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Algebraic Topology
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Sheet 13

Solutions are due on 13.07.18.

Problem 13.1

(a) Compute $H^2(\mathbb{S}^1 \times [0,1])$. Does this mean that Theorem 2.8.3 is wrong?

(b) Compute $H^2_c(\mathbb{S}^1 \times (0,1))$.

Problem 13.2

Let M be a compact, connected, orientable m-dimensional manifold.

(a) Show that $H_{m-1}(M)$ is torsion-free. (Can you show this in two slightly different ways?)

(b) Now suppose that $m = 2k$ is even. Show that if $H_{k-1}(M)$ is torsion-free, then $H_k(M)$ is torsion-free as well. (Use that homology groups of compact connected manifolds are finitely generated – see Hatcher, Cor. A.8, A.9.)

Problem 13.3

For connected n-manifolds M_1 and M_2, the *connected sum* $M_1 \sharp M_2$ is constructed by cutting out small n-dimensional balls from M_1 and M_2 and gluing the resulting boundary spheres via a homeomorphism. For example, the connected sum of a compact connected surface F_g of genus g with a compact connected surface F_h of genus h is a compact connected surface F_{g+h} of genus $g + h$.

(a) Determine under which conditions the manifold $M_1 \sharp M_2$ is orientable.
Hint: One possible way to do this is via the orientation covering from Problem 11.3. Recall that (for $\mathbb{Z}$-orientations) this is a two-sheeted covering $\pi \colon \tilde{M} \to M$ whose fibre over any point $x \in M$ consists of the two possible choices of local orientations at x.

(b) Compute the cohomology $H_*(M_1 \sharp M_2)$ in terms of $H_*(M_1)$ and $H_*(M_2)$ for M_1 and M_2 compact and connected. Check your result in the case of surfaces.
Hint: Contract the sphere along which M_1 and M_2 are glued to a point to compare $M_1 \sharp M_2$ to $M_1 \vee M_2$. Use that $H_m(M; R) = 0$ for compact m-dimensional manifolds M that do not admit an R-orientation (see Remark 2.6.14.2). You may also use that orientability with respect to $R = \mathbb{Z}_p$ for p odd is equivalent to $\mathbb{Z}$-orientability.

Problem 13.4

Let M be a non-orientable closed n-manifold and let $\pi\colon \tilde{M} \to M$ be the two-sheeted orientation cover of M. Let F be either the field $\mathbb{Q}$ or $\mathbb{Z}_p$, with p an odd prime (or even any field of characteristic $\neq 2$). Follow the instructions given below to show that

$$H^k(\tilde{M}; F) \cong H^k(M; F) \oplus H^{n-k}(M; F)\,.$$

(a) Prove that for an F-vector space V and a linear endomorphism $T\colon V \to V$ such that $T^2 = \mathrm{id}_V$, there is a splitting $V = V^+ \oplus V^-$ into a direct sum of eigenspaces of T for the eigenvalues ± 1.

(b) Use the non-trivial deck transformation $\tau\colon \tilde{M} \to \tilde{M}$ that interchanges the two sheets to define splittings of cohomology and homology groups

$$H_k(\tilde{M}; F) = H_k^+(\tilde{M}; F) \oplus H_k^-(\tilde{M}; F) \quad \text{and} \quad H^k(\tilde{M}; F) = H^{k+}(\tilde{M}; F) \oplus H^{k-}(\tilde{M}; F)\,.$$

(c) Again using τ, give a natural isomorphism $H_k(M; F) \to H_k^+(\tilde{M}; F)$. Proceed similarly in cohomology. Use these isomorphisms to identify the two vector spaces.

(d) Show that the Poincaré duality isomorphism induces isomorphisms

$$H^{k\pm}(\tilde{M}; F) \cong H^{\mp}_{n-k}(\tilde{M}; F)$$

To this end, use that τ is orientation-reversing, i.e. $\tau[\tilde{M}] = -[\tilde{M}]$ for the fundamental class $[\tilde{M}]$ of the orientable manifold $\tilde{M}$.