Universität Hamburg
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Algebraic Topology
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Sheet 5

Solutions are due on 11.05.18.

Problem 5.1

(a) Let $\overline{B_R(x)} \subset \mathbb{R}^n$ be the closed ball of radius $R > 0$, centred at an arbitrary point $x \in \mathbb{R}^n$. Prove Brouwer's Fixed Point Theorem:
Any continuous map $f\colon \overline{B_R(x)} \longrightarrow \overline{B_R(x)}$ has a fixed point.
(Hint: Consider rays starting at $z \in \overline{B_R(x)}$ and passing through $f(z)$.)

(b) Use Brouwer's Fixed Point Theorem to prove the Perron-Frobenius Theorem:
Any matrix $(a_{ij}) = A \in M(n \times n, \mathbb{R})$ with non-negative entries a_{ij} must have an eigenvector with non-negative coordinates, and the corresponding eigenvalue is positive.

Problem 5.2

Let $A \in \mathrm{O}(n+1)$ be an orthogonal matrix. Then multiplication by A induces a continuous self-map on $\mathbb{S}^n$. Compute its degree.

Problem 5.3

Consider the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \xrightarrow{\iota_1} & A & \xrightarrow{\pi_1} & A'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & \\
0 & \longrightarrow & B' & \xrightarrow{\iota_2} & B & \xrightarrow{\pi_2} & B'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \psi_3} & & \\
0 & \longrightarrow & C' & \xrightarrow{\iota_3} & C & \xrightarrow{\pi_3} & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Suppose that the columns are complexes. Show that if the first two columns or the right two columns are short exact sequences, then the other column is a short exact sequence as well.

Problem 5.4

(a) Let $f, g\colon X \to Y$ be two continuous maps. The *mapping torus* of f and g is defined as the quotient space

$$T(f,g) := \big(X \times [0,1] \sqcup Y\big)/\sim\,, \quad (x,0) \sim f(x) \quad \text{and} \quad (x,1) \sim g(x) \quad \forall\, x \in X\,.$$

Show that there is a long exact sequence

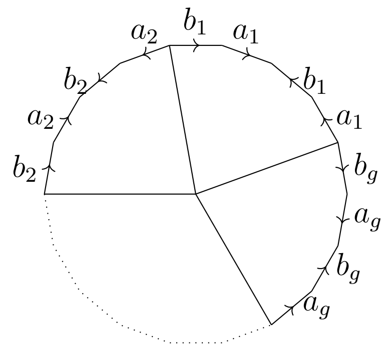
$$\dots \longrightarrow H_n(X) \xrightarrow{f_* - g_*} H_n(Y) \xrightarrow{\iota_*} H_n(T(f,g)) \xrightarrow{\delta} H_{n-1}(X) \xrightarrow{f_* - g_*} \dots$$

Hint:
Find a map $q\colon X \times [0,1] \to T(f,g)$ and consider the induced morphism of the long exact sequences in relative homology for the subspace $X \times \partial[0,1]$.

(b) Write the Klein bottle as a mapping torus and compute its homology in this way.

Problem 5.5

Let F_g denote the connected, closed, orientable surface of genus $g \in \mathbb{N}_0$, obtained by identifying edges with the same label according to the orientations indicated in the $4g$-gon

$F_g :=$ 

Use the Mayer-Vietoris Theorem to compute $H_*(F_g)$.
(Do not use the Hurewicz Theorem to compute H_1!)