ON THE GROWTH OF GROUPS AND AUTOMORPHISMS

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Abstract. We consider the growth functions $\beta_\Gamma(n)$ of amalgamated free products $\Gamma = A *_C B$, where $A \cong B$ are finitely generated, C is free abelian and $|A/C| = |A/B| = 2$. For every $d \in \mathbb{N}$ there exist examples with $\beta_\Gamma(n) \simeq n^{d+1}\beta_A(n)$. There also exist examples with $\beta_\Gamma(n) \simeq e^n$. Similar behaviour is exhibited among Dehn functions.

For Slava Grigorchuk, in friendship, with great respect.

The first purpose of this note is to present explicit examples of groups which show that if one amalgamates two groups with polynomial growth of degree δ along isomorphic subgroups of index two, then the growth of the resulting group may be polynomial of degree $\delta + 1$, polynomial of greater degree, or may be exponential. Similar *jump behaviour* is exhibited for Dehn functions of finite-index amalgams of both virtually abelian and virtually free groups.

Let G be a group with finite generating set S. The *growth function* $\beta_{G,S}(n)$ counts the number of elements of G that can be expressed as a word of length at most n in the generators S and their inverses. If S' is another finite generating set for G then there exists a constant $k > 0$ such that $\beta_{G,S}(n) \leq \beta_{G,S'}(kn)$. We write $f(n) \preceq g(n)$ when functions $f, g : \mathbb{N} \to \mathbb{N}$ are related by a constant in this way, and we write $f(n) \simeq g(n)$ if, in addition, $g(n) \preceq f(n)$. It is common to omit the subscript S from $\beta_{G,S}(n)$ and write $\beta_G(n)$ with the understanding that this function is only well-defined up to $\simeq$equivalence. Note that if $H \subset G$ is a subgroup of finite index, then $\beta_H(n) \simeq \beta_G(n)$.

All finitely generated groups have growth $\preceq e^n$, because $c^n \simeq e^n$ if $c > 1$, and if $c > 2|S|$ then there are less than c^n reduced words of length n over the alphabet $S^{\pm 1}$. In his landmark paper [9], R. Grigorchuk proved that there exist finitely generated groups G of intermediate growth, i.e. $e^{\delta n} \preceq \beta_G(n) \preceq e^{\eta n}$ where $0 < \delta < \eta < 1$ (see also [10], [11]). On the other hand, the growth of every known finitely presented group is either $\simeq e^n$ or else $\simeq n^d$ for some $d \in \mathbb{N}$. A celebrated theorem of M. Gromov [13] states that if $\beta_G(n) \preceq n^p$, then G has a nilpotent subgroup G_0 of finite index and $\beta_G(n) \simeq n^d$ where $d \leq p$ is an integer that can be calculated in terms of the ranks of the factors of lower central series of G_0. See [12] for a wide-ranging survey of results concerning the growth of groups; also [14] Chap. VI.

Let $\Gamma = A *_C B$ be an amalgamated free product of finitely generated groups. We assume that C has index at least 2 in both A and B. It follows immediately from the normal form theorem for amalgamated free products that Γ has exponential growth if $|A/C| \geq 3$.

We consider the case $|A/C| = |B/C| = 2$. Since C is normal in both A and B, it is normal in Γ. Thus we have a short exact sequence $1 \to C \to \Gamma \to D_\infty \to 1$, where $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$ is infinite dihedral.

Remark 1. Each of the constructions described below exploits the simple observation that, given any group C and any infinite dihedral group $D \subset \mathrm{Aut(C)}$, one can construct $C \rtimes D$ by amalgamating two groups of the form $C \rtimes \mathbb{Z}_2$ along the visible copies of C.

Given any exact sequence of of finitely generated groups $1 \to N \to \Gamma \to Q \to 1$ it is easy to see that $\beta_N(n)\beta_Q(n) \preceq \beta_\Gamma(n)$. Here are some examples where this bound is not sharp.

Example 1. Let $\{x_1, \dots, x_m\}$ be a basis for $\mathbb{Z}^m$ and let $|v|$ denote the distance of v from the identity in the corresponding word metric. Associated to each $\phi \in \mathrm{GL}_m(\mathbb{Z})$ one has the semi-direct product $\mathbb{Z}^m \rtimes_\phi \mathbb{Z} = \langle x_1, \dots, x_m, t \mid \forall i, j,\ [x_i, x_j] = 1,\ t x_i t^{-1} = \phi(x_i)\rangle$.

(i) If ϕ has an eigenvalue λ with $|\lambda| > 1$, then $\|\phi^n\| := \max_i |\phi^n(x_i)|$ grows as an exponential function of n. To see this, one observes that there exists a linear map $L : \mathbb{C}^m \to \mathbb{C}$ and vectors $v \in \mathbb{Z}^m$ such that $L \circ \phi = \lambda L$ and $L(v) \neq 0$; one then checks that for any integer $r > 1/|\lambda$, the elements t^r and v generate a free semigroup of rank 2 in $G = \mathbb{Z}^m \rtimes_\phi \mathbb{Z}$, and therefore G has exponential growth — see page 189 of [14].

(ii) By direct calculation (or by [1]) one sees that if $\psi_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ then the Heisenberg group $G_2 = \mathbb{Z}^2 \rtimes_{\psi_2} \mathbb{Z}$ has growth $\simeq n^4$. More generally, if $\psi_{2d} \in \mathrm{GL}_{2d}(\mathbb{Z})$ is the matrix with d blocks of ψ_2 down the diagonal and zeros elsewhere, then $G_{2d} = \mathbb{Z}^{2d} \rtimes_{\psi_{2d}} \mathbb{Z}$ has growth $\simeq n^{3d+1}$.

(iii) To obtain the maximum degree of polynomial growth among groups of the form $G = \mathbb{Z}^m \rtimes_\phi \mathbb{Z}$, one can take $\phi \in \mathrm{GL}_m(\mathbb{Z})$ to be any upper-triangular matrix with ones on the diagonal, non-negative entries above the diagonal, and strictly positive entries in the super-diagonal. The growth $n \mapsto \|\phi^n\|$ of such ϕ is polynomial of degree $m-1$, and $\beta_G(n) \simeq n^{\frac{1}{2}(m^2+m+2)}$.

We return to the consideration of $1 \to C \to A *_C B \to D_\infty \to 1$. After passing to the cyclic subgroup of index 2 in D_∞ we can split this sequence to obtain a subgroup $\Gamma_0 = C \rtimes_\phi \mathbb{Z}$ of index 2 in $\Gamma = A *_C B$. The above examples show that the growth of Γ depends very much on the automorphism $\phi \in \mathrm{Aut}(C)$.

Remark 2. If A and B have polynomial growth and C has index 2 in both A and B, then $A *_C B$ must have either polynomial or exponential growth. Indeed, by Gromov's Theorem [13], A and B are virtually nilpotent, hence $A *_C B$, being commensurable with $C \rtimes \mathbb{Z}$, is virtually solvable. Solvable groups cannot have intermediate growth [16].

Theorem 1. *For each integer $d > 2$ there exist groups of the form $\Gamma = A *_C \overline{A}$ with $\beta_\Gamma(n) \simeq e^n$, where $\overline{A} \cong A = \mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_2$ and $C \cong \mathbb{Z}^d$ has index 2 in A and $\overline{A}$.*

Proof. Fix a basis $\{x, y\}$ of $\mathbb{Z}^2$. Let $A_2 = \mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}_2$ where $\alpha \in \mathrm{GL}_2(\mathbb{Z})$ has matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Let $\phi = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. We amalgamate A_2 and $\overline{A}_2$ along $\mathbb{Z}^2$ by the identifications $x = \overline{\phi(x)}$ and $y = \overline{\phi(y)}$ to obtain $\Gamma = \mathbb{Z}^2 \rtimes D_\infty$, where $\mathbb{Z}^2 = \langle x, y\rangle$ and in the action of $D_\infty = \langle a, \overline{a}\rangle$, the involution a acts on $\mathbb{Z}^2$ as α while $\overline{a}$ acts as $\phi^{-1}\alpha\phi$, which has matrix $\begin{pmatrix} -2 & -3 \\ 1 & 2 \end{pmatrix}$. Thus Γ contains as a subgroup of finite index $\Gamma_0 = \mathbb{Z}^2 \rtimes_\psi \mathbb{Z}$ where $\psi = \alpha\phi^{-1}\alpha\phi$, which describes the action of $a\overline{a}$, has matrix $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$. This matrix is hyperbolic (its eigenvalues are $2 \pm \sqrt{3}$). It follows that Γ_0, and hence Γ, has exponential growth.

Entirely similar constructions yield examples with $A = \mathbb{Z}^d \rtimes \mathbb{Z}_2$. Indeed it suffices to extend the given $\alpha, \phi \in \mathrm{GL}_2(\mathbb{Z})$ to automorphisms of $\mathbb{Z}^d$ that act trivially on all but the first two elements of a basis. ☐

Proposition 2. *For each $d \in \mathbb{N}$ there exist groups of the form $\Gamma_d = B *_C \overline{B}$ with $\beta_{\Gamma_d}(n) \simeq n^{d+1}\beta_B(n)$, where $\overline{B} \cong B = \mathbb{Z}^{2d} \rtimes_{\alpha_d} \mathbb{Z}_2$ and $C \cong \mathbb{Z}^{2d}$ has index 2 in B and $\overline{B}$.*

Proof. Fix a basis for $\mathbb{Z}^{2d}$ and let α_d be the involution whose matrix has d copies of α (the involution of the previous proposition) down the diagonal and zeros elsewhere. As in the previous proposition, we amalgamate $\mathbb{Z}^{2d}$ and $\overline{\mathbb{Z}}^{2d}$ but this time replacing ϕ by the unipotent automorphism $\psi_{2d} \in \mathrm{GL}_{2d}(\mathbb{Z})$ of Example 1(ii). Arguing as above, we see that the resulting amalgam Γ has as a subgroup of index two $\mathbb{Z}^{2d} \rtimes_{\psi_{2d}^2} \mathbb{Z}$, because $\psi_{2d}^2 = \alpha_d(\psi_{2d}^{-1}\alpha_d\psi_{2d})$. And $\mathbb{Z}^{2d} \rtimes_{\psi_{2d}^2} \mathbb{Z}$ is a subgroup of index 2 in the group G_{2d} of Example 1(ii). ☐

One obtains torsion-free versions of the above phenomena by replacing $\mathbb{Z}_2$ with $\mathbb{Z}$ and by taking C to be $\mathbb{Z}^{2d} \rtimes 2\mathbb{Z}$.

Different choices of upper-triangular dihedral subgroups in $\mathrm{GL}_m(\mathbb{Z})$ can be used to vary the jump in the degree of growth obtained in Proposition 2; the examples given were chosen for their simplicity. The following example enjoys the greatest possible degree of polynomial growth among amalgams of the form $(\mathbb{Z}^m \rtimes_\alpha \mathbb{Z}_2) *_{\mathbb{Z}^m} (\mathbb{Z}^m \rtimes_{\alpha'} \mathbb{Z}_2)$.

Example 2. Let $U \in \mathrm{GL}_m(\mathbb{Z})$ be the upper-triangular matrix with all diagonal entries equal to 1 and all entries above the diagonal equal to 2. Let E be the diagonal matrix $\mathrm{diag}(1, -1, 1, -1, \dots)$. One calculates that $P := EU$ has order 2 in $\mathrm{GL}_m(\mathbb{Z})$. Thus $\mathbb{Z}^m \rtimes_U \mathbb{Z}$ is a subgroup of index 2 in $(\mathbb{Z}^m \rtimes_P \mathbb{Z}_2) *_{\mathbb{Z}^m} (\mathbb{Z}^m \rtimes_E \mathbb{Z}_2)$. As noted in Example 1(iii), the growth of $\mathbb{Z}^m \rtimes_U \mathbb{Z}$ is $\frac{1}{2}(m^2 + m + 2)$.

Remark 3. Groups of the form $\mathbb{Z}^m \rtimes_\phi \mathbb{Z}$ and their finite extensions are not asynchronously automatic if $\phi \in \mathrm{GL}_m(\mathbb{Z})$ has infinite order [8], [3]. Thus the above examples show that when one amalgamates a pair of automatic groups (indeed virtually abelian groups) along subgroups of index 2, the result need not even be asynchronously automatic. In contrast, groups of the form $\mathbb{Z}^n *_C \mathbb{Z}^m$ are always automatic. See [2] for further conditions that guarantee the (asynchronous) automaticity of amalgams.

Dehn Functions.

Recall that the *Dehn function* of a finitely presented group Γ measures the complexity of the word problem in Γ by giving a bound on the number of defining relations that one must apply in order to show that a word in the generators represents the identity; the bound is given as a function of the length of the word. The asymptotic nature of Dehn functions is unaffected by passage to finite-index subgroups. A finitely generated virtually abelian group is either virtually cyclic or else has a quadratic Dehn function. In [5] we proved that any group of the form $\mathbb{Z}^m \rtimes_\phi \mathbb{Z}$ has Dehn function $\simeq n^2\|\phi^n\|$. Our earlier examples are therefore sufficient to prove the following results, which are similar to constructions of N. Macura [15].

Proposition 3. *There exist groups A_1 and A_2 with quadratic Dehn functions and isomorphic subgroups $A_i \supset C_i \cong C$ of index 2, such that the Dehn function of $A_1 *_C A_2$ is exponential.*

Proposition 4. *For each $d \geq 2$, there exist groups $A_1 \cong A_2$ with quadratic Dehn functions and isomorphic subgroups $A_i \supset C_i \cong C$ of index 2, such that the Dehn function of $A_1 *_C A_2$ is polynomial of degree d.*

In contrast to $\mathbb{Z}^m \rtimes \mathbb{Z}$, free-by-cyclic groups have linear or quadratic Dehn functions [6]. On the other hand, it is proved in [4] that if F and L are finitely generated non-abelian free groups, then the Dehn function of a semi-direct product $F \rtimes_\Phi L$ depends heavily on the growth[1] of the automorphisms in $\Phi(L) \subset \mathrm{Aut(F)}$. If $\Gamma = F \rtimes_\Phi L$ and the image of Φ is cyclic $\langle\phi\rangle$, then the Dehn function of Γ is $\delta_\Gamma(n) \simeq n^2\|\phi^n\|$, where $\|\phi^n\|$ is defined to be the maximum of the lengths of the words $\phi^n(a_i)$ for a fixed basis $\{a_1, \ldots, a_m\}$ of F. It follows that $\delta_\Gamma(n)$ is polynomial of degree at most $m+1$ or else is exponential (see [4]).

Theorem 5. *Let F be a free group of rank m. Given any integer d with $2 \leq d \leq m+1$, one can construct a group of the form $\Gamma = A *_F B$ so that the F is normal in A and B, with $|A/F| = |B/F| = 4$, and the Dehn function of Γ is polynomial of degree d.*

There also exist such groups with exponential Dehn functions.

Proof. We shall describe in detail how to construct an example where the Dehn function is polynomial of degree $m+1$. Polynomials of lesser degree can be obtained by writing $F = F_k * F_l$ and restricting our construction of automorphisms to F_k then extending by the identity on F_l. Exponential examples are obtained by replacing the $\hat{U}$ we construct by any element of exponential growth that is contained in an infinite dihedral subgroup.

[1] There are several natural ways to define the growth of an automorphism of a finitely generated group; for abelian and free groups, these definitions are all equivalent [4].

The automorphism U of $\mathbb{Z}^m$ described in Example 2 lifts to the following automorphism $\hat{U}$ of the free group $F = \mathrm{Free}(a_1, \dots, a_m)$,

$$\hat{U}(a_1) = a_1 \text{ and } \hat{U}(a_i) = a_1 a_2 \dots a_{i-1} a_i a_{i-1} \dots a_2 a_1 \text{ for } i \geq 2.$$

Since $\hat{U}$ is a positive automorphism, the length of $\hat{U}(a_i)$ is the ℓ_1-norm of $U(a_i) \in \mathbb{Z}^m$ with respect to the basis $\{a_1, \dots, a_m\}$, and hence the growth of $\hat{U}$ is the same as that of $U \in \mathrm{GL_m}(\mathbb{Z})$, namely polynomial of degree $m-1$.

Let $\tau \in \mathrm{Aut(F)}$ be the involution $a_i \mapsto a_i^{\epsilon(i)}$ where $\epsilon(i) = (-1)^{i+1}$. One calculates that $\sigma = \tau\hat{U}$ is an involution. Thus $\hat{U}$ is a generator of the cyclic subgroup of index 2 in the dihedral group $\Delta = \langle \sigma, \tau \rangle \subset \mathrm{Aut(F)}$.

Let $A = F \rtimes_\sigma \mathbb{Z}_4$ where the generator of $\mathbb{Z}_4$ acts as σ, and let $B = F \rtimes_\tau \mathbb{Z}_4$ where the generator of $\mathbb{Z}_4$ acts as τ. By amalgamating the visible copies of F in A and B we obtain $\Gamma = A *_F B \cong F \rtimes_\Psi (\mathbb{Z}_4 * \mathbb{Z}_4)$, where the image of Ψ is Δ. By passing to a subgroup of index four in $\mathbb{Z}_4 * \mathbb{Z}_4$ we obtain a subgroup of finite index in Γ of the form $F \rtimes_\Phi L$ with L free and $\Phi(L) = \langle \hat{U} \rangle$. Since $\|\hat{U}^n\| \simeq n^{m-1}$, it follows from [4] that the Dehn function of Γ is polynomial of degree $m+1$. $\square$

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