Topology Advanced Class on

The AKSZ Construction in Derived Algebraic Geometry as an Extended Topological Field Theory

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The University of Oxford

In this TAC we will see how the AKSZ construction in derived algebraic geometry generates fully extended topological field theories. Our main reference for this is [CHS], but see also [Cal21, Cal] for some very nice introduction and overview material. These papers also contain more examples than our main reference [CHS]. For more background on the physics side, see, for instance, the original paper [AKSZ97], or [JRSW19, Cos11].

Prep. for everyone: read [CHS, Sec. 1].

Broadly speaking, Talks 1–4 cover the categorical and geometric foundations for the construction of the target categories of the field theories we aim to build, as well as for a significant part of the field theory construction itself. Talk 5 covers the background on the field-theory side. Talks 6–9 bring everything together and construct the AKSZ field theories. Note that we will skip Talk 7 because of the Hopkins conference in Oxford that week.

Talk 1. (24/04/23, Severin) **DAG 1.** We start by covering some ∞-categorical and algebro-geometric background in the first two talks. The AKSZ construction in derived algebraic geometry is a way of building shifted symplectic forms on derived mapping stacks (the 'classical spaces of fields on bordisms'). Our first goal is to understand these notions.

Briefly explain localisation of ∞-categories, (co)Cartesian fibrations and their relation to functors $\mathcal{C}^{(\mathrm{op})} \to \mathcal{C}\mathrm{at}_\infty$ (e.g. [Lan21, Secs. 1.2, 3.1, 3.3], in ptl. [Lan21, Prop. 2.4.8, Thms. 3.3.10 and 3.3.16]).

Then we use these to formulate derived algebraic geometry: survey [CHS, Secs. B.1–B.7] (see also [Cal21, Cal]). Introduce cotangent complexes of higher commutative algebras, the ∞-categories of (relative) derived stacks (mention Betti stacks and mapping stacks as examples), and quasicoherent sheaves (mention (co)tangent complexes).

We will later build field theories with values in cospans of Betti stacks and spans of mapping stacks (with additional geometric structures).

Talk 2. (01/05/23, Adri) **DAG 2.** We continue covering background from derived algebraic geometry. Survey [CHS, Secs. B.8–B.12]; see also [Cal21, Cal] for a different nice overview. Explain base change and the projection formula. Introduce cotangent complexes of derived stacks, de Rham complexes and shifted (closed) differential forms. See also [Cal21, Cal] for different presentations and examples (e.g. BG for nice algebraic groups or Higgs bundles). Mention shifted symplectic forms as important examples.

The target categories of the field theories we will construct will, in particular, be built from derived mapping stacks endowed with shifted symplectic forms.

Talk 3. (08/05/23, Thibault) **Spans 1.** In order to properly formulate the target categories of our field theories, and thus also the field theories themselves, we need (∞, n)-categories of (co)spans. This is developed in [Hau18], actually even with an eye towards 'iterated Lagrangian correspondences', which are the targets we need.

Cover [Hau18, Secs. 3–9]. Introduce complete Segal objects and iterated complete Segal objects in ∞-categories. Define iterated spans, add 'local systems', and show Segal properties and completeness. Use cospans of spaces Σ and their Betti stacks Σ_B, as well as mapping stacks $\mathrm{Map}(\Sigma_B, X)$ (for X fixed) as examples.

Talk 4. (15/05/23, Nivedita) **Spans 2.** For field theories, we need symmetric monoidal (∞, n)-categories as our targets. If we would like to use the Cobordism Hypothesis to understand and classify our field theories, we further need to understand their fully dualisable objects.

Cover [Hau18, Secs. 7–14]. Give the symmetric monoidal structures on iterated spans, review adjoints and duals in iterated Segal spaces (see also [Lur09]), state the Cobordism Hypothesis as motivation. Show that each object in (∞, n)-categories of n-fold spans is full dualisable; sketch the case of Lagrangian correspondences of shifted symplectic derived stacks [Hau18, Sec. 14].

Talk 5. (22/05/23, Filippos) **Bordisms and cospans.** Now that we have an idea of (∞, n)-categories of iterated spans (i.e. the codomain side of our field theories), we work on the domain side of our field theory functors: we recall the construction of extended bordism categories and the Cobordism Hypothesis in more detail.

Review the construction of bordism (∞, n)-categories form [CS19, Secs. 4–7] (see also [Lur09, CS19] for intuition and background, as well as the summary in [CHS, Sec. 4.1]). Recall the Cobordism Hypothesis and explain $\mathrm{O}(n)$-actions in Cobordism Hypothesis [Lur09, Cor. 2.4.10, Thm. 2.4.18]. Build the canonical/tautological TQFT in cospans of spaces given by cutting bordisms [CHS, Sec. 4] and show that the $\mathrm{O}(n)$-action for field theories valued in iterated spans is trivial [CHS, Cor. 4.4.3].

Talk 6. (29/05/23 Chenjing) **Lagrangian correspondences.** We now start to properly combine the categorical and derived geometric structures we have encountered so far.

Cover [CHS, Secs. 2.1–2.9]. Introduce orientations, recall shifted symplectic structures, and define Lagrangian correspondences and oriented cospans. The main goal is to build (∞, n)-categories of iterated Lagrangian correspondences and iterated oriented cospans. Both of these feature in the AKSZ construction/transgression formalism which underlies the field theory construction.

Talk 7. (05/06/23) A Panorama of Homotopy Theory/Hopkins conference, highly recommended—no talk here, instead in Week 9.

Talk 8. (12/06/23, Thomas) **The AKSZ construction in DAG.** This talk will complete the construction of the (∞, n)-categories of Lagrangian correspondences and oriented cospans by adding their symmetric monoidal structure and showing the full dualisability of Lagrangian correspondences; this is [CHS, 2.10–12].

Then, introduce the fibre integration/pushforward of shifted forms, and thus the AKSZ construction for derived shifted symplectic stacks. Explain its bivariant functoriality properties and show its non-degeneracy: it produces iterated Lagrangian correspondences [CHS, Secs. 2.9–3.4].

Talk 9. (15/06/23 Severin) **The main theorem.** This talk will sketch the proof of the main result [CHS, Cor. 5.6.3]: 'the AKSZ construction produces fully extended n-dimensional topological field theories, valued in n-fold Lagrangian correspondences'. More precisely, cutting oriented bordisms gives n-fold oriented cospans of spaces, then taking Betti stacks produces oriented derived stacks, then taking $\mathrm{Map}(-, X)$ for some shifted symplectic stack X and applying AKSZ fibre integration gives n-fold spans of Lagrangian correspondences. This is [CHS, Sec. 5].

Talk 10. (05/07/23, Sujay) **The AKSZ construction in mathematical physics.**

Background on dg manifolds, cover parts of [AKSZ97, Roy07] as desired, applications to Chern-Simons theory and possibly mirror symmetry.

References

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