Universität Hamburg
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Algebraic Topology
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Sheet 3

Solutions are due on 27.04.18.

Problem 3.1

(a) How many exact sequences of the form

$$\ldots \xrightarrow{f_3} \mathbb{Z}/4\mathbb{Z} \xrightarrow{f_2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{f_1} \mathbb{Z}/4\mathbb{Z} \xrightarrow{f_0} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

exist?

(b) Is there a short exact sequence of $\mathbb{Z}$-modules of the form

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow 0 \quad ?$$

Problem 3.2

Let R be a unital ring. A short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of R-modules is called <u>split</u> if there is an isomorphism of R-modules $\Phi\colon M_2 \to M_1 \oplus M_3$ such that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & M_1 & \xrightarrow[\iota_1]{} & M_1 \oplus M_3 & \xrightarrow[\mathrm{pr}_{M_3}]{} & M_3 & \longrightarrow & 0
\end{array}$$

commutes, where ι_1 is the canonical injection and pr_{M_3} the canonical projection. Show that the following statements are equivalent:

(a) The short exact sequence splits.

(b) The injection $f_1\colon M_1 \to M_2$ has a left inverse, i.e. there exists a morphism $g_1\colon M_2 \to M_1$ such that $g_1 \circ f_1 = \mathrm{id}_{M_1}$.

(c) The surjection $f_2\colon M_2 \to M_3$ has a right inverse, i.e. there exists a morphism $g_2\colon M_3 \to M_2$ such that $f_2 \circ g_2 = \mathrm{id}_{M_3}$.

Problem 3.3

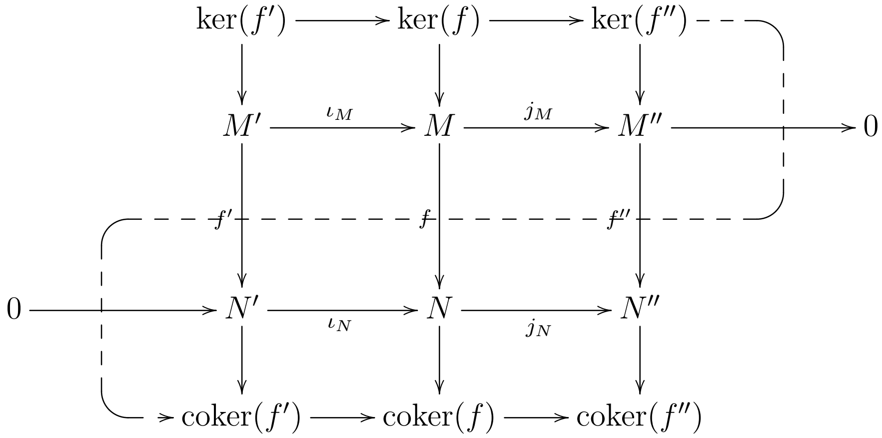
Prove the famous snake lemma: if

$$\begin{CD} @. M' @>{\iota_M}>> M @>{j_M}>> M'' @>>> 0 \\ @. @V{f'}VV @V{f}VV @V{f''}VV \\ 0 @>>> N' @>>{\iota_N}> N @>>{j_N}> N'' \end{CD}$$

is a commutative diagram of R-modules with exact rows, then there is an exact sequence

$$\ker(f') \to \ker(f) \to \ker(f'') \xrightarrow{\partial} \operatorname{coker}(f') \to \operatorname{coker}(f) \to \operatorname{coker}(f'')$$

with a morphism ∂, which you are supposed to construct in the proof (see also Proposition 1.5.5). All other morphisms are induced by the morphisms in the diagram. The name 'snake lemma' is explained by the diagram in which kernels and cokernels are added:



Additionally, observe that if $M' \to M$ is a monomorphism, then $\ker(f') \to \ker(f)$ is a monomorphism as well, and if $N \to N''$ is an epimorphism, then $\operatorname{coker}(f) \to \operatorname{coker}(f'')$ is an epimorphism as well.

Problem 3.4

(a) Let F_g be a closed orientable surface of genus $g \in \mathbb{N}_0$. Use the Seifert van Kampen Theorem to determine the fundamental group $\pi_1(F_g)$ and the Hurewicz Theorem to calculate the first homology group $H_1(F_g)$.

(b) Compute the fundamental group and the first homology of the Klein bottle.