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Algebraic Topology
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Sheet 10

Solutions are due on 22.06.18.

Problem 10.1

Let X, Y be topological spaces whose homology groups are finitely generated and trivial except in finitely many degrees. Prove the multiplicativity of the Euler characteristic, i.e. show that

$$\chi(X \times Y) = \chi(X)\,\chi(Y)\,.$$

Problem 10.2

Recall that the Moore space $M(\mathbb{Z}_m, n)$ for the cyclic group $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ is obtained by attaching an $(n+1)$-cell to $\mathbb{S}^n$, using a map $\mathbb{S}^n \to \mathbb{S}^n$ of degree m.

(a) Compute the homology and the cohomology of $M(\mathbb{Z}_m, n)$ with values in an arbitrary abelian group G.

(b) Consider the map $f\colon M(\mathbb{Z}_m, n) \to \mathbb{S}^{n+1}$ which collapses the n-skeleton of $M(\mathbb{Z}_m, n)$ to a point. Show that f induces the trivial map on $H_i(M(\mathbb{Z}_m, n), \mathbb{Z})$ precisely for $i \neq 0$ and on $H^i(M(\mathbb{Z}_m, n), \mathbb{Z})$ precisely when $i \neq 0, n+1$.

(c) Conclude that there does not exist an isomorphism

$$i_X\colon H^n(X, \mathbb{Z}) \longrightarrow \mathrm{Ext}\big(H_{n-1}(X), \mathbb{Z}\big) \oplus \mathrm{Hom}\big(H_n(X), \mathbb{Z}\big)$$

for every space X that is natural in X, i.e. such that for any continuous map $f\colon X \to Y$ there is a commutative diagram

$$\begin{array}{ccc} H^n(X,\mathbb{Z}) & \xrightarrow{\ i_X\ } & \mathrm{Ext}\big(H_{n-1}(X),\mathbb{Z}\big) \oplus \mathrm{Hom}\big(H_n(X),\mathbb{Z}\big) \\ {\scriptstyle H^n(f)}\uparrow & & \uparrow{\scriptstyle \mathrm{Ext}(H_{n-1}(f),\mathrm{id}_{\mathbb{Z}})\oplus\mathrm{Hom}(H_n(f),\mathrm{id}_{\mathbb{Z}})} \\ H^n(Y,\mathbb{Z}) & \xrightarrow[\ i_Y\ ]{} & \mathrm{Ext}\big(H_{n-1}(Y),\mathbb{Z}\big) \oplus \mathrm{Hom}\big(H_n(Y),\mathbb{Z}\big) \end{array}$$

In other words, the split short exact sequence from the Universal Coefficient Theorem yields a description of the cohomology groups of any *individual* topological space X up to *some* isomorphism, but there is no *coherent* choice of such isomorphisms for *all* topological spaces.

Problem 10.3

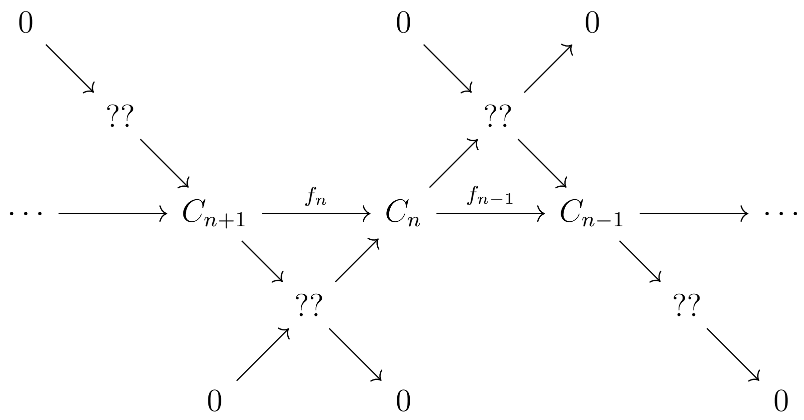
Let G be an abelian group such that $\mathrm{Hom}(-, G)$ is exact (e.g. any of the cases in Example 2.2.2.7). Show that $\mathrm{Hom}(-, G)$ preserves long exact sequences: if

$$\cdots \longrightarrow C_{n+1} \xrightarrow{f_n} C_n \xrightarrow{f_{n-1}} C_{n-1} \longrightarrow \cdots$$

is a long exact sequence of abelian groups, then so is

$$\cdots \longrightarrow \mathrm{Hom}(C_{n-1}, G) \xrightarrow{f_{n-1}^*} \mathrm{Hom}(C_n, G) \xrightarrow{f_n^*} \mathrm{Hom}(C_{n+1}, G) \longrightarrow \cdots .$$

Hint: consider an augmented diagram



where all diagonals are short exact sequences and all triangles commute.

Problem 10.4

Show that the family $\{h^n\}_{n\in\mathbb{N}}$ of contravariant functors from pairs of topological spaces and continuous maps of pairs to abelian groups and group homomorphisms, defined by

$$(X, A) \mapsto h^*(X, A) \coloneqq \mathrm{Hom}\big(H_*(X, A; \mathbb{Z}), \mathbb{Z}\big) ,$$
$$\big((X, A) \xrightarrow{f} (Y, B)\big) \mapsto h^*(f) \coloneqq \mathrm{Hom}\big(H_*(f), \mathrm{id}_{\mathbb{Z}}\big) ,$$

does not define a cohomology theory. Which axioms are satisfied, which are violated? Give proofs or counterexamples.

What if we replace $\mathrm{Hom}(-, \mathbb{Z})$ in the definition of h^* by $\mathrm{Hom}(-, G)$ for some abelian group G as in Problem 10.3?