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Algebraic Topology
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# Sheet 2

Solutions are due on 20.04.18.

## Problem 2.1

Consider an exact chain complex

$$\dots \longrightarrow 0 \xrightarrow{d_{N+1}} V_N \xrightarrow{d_N} \dots \xrightarrow{d_2} V_1 \xrightarrow{d_1} V_0 \xrightarrow{d_0} 0 \longrightarrow \dots$$

of finite-dimensional vector spaces which is bounded above and below as indicated. Here the differentials $d_*$ are linear maps. Compute the Euler characteristic

$$\chi(V_*) := \sum_{i \in \mathbb{Z}} (-1)^i \dim(V_i)\,.$$

## Problem 2.2

(a) Let $X$ and $Y$ be topological spaces. Is every chain map $f_* \colon S_*(X) \to S_*(Y)$ induced by a map of topological spaces?

(b) Let $p \colon \tilde{X} \to X$ be a covering map. We know that the induced map $\pi_1(p)$ on fundamental groups is a monomorphism. Is that also true for the map $H_1(p)$ induced on homology?

## Problem 2.3

Let $\Delta^n$ be the standard topological $n$-simplex, i.e.

$$\Delta^n := \Big\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \,\Big|\, \sum_{i=0}^n t_i = 1,\, t_i \geq 0 \ \ \forall\, i = 0, \dots, n \Big\} \subset \mathbb{R}^{n+1}\,, \tag{1}$$

endowed with the subspace topology of $\mathbb{R}^{n+1}$. For $i \in \{0, \dots, n-1\}$ we define the *degeneracy maps*

$$s_i \colon \Delta^n \to \Delta^{n-1}\,, \quad (t_0, \dots, t_n) \mapsto (t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_n)\,.$$

Check that the face and degeneracy maps together satisfy the *cosimplicial identities*

$$\begin{cases} d_j \circ d_i = d_i \circ d_{j-1}\,, & 0 \leq i < j \leq n\,, \\ s_j \circ d_i = d_i \circ s_{j-1}\,, & 0 \leq i < j \leq n\,, \\ s_j \circ d_j = \mathrm{id} = s_j \circ d_{j+1}\,, & 0 \leq j \leq n\,, \\ s_j \circ d_i = d_{i-1} \circ s_j\,, & 1 \leq j+1 < i \leq n\,, \\ s_j \circ s_i = s_i \circ s_{j+1}\,, & 0 \leq i \leq j \leq n\,. \end{cases}$$

Observe that the first of these identities has already been shown in Lemma 1.2.3.

**Problem 2.4**

Let $n \in \mathbb{N}_0$ and $k \in \{0, \dots, n\}$ be arbitrary. Let $E_n \subset \mathbb{R}^{n+1}$ be the unique $n$-dimensional affine subspace of $\mathbb{R}^{n+1}$ that contains the standard basis vectors $(e_i)_{i=0,\dots,n}$. We let $\partial\Delta^n$ denote the topological boundary of the standard topological $n$-simplex $\Delta^n$, seen as a subspace of $E_n$. Further, the *$k$-th horn of* $\Delta^n$ is defined as the union

$$\Lambda^n_k := \bigcup_{i \in \{0,\dots,n\} \setminus \{k\}} d_i(\Delta^{n-1}) \quad \subset \Delta^n$$

of the images of the face maps $d_i$ for $i \in \{0, \dots, n\} \setminus \{k\}$.

(a) Give explicit expressions of the form of Equation (1) for $\partial\Delta^n$ and for $\Lambda^n_k$. (Why is the horn called horn? Can you explain why $\Delta^n$ and $\Lambda^n_k$ are very intuitive choices of notation?)

(b) Show that $\Delta^n$ deformation retracts onto any of its faces $d_k\Delta^{n-1}$. Do so by constructing a deformation retraction $h\colon [0,1] \times \Delta^n \to \Delta^n$ whose restriction to $\Lambda^n_k$ yields a homeomorphism $h_{1|\Lambda^n_k}\colon \Lambda^n_k \to d_k\Delta^{n-1}$.

(c) Let $X$ be a topological space. A *$\Lambda^n_k$-horn on* $X$ is a continuous map $\alpha\colon \Lambda^n_k \to X$. Use the statement of part (b) to prove that any $\Lambda^n_k$-horn $\alpha\colon \Lambda^n_k \to X$ on $X$ can be extended to an $n$-simplex $\hat{\alpha}\colon \Delta^n \to X$ on $X$.

(d*) The insight from part (c) can be used to concatenate 1-simplices "up to 2-simplices". That is, given two 1-simplices $\alpha_0, \alpha_2\colon \Delta^1 \to X$ such that $\partial_0\alpha_2 = \partial_1\alpha_0$, show that there exists some 2-simplex $\beta\colon \Delta^2 \to X$ such that $\partial_j\beta = \alpha_j$ for $j = 0, 2$. We may then call $\alpha_1 := \partial_1\beta\colon \Delta^1 \to X$ *a* (not *the*!) concatenation of $\alpha_0$ and $\alpha_2$; in general the 1-simplex $\alpha_1\colon \Delta^1 \to X$ depends on the choice of $\beta$. Given another 2-simplex $\beta'$ with the above properties, i.e. defining another choice $\alpha' := \partial_1\beta'$ of concatenation of $\alpha_0$ and $\alpha_2$, show that $[\alpha_1] = [\alpha'_1]$ in homology.