A NOTE ON THE GRAMMAR OF COMBINGS

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ABSTRACT. Explicit constructions are given to illustrate the loss of control that ensues when one replaces the regular languages of automatic group theory with larger families of languages.

The results of [2] show that one can distinguish between various classes of combable groups by analyzing the grammatical complexity of the combings that the groups admit. That work clarifies the boundaries of automatic group theory [8] and complements the results of [4], in which Gilman and I proved that one can bring all 3-manifolds groups within the compass of automatic group theory by allowing indexed languages in place of regular languages.

The purpose of the present note is to record some observations in the spirit of [2] and [3]. Although these observations are not difficult to prove, I feel that they are worth recording because they provide clear illustrations of the somewhat wild behaviour that one encounters when one replaces the regular languages of automatic group theory with larger classes such as context-free languages or indexed languages. In contrast to the results of [3], the wildness that we exemplify here does not concern the structure of the class of groups that admit the specified type of combing. Rather, it concerns the range of combings that individual groups (even well-understood ones) admit when one relaxes the regularity assumption.

For example, we shall prove that although infinite hyperbolic groups have a *unique* automatic structure up to equivalence, all such groups admit infinitely many inequivalent context-free combings.

1. COMBINGS

A choice of generating set for a group G is a surjective monoid homomorphism $\mu : \Sigma^* \to G$, where Σ^* is the free monoid on the set Σ. We shall always assume that Σ is finite. It is convenient to assume that Σ is equipped with an involution, written $s \mapsto s^{-1}$, such that $\mu(s^{-1}) = \mu(s)^{-1}$, and we shall do so without further comment. We write $|w|$ to denote the length of a word $w \in \Sigma^*$.

Date: 1 June 2003,

2000 Subject Classification: 20F65, 68Q45.

Key words and phrases. finitely presented groups, combings, automatic groups, context-free languages.

This work was supported in part by an EPSRC Advanced Fellowship.

We consider maps $\sigma : G \to \Sigma^*$ such that $\mu \circ \sigma(g) = g$ for all $g \in G$. It is often fruitful to regard $\sigma(g)$ as a choice of discrete path from $1 \in G$ to g: at integer times $t \leq |\sigma(g)|$ this path visits the group element $\mu(\sigma(g)_t)$, where w_t denotes the prefix of length t in a word w.

The map σ is called a *combing* of G with the *(synchronous) fellow-traveller property* if the paths to adjacent vertices stay uniformly close together. More precisely, there should exist a constant $K > 0$ such that for all $g, g' \in G$,

$$d(\mu(\sigma(g)_t), \mu(\sigma(g')_t)) \leq K\, d(g, g') \tag{1.1}$$

for all $t \leq \max\{|\sigma(g)|,\, |\sigma(g')|\}$, where d is the word metric associated to our choice of generators, i.e. the unique left-invariant metric on G such that $d(1, g)$ is the length of the shortest word in $\mu^{-1}(g)$.

One calls σ a *bicombing* if it satisifes the additional condition:

$$d(\mu(s.\sigma(s^{-1}gs')_t), \mu(\sigma(g)_t)) \leq K \tag{1.2}$$

for all $s, s' \in \Sigma$ and $g \in G$.

Let $\mathcal{A}$ is a full abstract family of languages, such as the regular languages Reg, the context-free languages CF, or the indexed languages Ind. One says that σ is an $\mathcal{A}$-*combing* if it satisfies the fellow-traveller property and the image of σ is a language in the class $\mathcal{A}$. In the special case $\mathcal{A} = \text{Reg}$, an $\mathcal{A}$-combing is called an *automatic structure* for G. Correspondingly, one has $\mathcal{A}$-*bicombings* and *biautomatic structures.*

We refer the reader to Hopcroft and Ullman [10] for an introduction to formal language theory, and to Epstein *et al.* [8] for a thorough account of how the study of such languages came to play an important role in group theory and geometry/topology.

A celebrated theorem of Muller and Schupp [11] (which relies on Dunwoody's accessibility Theorem [7]) states that, given a finite generating set for a group G, the set of words representing the identity in G is a context-free language if and only if G contains a free subgroup of finite index. We need an easy part of this result:

Lemma 1.1. *If F is a free group with finite generating set $\Sigma = \Sigma^{-1}$, then the set of words in Σ^* that represent the identity in F is a context-free language.*

Proof. We'll describe a pushdown automaton $\mathcal{M}$ over the alphabet Σ whose accepted language is the given set of words. The stack alphabet of $\mathcal{M}$ is $S \cup S^{-1}$, where S is a fixed basis for F.

Let k be the maximum of the lengths of the reduced words $\{\sigma_a : a \in \Sigma\}$ in the letters $S^{\pm 1}$ such that $\sigma_a = a$ in F. When the head of $\mathcal{M}$ reads an input letter $a \in \Sigma$, it pops the word w of length $\overline{k}$ written at the top of the stack and replaces it with the freely reduced form of the word $w\sigma_a$, where $\overline{k}$ is the minimum of k and the height of the stack. $\mathcal{M}$ accepts on empty stack. $\square$

Corollary 1.2. *Let G be a group with a finite set of generators $\mu : \Sigma^* \to G$. Let $L \subset \Sigma^*$ be a regular language. If $\phi : G \to F$ is an epimorphism to a finitely generated free group, then $\{w \in L \mid \mu(w) \in \ker \phi\}$ is context-free.*

Proof. According to the lemma, the language $\{w \in \Sigma^* \mid \phi \circ \mu(w) = 1\}$ is context-free. The intersection of a context-free language and a regular language is always context-free (see [10]). ☐

2. COMBINGS OF HYPERBOLIC AND AUTOMATIC GROUPS

Rather than indulging in a general discussion, let us explore the point made in the abstract of this paper by looking at Reg-combings and CF-combings for hyperbolic and automatic groups.

We work modulo Hausdorff equivalence: two combings of a finitely generated group G, say $\sigma : G \to \Sigma^*$ and $\sigma' : G \to \Sigma^*$, are said to be equivalent is there exists a constant $C > 0$ such that the Hausdorff distance between $\sigma(g)$ and $\sigma'(g)$ is less than C for all $g \in G$, where we regard words as discrete paths in G and distance is measured with respect to an arbitrary word metric. (This is an easy notion of equivalence to state and it is sufficient for our purposes, but in many settings it is wiser to consider more refined notions of equivalence — cf. [1] and [12].)

Associated to any combing one has the *length function*

$$L_\sigma(n) := \max\{|\sigma(g)| \mid d(1,g) \leq n\}.$$

In general one wishes to remove the dependence of L_σ on the choice of word metric and to concentrate on the qualitative growth of L_σ; this is achieved by focusing on the $\simeq$ equivalence class of L_σ, where $\simeq$ is the standard relation of geometric group theory. We shall be concerned only with the condition $L_\sigma(n) \simeq n$, which is equivalent to the statement that there exists a constant $K > 0$ such that $|\sigma(\gamma)| \leq K\, d(1,\gamma)$ for every $\gamma \in \Gamma$.

Theorem 2.1. *Let Γ be a hyperbolic group.*

(1) Up to equivalence, Γ admits a unique Reg*-combing*[1].

(2) If Γ is non-elementary and $F \neq 1$ is a free group, then every epimorphism $\Gamma \to F$ with infinite kernel gives rise to infinitely many inequivalent CF*-combings σ of Γ, each with $L_\sigma(n) \simeq n$.*

We should offset the second part of the theorem by pointing out that there exist infinitely many equivalence classes of CF-combings on any infinite automatic group for rather trivial reasons; see Proposition 2.5.

Proof. The first part of this theorem is a well-known result due to Jim Cannon — indeed this is the result that began automatic group theory [6]. We recall

[1] i.e. automatic structure

the essential points of the proof for the reader's convenience (see Sections 3.2 to 3.5 of [8] or Sections III.H.1 and III.Γ.2 of [5] for details).

First, one uses Cannon's *finite cone types* argument to prove that for any fixed generating set $\mu : \Sigma \to \Gamma$, the set of geodesic words in Σ^* is a regular language. One then obtains a regular combing by choosing the "best" geodesics: linearly order Σ and give Σ^* the induced lexicographical ordering — $u \prec w$ if u is shorter than w or has the same length and precedes it in the $\prec$-dictionary; because the language of geodesics is regular, $\{w \in \Sigma^* \mid \mu(u) = \mu(w) \text{ implies } u \prec w\}$ is as well.

It follows easily from the definition of a hyperbolic group that any combing of Γ by geodesics satisfies condition (1.2), and hence is biautomatic. Indeed something much stronger than (1.2) is true: there is a constant $K = K(\delta, \lambda, \varepsilon)$ such that if two (λ, ε)-quasi-geodesics in a δ-hyperbolic geodesic space begin and end a distance 1 apart, then the Hausdorff distance between their images is less than K. (Recall that a path $c : [0, T] \to X$ in a metric space is a (λ, ε)-quasi-geodesic if $\frac{1}{\lambda}|t - t'| - \varepsilon \leq d(c(t), c(t')) \leq \lambda|t - t'| + \varepsilon$ for all $t, t' \in [0, T]$.)

Since the combing lines of an automatic structure are always (λ, ε)-quasi-geodesics (where λ and ε depend only on the automatic structure — Lemma 2.3.9 of [8]), the above stability property shows that, up to equivalence, every automatic structure on a hyperbolic group is equivalent to the geodesic one described above.

We now turn to the second part of the theorem.

Lemma 2.2. *Let F be the free group with finite basis $\mathcal{B}$, and let $\Gamma = N \rtimes F$ be a group with biautomatic structure $\sigma : \Gamma \to \Sigma^*$. Given $\gamma \in \Gamma$, write $\gamma = wn$, where w is a freely-reduced word over $\mathcal{B}^{\pm 1}$ and $n \in N$.*

Then $\tilde{\sigma}(\gamma) := w\sigma(n)$ is a CF*-combing $\Gamma \to (\Sigma \cup \mathcal{B}^{\pm 1})^*$, and $L_{\tilde{\sigma}}(m) \simeq m$.*

Proof. First we check that $\tilde{\sigma}$ satisfies condition (1.1). If $\gamma = wn$ and $b \in \mathcal{B}^{\pm 1}$ then $\tilde{\sigma}(\gamma b) = \overline{wb}\sigma(b^{-1}nb)$, where $\overline{wb}$ is the freely reduced form of wb. Thus $\tilde{\sigma}(\gamma)$ and $\tilde{\sigma}(\gamma b)$ have a common prefix (of length $|w| \pm 1$) followed by the suffixes $b\sigma(b^{-1}nb)$ and $\sigma(n)$, or else $\sigma(b^{-1}nb)$ and $b^{-1}\sigma(n)$. In either case, the paths from $1 \in \Gamma$ determined by $\tilde{\sigma}(\gamma)$ and $\tilde{\sigma}(\gamma b)$ fellow-travel because σ satisifes condition (1.2).

In order to prove that $L_{\tilde{\sigma}}(m) \simeq m$, we need the following observations. First, in the word metric associated to $\Sigma \cup \mathcal{B}^{\pm 1}$, for any $w \in F$ and $n \in N$ one has $d(1, wn) \geq k^{-1}\,|w|$ where k is the maximum length of the projections of the $s \in \Sigma$ to F. Secondly, because σ is a biautomatic structure, $L_\sigma(m) \simeq m$, so there is a constant K such that $\sigma(\gamma) \leq K\, d(1, \gamma)$ for all γ. Finally, by the triangle inequality, $d(1,n) = d(w,wn) \le d(w,1) + d(1,wn) \le |w| + d(1,wn)$.

$$\begin{aligned}|\tilde\sigma(wn)| &= |w| + |\sigma(n)| \\ &\le |w| + K\, d(1,n) \\ &\le |w| + K\,(|w| + d(1,wn)) \\ &= (1+K)|w| + K\, d(1,wn) \le [k(1+K)+K]\, d(1,wn).\end{aligned}$$

The product L_1L_2 of a regular language L_1 and a context-free language L_2 is context-free. The language of reduced words over $\mathcal{B}^{\pm 1}$ is regular, and according to Corollary 1.2 the language $\sigma(N)$ is context-free. □

Returning to the proof of the theorem, we consider a map $h : \Gamma \twoheadrightarrow F$, choose a basis $\{b_1, \dots, b_n\}$ for F, and choose a splitting $b_i \mapsto \tilde b_i \in \Gamma$ to get $\Gamma = N \rtimes F$. Using the resulting factorisations $\gamma = wn$, we define $\tilde\sigma(\gamma) = w\sigma(n)$, where σ is the geodesic combing of Γ described in the proof of (1) and w is a reduced word in the letters $\tilde b_i^{\pm 1}$. The lemma tells us that $\tilde\sigma$ is a CF-combing of Γ, with $L_{\tilde\sigma}(m) \simeq m$.

By varying the splitting of h, one obtains inequivalent combings of Γ. For example, since N is infinite it contains an element, s say, such that s and $\tilde b_1$ generate a free subgroup of rank 2 (see [9] or Section III.Γ.2 of [5]). For each $i \in \mathbb{Z}$ we may change our splitting of h by replacing $\tilde b_1$ with $\hat b_1 = s^i \tilde b_1$; this will result in a CF-combing where the combing line for $\tilde b_1^r$ begins $(s^i\tilde b_1)^r \dots$. If $i \ne j$, the prefixes $(s^i\tilde b_1)^r$ and $(s^j\tilde b_1)^r$ can be forced arbitrarily far apart by increasing r. □

Remarks 2.3. (1) Lemma 2.2 may be viewed in the following general context: if $G = N \rtimes Q$ is bicombable with combing σ^G, and Q is combable with a combing σ^Q that has the comparable lengths property[2] (σ^Q may be totally unrelated to σ^G), then $\sigma(qn) := \sigma^Q(q)\sigma^G(n)$ defines a combing of G that satisfies condition (1.1) and has length $\preceq L_{\sigma^G}(n) + L_{\sigma^Q}(n)$.

(2) Although the non-regular combings in the above theorem have linear length functions, their combing lines are never quasi-geodesic: certain subpaths of the combing lines will be extremely (metrically) inefficient. This behaviour is manifest in the first of the following examples.

Examples 2.4. (1) We apply the construction of Lemma 2.2 with the role of Γ played by the free group $F(a,b)$, decomposed by the obvious splitting of the map $F(a,b) \to \mathbb{Z}$ that sends a to a generator and b to the identity. In this case we obtain a CF-combing $\sigma : F \to \{a, a^{-1}, b, b^{-1}\}^*$ in which $\sigma((ab)^n) = a^n(a^{-1})^{n-1}(ba)^{n-1}b$.

[2] e.g. if σ^Q is an automatic structure; see [2] for the general definition.

(2) $\mathbb{Z} = \langle t \rangle$ admits infinitely many (uninteresting) inequivalent CF-combings σ with $L_\sigma(n) \simeq n$. Indeed for any positive integers $m \neq m'$, the following CF-combings $\sigma^m, \sigma^{m'} : \mathbb{Z} \to \{t, t^{-1}\}^*$ are inequivalent: $\sigma^m(t^i) = t^{mi}(t^{-1})^{(m-1)i}$.

Continuing the theme of the above example, we note that on any infinite automatic group one can construct infinitely many (pathological) inequivalent CF-combings of linear length.

Proposition 2.5. *Let $\sigma : \Gamma \to \Sigma^*$ be an automatic structure, let $t \in \Gamma$ be an element of infinite order, and let m be a positive integer. Define $\sigma^m : \Gamma \to (\Sigma \cup \{t, t^{-1}\})^*$ by $\sigma^m(g) = \sigma(g) t^{m\,|\sigma(g)|} t^{-m\,|\sigma(g)|}$.*

Then, σ^m is a CF-combing with $L_{\sigma^m}(n) \simeq n$. If $m_1 \neq m_2$, then σ^{m_1} and σ^{m_2} are not equivalent.

Proof. We shall prove that $\sigma^m(\Gamma)$ is context-free by describing a pushdown automaton $\mathcal{M}$ over the alphabet $(\Sigma \cup \{t, t^{-1}\})$ that accepts precisely this language.

$\mathcal{M}$ consists of a finite state automaton $\mathcal{M}_0$ whose accepted language is $\sigma(\Gamma)$ and a stack with a 1-letter stack alphabet. We decree that $\mathcal{M}$ should stop and reject a word if in the course of reading it any letter $s \in \Sigma$ occurs after an occurrence of $t^{\pm 1}$. Also, $\mathcal{M}$ rejects the word if it reads a t after a t^{-1}. Thus all accepted words will be of the form uv with $u \in \Sigma^*$ and $v \in \{t\}^*\{t^{-1}\}^*$.

As $\mathcal{M}$ reads the word u from the input tape, the FSA $\mathcal{M}_0$ computes whether or not u is in the image of σ; at the same time m letters are pushed on the stack for each letter of u that is read. When u has been read completely, the computation continues if and only if u is in the image of σ.

If the computation continues, as $\mathcal{M}$ reads each letter t from v it pushes a further m letters on the stack. Finally, as $\mathcal{M}$ reads each letter t^{-1}, it pops $2m$ letters from the stack. The word uv is accepted by $\mathcal{M}$ only if the stack is empty when the tape has been read completely. $\square$

Acknowledgement: I wrote this note in preparation for my lecture on combings of groups at the 'International Conference on Geometric and Combinatorial Methods in Group Theory and Semigroup Theory' held at the University of Nebraska in Lincoln, May 2000. If I not been so slow in producing the final version of the manuscript, it would have appeared in the proceedings of that conference.

I would like to take the organisers, Susan Hermiller, John Meakin and Mark Sapir, for arranging this stimulating and enjoyable meeting. I particularly thank John Meakin for his warm hospitality.

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