\mathcal{B}_\rho(\mathcal{P})$ need not inject into any finite quotient of $|\mathcal{P}|$, even if such quotients exist. For instance, $\mathcal{P}$ may present a non-cyclic group whose finite quotients are all cyclic, such as the example of Baumslag [2].

5. Rigid developments and pseudogroups

Pseudogroups play an important role in many geometric contexts. A pseudogroup of local homeomorphisms on a topological space X is a collection $\mathcal{H}$ of homeomorphisms $h : U \to V$ of open sets of X such that:

1. if $h : U \to V$ and $h' : U' \to V'$ are in $\mathcal{H}$ then h^{-1} and the composition $h'h : h^{-1}(V \cap U') \to h'(V \cap U')$ belong to $\mathcal{H}$;
2. the restriction of h to any open subset of U belongs to $\mathcal{H}$;
3. $\mathrm{id}_X \in \mathcal{H}$;
4. if a homeomorphism between open subsets of X is the union of elements of $\mathcal{H}$, then it too belongs to $\mathcal{H}$.

We shall concentrate on the case where X is a finite set with the discrete topology.

A set Π of partial permutations of a set X determines a pseudogroup denoted $\mathcal{H}_\Pi$, namely the pseudogroup generated by all restrictions of the elements $p \in \Pi$. For example, the pseudogroup associated to a Cameron permutoid $(\Pi_\rho; B_{2\rho})$ consists of all maps $U \to V$, with $U, V \subseteq B_{2\rho}$, that are restrictions of left-muliplications $x \mapsto g.x$ on G.

If $(\Pi; X)$ is a permutoid, then by passing from Π to $\mathcal{H}_\Pi$ one loses the crucial condition 1.1(2). Correspondingly, $\mathcal{H}_\Pi$ can always be embedded in the pseudogroup $\mathcal{H}_{\Pi'}$ associated to a set Π' of permutations of X: take any choice of extension $p' \in \mathrm{Perm}(X)$ for $p \in \Pi$.

A more substantial analogue of Problem 1.1 in the context of pseudogroups arises when one restricts attention to pseudogroups that are *rigid* in the sense that maps are defined by their value at any point.

Definition 5.1. A permutoid $(\Pi; X)$ is *rigid* if for all $p \neq q \in \Pi$, there is no $x \in X$ such that $p(x) = q(x)$.

A pseudogroup $\mathcal{H}$ is *rigid* if $f \cup g \in \mathcal{H}$ whenever $f, g \in \mathcal{H}$ and $f(x) = g(x)$ for some $x \in X$ (equivalently, every element of $\mathcal{H}$ has a unique maximal extension).

A basic example of a rigid pseudogroup is the pseudogroup $G \ltimes X$ associated to a free action of a group on a space X (in our case a finite set with the discrete topology). The elements of this pseudogroup are the restrictions of the transformations in the action. In close analogy with Problem 1.1, one would like to know, given a finite rigid pseudogroup, $\mathcal{H}$ on X, if one can embed X in a finite set Y so that the elements of $\mathcal{H}$ are restrictions of transformations of Y in a free action of a finite group G; in other words we wish to embed $\mathcal{H}$ in some $G \ltimes Y$. When this can be done, we say that $\mathcal{H}$ is *developable*.

Theorem 5.2. *There does not exist an algorithm that can determine whether or not a finite, rigid pseudogroup has a finite development.*

The proof of this theorem is implicit in our earlier arguments; to translate them we need the following lemma.

Lemma 5.3. *Let $\mathcal{H}$ be a rigid pseudogroup on a finite set X and let $\Lambda \subset \mathcal{H}$ be the set of maximal elements.*

(1) $(\Lambda; X)$ *is a rigid permutoid.*
(2) $\mathcal{H}_\Lambda = \mathcal{H}$.
(3) *If $\mathcal{H} = \mathcal{H}_\Pi$ for some permutoid* $(\Pi; X)$, *then the map that assigns each $p \in \Pi$ to its maximal extension in $\mathcal{H}$ defines an extension of permutoids* $(\Pi; X) \to (\Lambda; X)$.
(4) *If $\mathcal{H}$ is developable, then so is* $(\Lambda; X)$ *(and hence* $(\Pi; X)$*).*

Proof. The first three items follow easily from the definitions. For example, if $p, q \in \Pi$ and $q(x) \in \text{ran}(q) \cap \text{dom}(p)$, then the unique maximal element $r \in \mathcal{H}$ with $r(x) = pq(x)$ is the unique element of Π such that r extends $p \cdot q$.

(4). If $G \ltimes Y$ is a finite development of $\mathcal{H}$, then each maximal element $p \in \mathcal{H}$ is the restriction of the action of a unique $\hat{p} \in G$, so $\hat{r} = \hat{p}\hat{q}$ if r extends $p \cdot q$. Thus $\Phi : X \hookrightarrow Y$ and $\phi(p) := \hat{h}$ define a development of $(\Pi; X)$. □

Let G be a group with finite generating set A and let B_r denote the ball of radius r about $1 \in G$ in the corresponding word metric. Earlier, we considered the permutoid $\mathcal{B}_\rho(\mathcal{P}) = (\Pi_\rho; B_{2\rho})$. The associated pseudogroup $\mathcal{H}_{\Pi_\rho}$ consists of all maps $U \to V$, with $U, V \subseteq B_{2\rho}$, that are restrictions of left multiplications $\lambda_g : G \to G$.

Proposition 5.4. *Let G be a group with finite presentation $\langle A \mid R\rangle$, let $\rho > \frac{1}{2}\max\{|r| : r \in R\}$ be an integer and consider the permutoid $\mathcal{B}_\rho(\mathcal{P}) = (\Pi_\rho; B_{2\rho})$. The following conditions are equivalent.*

(1) *G has a non-trivial finite quotient;*
(2) *$\mathcal{B}_\rho(\mathcal{P})$ has a quotient permutoid that is developable;*
(3) *$\mathcal{B}_\rho(\mathcal{P})$ has a quotient permutoid that has a* rigid *finite development;*
(4) *$\mathcal{B}_\rho(\mathcal{P})$ has a quotient permutoid whose associated pseudogroup is rigid and developable.*

Proof. We proved the equivalence of (1) and (2) in the proof of Proposition 4.3. (3) implies (2), trivially, and (1) implies (3) because if Q is a finite quotient of G, then the action of Q by left-multiplication on itself provides a rigid development for some quotient P of $\mathcal{B}_\rho(\mathcal{P})$.

Moreover, the rigid pseudogroup $Q \ltimes Q$ associated to this action (where Q acts on itself by left multiplication) is a development of the pseudogroup defined by P, and therefore (1) implies (4). Finally, Lemma 5.3(4) tells us that (4) implies (3). □

Proof of Theorem 5.2. We follow the proof of Proposition 4.3. Taking finite presentations in a class $\mathfrak{P}$ where there is a uniform solution to the word problem but no algorithm that can determine if the groups presented have finite quotients or not, we construct the permutoids $\mathcal{B}_\rho(\mathcal{P})$ as above, list the finitely many quotients of each $\mathcal{B}_\rho(\mathcal{P})$, then pass to the pseudogroups defined by these quotients, retaining only those pseudogroups that are rigid (an easy check). If there were an algorithm that could determine developability for rigid pseudogroups, then we would apply it to the members of the resulting list and thereby (in the light of Proposition 5.4) determine which of the groups with presentations in $\mathfrak{P}$ have finite quotients. This would contradict our choice of $\mathfrak{P}$, and therefore no such algorithm exists. □

Remark 5.5. The undecidability phenomena that we have articulated in the language of permutoids and pseudogroups can equally be expressed in the language of groupoids or inverse semigroups (cf. [8]). In the context of inverse semigroups, Steinberg [11] has proved a result similar to Theorem 5.2 (see also [6]). Also, instead of considering finite sets, one could consider sets of partial automorphisms of simplicial complexes, for example.

References

[1] R. Baer, *Free sums of groups and their generalizations II, III*, Amer. J. Math. **72** (1950), 625–670.

[2] Gilbert Baumslag. A non-cyclic one-relator group all of whose finite quotients are cyclic. *J. Austral. Math. Soc.*, 10:497–498, 1969.

[3] M.R. Bridson and H. Wilton, *The triviality problem for profinite completions*, arXiv:1401.2273.

[4] Peter Cameron. Extending partial permutations. http://www.maths.qmul.ac.uk/~pjc/odds/partial.pdf, 2004.

[5] David B. A. Epstein, James W. Cannon, Derek F. Holt, Silvio V. F. Levy, Michael S. Paterson, and William P. Thurston, *Word processing in groups*, Jones and Bartlett Publishers, Boston, MA, 1992.

[6] T.E. Hall, S.I. Kublanovskii, S. Margolis, M.V. Sapir, P.G. Trotter, *Algorithmic problems for finite groups and finite 0-simple semigroups*, J. Pure Appl. Algebra **119** (1997), 75–96.

[7] E. Hrushovski, *Extending partial isomorphisms of graphs*, Combinatorica **12** (1992), 411–416.

[8] M.V. Lawson, *Inverse Semigroups: The Theory of Partial Symmetries*, World Scientific, 1998.

[9] F. Rimlinger, *Pregroups and Bass-Serre theory*, Mem. Amer. Math. Soc. **65** (1987), no. 361.

[10] J.R. Stallings, *Group theory and three-dimensional manifolds*, Yale Univ. Press, New Haven, Conn., 1971.

[11] B. Steinberg, *The uniform word problem for groups and finite Rees quotients of E-unitary inverse semigroups*, J. Algebra **266** (2003), 1–13.

Mathematical Insitute, Andrew Wiles Building, Woodstock Road, Oxford, OX2 6GG, U.K.

E-mail address: bridson@maths.ox.ac.uk

Department of Mathematics, University College London, Gower Street, London WC1E 6BT, UK

E-mail address: hwilton@math.ucl.ac.uk