Advanced Class on
Supersymmetric Field Theories and Elliptic Cohomology
or the Stolz-Teichner Program

Hilary term 2022

Talk 1. (André) *Introduction and overview.*

This is going to be an introduction to this term's topic, a conjectural relation between supersymmetric quantum field theories and cohomology theories, and in particular elliptic cohomology. Introduce/sketch these notions (SuSy QFTs and (topological) modular forms), give an idea of Stolz-Teichner's main conjecture, and possibly comment on relations to physics (see also talks 7 and 8).

References (none of these lists are exhaustive): [ST04, Seg87, Seg07]

Talk 2. *Supermanifolds and de Rham cohomology*

Throughout this seminar, we will meet supermanifolds frequently. Introduce supermanifolds via two approaches: via ringed spaces and via their functor of points. Show how deformation classes of smooth supersymmetric quantum field theories of dimension $(0|1)$ on a manifold M reproduce the de Rham cohomology of M, and comment on the fact that there is only a single deformation class of $(0|0)$-dimensional field theories on M.

References: The result on $(0|1)$-dimensional field theories are in [HKST11], References on supermanifolds include [Rog07, DM99, Fre99, Sac08]

Talk 3. $(1|1)$-*dimensional field theories and K-theory I*

The goal of this and the next talk is to survey the relation between K-theory and $(1|1)$-dimensional Euclidean field theories [HST10] (please use this version!). This talk will cover several models for K- and KO-theory and their equivalence. Recall Clifford algebras and their periodicity and survey sections 1–5 of [HST10] (if short on time, focus on models (1), (4) and (5) for KO-theory).

Talk 4. (Filippos?) $(1|1)$-*dimensional field theories and K-theory II*

This talk will explain the notion of supersymmetric field theories with geometric data [HST10, Section 6] and how these form a topological space. The main goal is to explain that this topological space is a classifying space for KO-theory [HST10, Theorem 6.29]. If time permits, say a few words about the models for K-theory spectra from sections 7 and 8.

Talk 5. *Integrality and holomorphicity of the RR partition function*

In this talk we start to uncover the relation between $(2|1)$-CFTs and topological modular forms. Introduce the notion of a CFT of degree n [ST04, Def. 2.3.16] and its partition function [ST04, Def. 3.3.5]. Explain the statement that this partition function is a weak integral modular form [ST04, Thm. 3.3.14], comment on the relation to K-theory [ST04, Thm. 1.0.2].

Talk 6. (Severin) *Sigma models with values in string manifolds*

Explain/sketch Witten's computation of the index of the Dirac operator on the free loop space of a manifold M, the original approach to the Witten genus [Wit88, Wit87]. Explain that when M is spin, the Witten genus is integral (see also [Sto96]). Explain the notion of a string structure (e.g. [Sto96]) and its relation to the Witten genus. For a modern point of view, related to the Stolz-Teichner programme, towards the Witten genus and its modularity on string manifolds, survey [BE19].

Talk 7. *Detecting* $\pi_3(\mathrm{tmf}) = \mathbb{Z}/24\mathbb{Z}$

The elliptic genus is a map (see also Talk 5)

$$\{\text{Minimally SuSy 2d QFTs with anomaly } n\} \longrightarrow \{\text{Modular forms of weight } n/2\}$$

A lot of the subtle information about tmf is contained in its torsion groups, which are lost when passing to modular forms. Conjecturally, the above map has a lift

$$\{\text{Minimally SuSy 2d QFTs with anomaly } n\} \longrightarrow \pi_n(\mathrm{tmf}),$$

and thus field theories should detect this torsion. For $n = 3$, we have $\pi_3(\mathrm{tmf}) = \mathbb{Z}/24\mathbb{Z}$, and a map from field theories exhibiting this has been constructed; survey this construction from [GJFW19, GJF19].

Talk 8. *The spectrum of supersymmetric QFTs*

Explain how the collection of supersymmetric QFTs assembles into a spectrum, which is conjectured to be equivalent to tmf, and survey how various field theories represent classes in tmf [JF20].

References

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