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Algebraic Topology
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Sheet 4

Solutions are due on 04.05.18.

Problem 4.1

Consider the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow & & {\scriptstyle f_5}\downarrow \\ B_1 & \xrightarrow[\beta_1]{} & B_2 & \xrightarrow[\beta_2]{} & B_3 & \xrightarrow[\beta_3]{} & B_4 & \xrightarrow[\beta_4]{} & B_5 \end{array}$$

Under which conditions on f_1, f_2, f_4 and f_5 can we deduce that the map f_3 is a monomorphism or an epimorphism? Prove your claims!

Problem 4.2

(a) Let $x \in \mathbb{R}^n$ be an arbitrary point. Express $H_*(\mathbb{R}^n \setminus \{x\})$ in terms of homology groups of spheres.

(b) Let $[y] \in \mathbb{R}P^2$ be an arbitrary point. Express the homology groups $H_*(\mathbb{R}P^2 \setminus \{[y]\})$ in terms of homology groups of spheres.

(c) For $n \in \mathbb{N}$, let $\mathbb{C}P^n := (\mathbb{C}^{n+1} \setminus \{0\})/\sim$, where $z \sim z'$ if there exists a $\lambda \in \mathbb{C} \setminus \{0\}$ such that $z = \lambda\, z'$. The spaces $\mathbb{C}P^n$ are called the *complex projective spaces*. Consider an arbitrary point $[z] \in \mathbb{C}P^2$. Express $H_*(\mathbb{C}P^2 \setminus \{[z]\})$ in terms of homology groups of spheres.

Problem 4.3

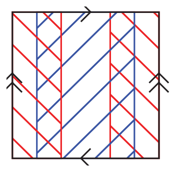
A topological space M is called *locally Euclidean of dimension* $n \in \mathbb{N}_0$ if every point $x \in M$ has an open neighbourhood that is homeomorphic to some open subset $V \subset \mathbb{R}^n$.

(a) Let M be Hausdorff and locally Euclidean of dimension $n \geq 1$ (for example a smooth n-dimensional manifold without boundary).
Use excision to compute $H_n(M, M\setminus\{x\})$ for any point $x \in M$. These groups are sometimes referred to as the *local homology groups of* M *at* $x \in M$.

(b) Consider the case when M is an open Möbius strip, i.e. has no boundary. Pick a generator $\mu_x \in H_n(M, M \setminus \{x\}) \cong \mathbb{Z}$ and describe what happens with μ_x if one walks along the meridian of the Möbius strip.

Problem 4.4

Cover a Klein bottle by two open subsets that are each homeomorphic to an open Möbius strip (see the parts shaded in blue and red in the figure). Compute the homology of the Klein bottle with the help of this cover.



Problem 4.5

Let (X, A) be a pair of topological spaces, and let $\iota\colon A \hookrightarrow X$ denote the inclusion of A into X.

(a) Show that the short exact sequence

$$0 \longrightarrow S_n(A) \xrightarrow{S_n(\iota)} S_n(X) \xrightarrow{p_n} \frac{S_n(X)}{S_n(A)} \longrightarrow 0$$

is split (cf. Problem 3.2) for every $n \in \mathbb{N}_0$.

(b) Why does this, in general, not induce a splitting $H_n(X) \cong H_n(A) \oplus H_n(X, A)$ in homology?

(c) Give an example of a pair (X, A) such that there cannot be an isomorphism in homology as in part (b).