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Algebraic Topology
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Sheet 1

Solutions are due on 13.04.18.

Problem 1.1

Let $\mathbb{D}^n$ be the chain complex of abelian groups whose only non-trivial entries are in degrees n and $n-1$, with $\mathbb{D}^n_n = \mathbb{D}^n_{n-1} = \mathbb{Z}$. Its only non-trivial boundary operator is the identity:

$$\mathbb{D}^n_* := \quad \ldots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \longrightarrow 0 \longrightarrow \ldots .$$

Similarly, let $\mathbb{S}^n$ be the chain complex whose only non-trivial entry is $\mathbb{S}^n_n = \mathbb{Z}$, i.e.

$$\mathbb{S}^n_* := \quad \ldots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \ldots .$$

(a) What is the homology of the chain complexes $\mathbb{D}^n_*$ and $\mathbb{S}^m_*$?

(b) Assume that (C_*, d) is an arbitrary chain complex of abelian groups. Describe the chain maps from $\mathbb{D}^n_*$ to C_* and from $\mathbb{S}^n_*$ to C_* in terms of elements of the groups C_*.

(c) Are there chain maps between $\mathbb{D}^n_*$ and $\mathbb{S}^m_*$?

(d) Let $f_* \colon C_* \to C'_*$ be a chain map and assume that f_n is a monomorphism for all n. Do we then know that the maps $H_n(f_*)$ induced on homology are also monomorphisms?

Problem 1.2

(a) What are the homology groups of the chain complex

$$C_* := \quad \ldots \longrightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{2\cdot(-)} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2\cdot(-)} \ldots$$

(b) Is there a chain homotopy from the identity of C_* to the zero map, i.e. are there maps $s_n \colon C_n \to C_{n+1}$ with $d_{n+1} \circ s_n + s_{n-1} \circ d_n = \mathrm{id}_{C_n}$ for all $n \in \mathbb{Z}$?

Problem 1.3

Let $(A_n)_{n\in\mathbb{Z}}$ be an arbitrary family of finitely generated abelian groups. Is there a chain complex F_* with F_n a finitely generated, free abelian group as well as with $H_n(F_*) \cong A_n$ for all $n \in \mathbb{Z}$? (Recall the structure theorem for finitely generated abelian groups for this problem.)

Problem 1.4

Let (C_*, d_*^C) and (D_*, d_*^D) be chain complexes of abelian groups. For $k \in \mathbb{Z}$ we define a new chain complex $(D_*[k], d_*^{D[k]})$ by

$$D_n[k] := D_{n+k}\,, \quad d_n^{D[k]} := d_{n+k}^D\,.$$

We let $\mathrm{Hom}_{\mathrm{Ch}}(C_*, D_*)$ denote the abelian group of families $(f_n \colon C_n \to D_n)_{n\in\mathbb{Z}}$, where each $f_n \colon C_n \to D_n$ is a morphism of abelian groups, and we set

$$\mathrm{Hom}_{\mathrm{Ch},k}(C_*, D_*) := \mathrm{Hom}_{\mathrm{Ch}}(C_*, D_*[k])$$

for all $k \in \mathbb{Z}$.

Define morphisms of abelian groups

$$d_k^{\mathrm{Hom}} \colon \mathrm{Hom}_{\mathrm{Ch},k}(C_*, D_*) \to \mathrm{Hom}_{\mathrm{Ch},k-1}(C_*, D_*)$$

such that $(\mathrm{Hom}_{\mathrm{Ch},*}(C_*, D_*), d_*^{\mathrm{Hom}})$ becomes a chain complex with the two properties

(1) an element $f_* \in \mathrm{Hom}_{\mathrm{Ch},0}(C_*, D_*)$ is a chain map from C_* to D_* if and only if it is a 0-cycle in the complex $(\mathrm{Hom}_{\mathrm{Ch},*}(C_*, D_*), d_*^{\mathrm{Hom}})$,

(2) if $f_*, g_* \in \mathrm{Hom}_{\mathrm{Ch},0}(C_*, D_*)$ are chain maps and $h_* \in \mathrm{Hom}_{\mathrm{Ch},1}(C_*, D_*)$, then h_* is a chain homotopy from f_* to g_* if and only if $d_1^{\mathrm{Hom}}(h_*) = f_* - g_*$.

The complex $(\mathrm{Hom}_{\mathrm{Ch},*}(C_*, D_*), d_*^{\mathrm{Hom}})$ is also called the *mapping complex of* (C_*, d_*^C) *and* (D_*, d_*^D). Condition (2) says that there exists a chain homotopy from a chain map f_* to a chain map g_* if and only if the chain map $f_* - g_*$ is a boundary in the mapping complex.