

• The Main Theorem [CHS, §5]

- Last talk (Thomas): Each obj. in $\mathcal{O}_{\Gamma_n}^{B,d}$ and $\text{Lag}_n^{B,s}$ is fully dualisable.

\Rightarrow Cb. Hyp.: $Z_{(T,[T])} : \text{Bord}_{\sigma,n}^{\text{fr}} \longrightarrow \mathcal{O}_{\Gamma_n}^{B,d}$, $\forall (T,[T])$ d-or. $dSt/8$,
 $Z_{(X,\omega)} : \text{Bord}_{\sigma,n}^{\text{fr}} \longrightarrow \text{Lag}_n^{B,s}$, $\forall (X,\omega)$ s-symp. $dSt/8$.

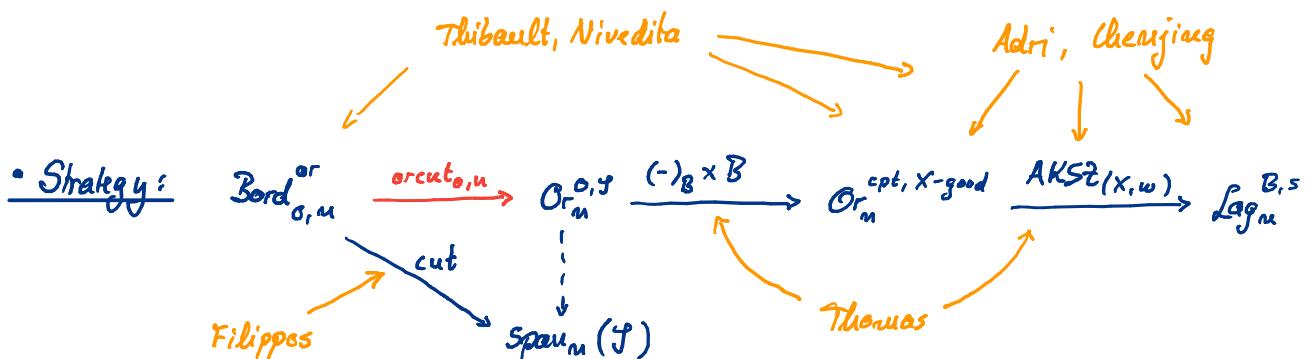
- Goal: Actually construct the TQFT

$$Z_{(X,\omega)} : \text{Bord}_{\sigma,n}^{\text{or}} \longrightarrow \text{Lag}_n^{B,s}$$

\uparrow

$\text{Bord}_{\sigma,n}^{\text{fr}}$ - - - \uparrow

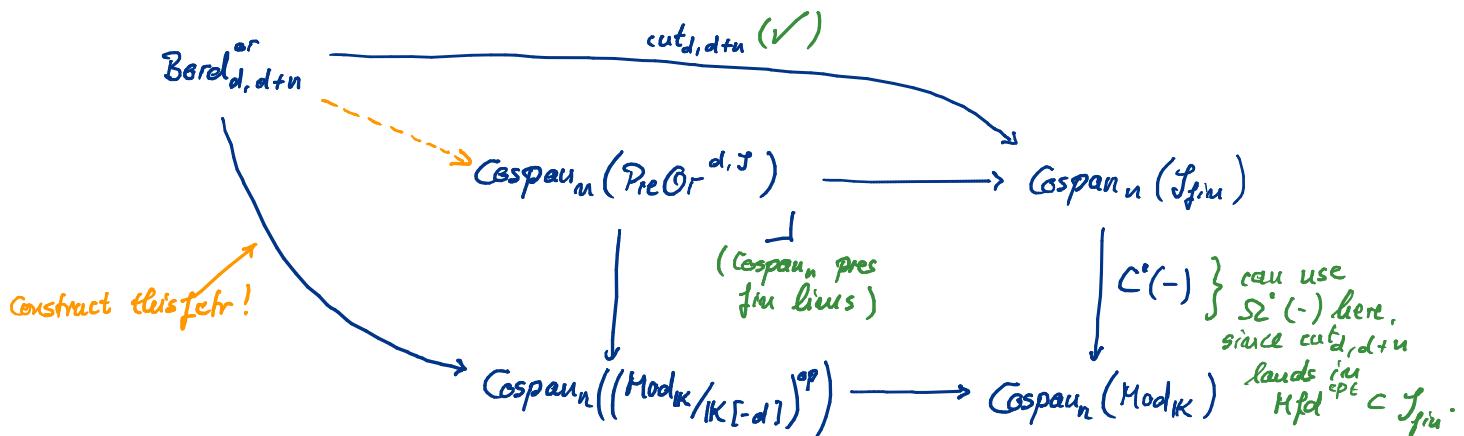
n -fold s-shifted Lag. corr. over B .



\Rightarrow Task: Construct the TQFT $\text{arcut}_{d,d+n} : \text{Bord}_{d,d+n}^{\text{or}} \longrightarrow \mathcal{O}_{\Gamma_n}^{d,j}$.

- Steps:
 - (1) Construct an (σ,n) -fctr. $\text{arcut}_{\sigma,d+n} : \text{Bord}_{\sigma,d+n}^{\text{or}} \longrightarrow \text{Cspau}_n(\text{PreGr}^{d,j})$ of (σ,n) -lats, i.e. of n -fold CSOs in (the wlat) \mathcal{I} .
 - (2) Show that this is sym. mon.
 - (3) Show that $\text{arcut}_{d,d+n}$ factors through $\mathcal{O}_{\Gamma_n}^{d,j} \subset \text{Cspau}_n(\text{PreGr}^{d,j})$.

- (1) : Strategy: Use CSS_m , as an *ω-lab!*



• Idea: $H \xrightarrow{\left((\Sigma^*(H), d) \xrightarrow{f_H(-)} R[-d] \right)} \simeq (C(H; R), d) \simeq \mathcal{O}_{M_R}$ gives desired or. on M_R .

• Problem 0: ∞ -functors hard to construct (∞ -coherence).

• (One) way out: Strictify.

↳ Here, some ω -labs we use are localisations of 1-cats:

$\overset{\text{Sets}}{\Delta}$ will omit this for now

$$\bullet \quad \mathcal{G} = S[\text{whe}^{-1}], \quad \text{CSS}_m^{(\infty)} = \underbrace{\text{Fun}(\Delta^{n,\text{op}} \times \mathbb{N}, S)[W_{\text{CSS}_m}^{-1}]}_{\text{CSS}_m^{\text{str}}},$$

$$\text{Mod}_K = \text{Ch}_K[\text{qiso}^{-1}], \dots$$

⇒ Try to work in $\text{CSS}_m^{\text{str}}$.

• Problem 1: Coherent maps to $K[-d]$:

• E.g.: Let $M_{0,1} \xrightarrow{M_{0,1}} M_{1,1}$ be a $(d+1)$ -dim bordism.

$$\begin{array}{ccccc} \Sigma^*(M_{0,0}) & \leftarrow & \Sigma^*(M_{0,1}) & \longrightarrow & \Sigma^*(M_{1,1}) \\ & \searrow & \downarrow \text{X} & \swarrow & \\ & & R[-d] & & \end{array}$$

Does not commute in Ch_K
⇒ In Mod_K , it does not need to (has 2-cells).

↳ Can't just consider \mathbb{Z}^I

$$\begin{array}{ccc} \Sigma^*(M_{0,0}) & \xrightarrow{\text{const}(K[-d])} & \text{Ch}_K \\ \downarrow & & \end{array}$$

↪ Way out: encode the 2-cells by replacing $\text{const}(\mathbb{K}[-d]) : \mathbb{Z}^{\bar{J}} \rightarrow \mathcal{C}_{\mathbb{K}}$ ↓
invertible in
 $\text{Fun}(N\mathbb{D}^{\text{op}}, \text{Mod}_{\mathbb{K}})$
by a weakly equiv. diag. $P_{\bar{J}} : \mathbb{Z}^{\bar{J}} \rightarrow \mathcal{C}_{\mathbb{K}}$, $P_{\bar{J}} \xrightarrow{\sim} \text{const}(\mathbb{K}[-d])$,
functionally in $\bar{J} \in \Delta^{n,\text{op}}$ (and, later $P_{\bar{J},l}$, l in the spatial copy of Δ^{op}).

Idea:

• E.g.: $\Omega^*(M_{00}) \xleftarrow{(-)|_{M_{00}}} \Omega^*(M_{01}) \xrightarrow{(-)|_{M_{11}}} \Omega^*(M_{11})$

$\int_{M_{00}} \downarrow \quad \begin{matrix} M_p \text{ in } \mathcal{C}_{\mathbb{K}} \\ \text{by Stokes' Thm!} \end{matrix} \quad \downarrow \quad \int_{M_{11}} \downarrow$

$\begin{array}{c} \text{by } // \text{ constr.} \\ \left(\int_{M_{01}}, \int_{M_{11}} (-) |_{M_{11}} \oplus (-) |_{M_{00}} \right) \\ \left(\begin{array}{c} R[-d-1] \\ \uparrow^{(x,y)} \\ R[-d] \oplus R[-d] \end{array} \right) \end{array} \xrightarrow{\quad \text{by } // \text{ constr.} \quad}$

$R[-d] \xleftarrow{\quad} R[-d]$

• Generally: $P_{\bar{J}} = \prod_{a=1}^n C^*(Sp^{j_a})$

(• combinatorics problem
• given by inert part / $M^{\bar{J}} \subset \mathbb{Z}^{\bar{J}}$!)

e.g. $\begin{array}{ccccc} x \otimes b & & & & \\ \uparrow & \longrightarrow & \uparrow & \longrightarrow & \uparrow \\ (xy) \otimes ab & & (yz) \otimes ab & & \\ \uparrow & \longrightarrow & \uparrow & \longrightarrow & \uparrow \\ x \otimes a & y \otimes a & z \otimes a & & \\ (xy) \otimes a & & (yz) \otimes a & & \end{array}$

• Recall (Filippou's talk) [CS]

• $\text{Bord}_{d,d+n}^{\text{or}} \in \text{CSS}_n^{\text{str}}$ (vald. in $s^n\text{Kau}$),

$\bar{J} \longmapsto (\text{Bord}_{d,d+n+\bar{J}}^{\text{or}} \in S)$, with l -simplices

(M, \bar{I}) , $M \hookrightarrow \Delta_e^l \times \mathcal{B}(\bar{I}) \times R^\infty$ s.t. (...)

$\begin{array}{c} \text{'space' str.} \nearrow \Delta_e^l \times \mathcal{B}(\bar{I}) \times R^\infty \searrow \text{for hotype } \mathcal{BDiff} \\ \uparrow \text{cutting /} \\ \text{composing} \\ \text{in } l \text{ directions} \end{array}$

• Want: $\text{Bord}_{d,d+n/\bar{j}}^{\text{or}} \longrightarrow N\left(\text{Fun}\left(\Sigma^{\bar{j}}, \text{Ch}_{K/K[-d]}\right)^{\sim}\right)$

should be $P_{\bar{j}}(-)$, see above
only nat. ways.

models space $\text{Cospun}\left(\text{Ch}_{K/K[-d]}\right)_{\bar{j}} \in S$.

↳ Equiv.: $\text{Bord}_{d,d+n/\bar{j},l} \times N\Sigma^{\bar{j}} \times \Delta^l \longrightarrow NT$

aug target, e.g. $\text{Ch}_{K/K[-d]}$

• Problem 2: Spatial simplices:

$$\begin{array}{ccc} M_0 & \xrightarrow{\sim} & M \xleftarrow{\sim} M_1 \\ \downarrow & & \downarrow \\ \Delta_e^{0\#} & \longrightarrow & \Delta_e^1 \xleftarrow{\sim} \Delta_e^{1\#} \end{array}$$

→ # no canonical way to turn this into a map/when $M_0 \rightarrow M_1$, sim. for higher simplices.

↳ Naturally gives not $N\text{Fun}\left(\Sigma^{\bar{j}}, \gamma\right)_e^{\sim} = S(N\Sigma^{\bar{j}} \times \Delta^l, \gamma)^{\sim \#}$
 ↓ \Rightarrow can. equiv obj's in \mathcal{J} !

but elements of $(\text{Ex } N\text{Fun}\left(\Sigma^{\bar{j}}, \gamma\right))^{\sim}_e = S(N\Sigma^{\bar{j}} \times \underset{\substack{\uparrow \\ \text{subdivision}}}{\text{sd}(\Delta^l)}, NT)^{\sim \#}$

• Problem 3: Spatial degeneracies:

• s° in γ gives $x \mapsto (x=x)$, and $x=x=x$ in $\text{Ex}(NT)$.

• s° in $\text{Bord}_{d,d+n}^{\text{or}}$ gives $\begin{array}{c} M \\ \downarrow \\ \Delta_e^0 \end{array} \longmapsto \left(\begin{array}{ccc} M & \xrightarrow{\sim} & M \times \Delta_e^1 \xleftarrow{\sim} M \\ \downarrow & & \downarrow \\ \Delta_e^{0\#} & \longrightarrow & \Delta_e^1 \xleftarrow{\sim} \Delta_e^{1\#} \end{array} \right)$

⇒ Construct only maps of semi-spl. sets + use that [CHS, Cor. A.3.12]

$$\begin{array}{ccc} \text{Fun}(C, S^{\text{semi}}) [\text{whe}^{-1}] & \begin{array}{c} \xleftarrow{l_s!} \\ \xrightarrow{l_s*} \end{array} & \text{Fun}(C, S) [\text{whe}^{-1}] \\ & \searrow \xleftarrow{\sim} \swarrow & \\ & \text{Fun}(NC, \mathcal{G}) & \end{array}$$

\hookrightarrow CHS build maps of semi-sSets , natural in $(\bar{j}, l) \in \Delta^{\text{ret}, \text{op}}$,

$$\text{Bord}_{d,d+1}^{\text{or}}/\bar{g}, \ell \times \iota_s^* N\Sigma^{\bar{l}} \times \iota_s^* \text{Sd}(\Delta^\ell) \times \iota_s^* \Delta^1 \longrightarrow \text{Ch}_K$$

Δ^{tot} gives $S\mathbb{Z}(-)$
 Δ^{ext} gives $P_{\bar{g}, \ell}$

exactly as in the example above

↳ Idea exactly as in the example above.

\Rightarrow This presents a resp.

$$\text{orcut}_{d,d+m} : \text{Bord}_{d,d+m}^{\text{or}} \longrightarrow \text{Cospan}_m(\text{PreGr}^{d,j})$$

im $CSS_{\alpha u} = \text{Lat}_{\alpha u}$ as desired. \Rightarrow Step (1) ✓

• Step (2): Show that this is sym. mon.

- Will ship this — fiducially, but exactly analogous to how Thomas worked yesterday.
 - ↳ Show compat. with deloopings, as always so far.
 - ↳ for deloopings of $\text{Bord}_{d,d+n}$, see [CS, §6.1, 7.2].

Step (3): Non-degeneracy, i.e. circuit factors through $\mathcal{O}_m^{d,g}$.

Recall: Let $A \in \text{CAlg}_{\mathbb{K}}$. A local system of A -modules on $X \in \mathcal{I}$ is an ∞ -fctn.

$$\mathcal{E} : X \longrightarrow \text{Mod}_A.$$

$$\text{Set} \quad C_*(X; E) := \underset{x \in X}{\operatorname{colim}} E_x, \quad C^*(X; E) := \underset{x \in X}{\operatorname{lim}} E_x.$$

$$\bullet \quad \mathbb{Q}\text{-Coh}(X_B) \simeq \mathbb{Q}\text{-Coh}(\operatorname{colim}_{x \in X} \operatorname{Spec} K_x) = \lim_{x \in X} \operatorname{Mod}_K$$

$$\Gamma(\mathcal{O}_{X_B}) = \lim_{\leftarrow} \underset{x \in X}{\Gamma}(\text{Spec } k) = \lim_{\leftarrow} \underset{x \in X}{k} = C^*(X; k) =: C^*(X).$$

- $[X]$ is a d -orientation : $\Leftrightarrow \forall A \in \text{CAlg}_k, E \in \text{Fun}(X, \text{Mod}_A)$ dualisable

$$C^*(X; \mathcal{E}^k) \xrightarrow{\sim} C_*(X, E[-d])$$

i.e., $(C_*(X, E))^\vee_{H^0}$

$$C^*(x) \otimes_{\mathbb{K}} A$$

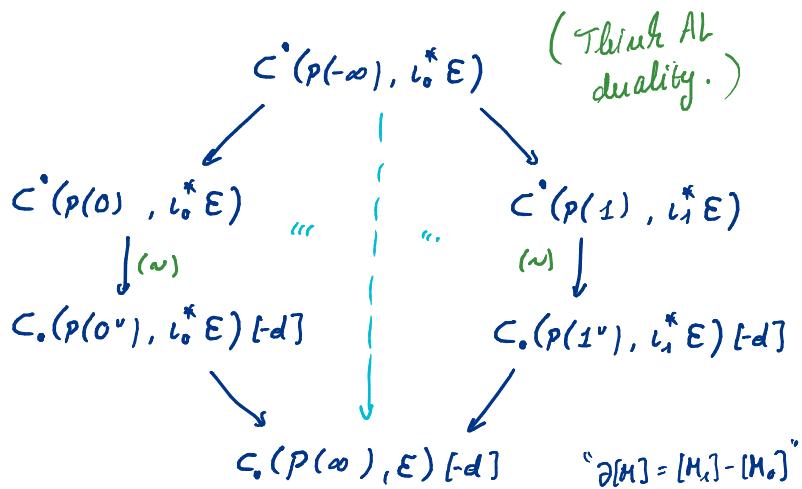
$$\Leftrightarrow C^*(X; \mathcal{E}^*) \otimes C^*(X; \mathcal{E}) \longrightarrow C^*(X; \mathcal{E}^* \otimes \mathcal{E}) \xrightarrow{\text{is non-deg. pairing.}} C^*(X; A) \xrightarrow{\text{in}} \mathbb{K}[d]$$

- E.g.: A closed d-dim nufol w. orientation (classically) is d-or. by Poincaré duality (/ P.-Verdier duality).

- d-preor. cospan $p: \tilde{Sp}^{\times u} \rightarrow \text{PreOr}^{g,d}$, $\mathcal{E} \in \text{Fun}(\mathcal{P}(-\infty), \text{Mod}_A)$
induces $\hat{p}_{A,\mathcal{E}}: \tilde{Sp}^{\times u} \rightarrow \text{Mod}_A$

E.g.:

$$\left(\begin{array}{ccc} p(0^\nu) & & \\ \downarrow \iota_0 & & \\ p(-\infty) & \xleftarrow{\iota_+} & p(1^\nu) \end{array} \right) \longmapsto$$



- Let $\vec{i} \in \Delta^n$. $F \in \text{Fun}(\Sigma^{\vec{i}}, \text{PreOr}^{g,d})$ lies in $(\text{Or}_n^{g,d})_{\vec{i}}$

\Leftrightarrow (1) F is Cocart ("all squares are pushouts (w. preors!)") "interval inclusion"

(2) $\forall A \in \text{CAlg}, \mathcal{E}: \mathcal{P}(-\infty) \rightarrow \text{Mod}_A, \vec{i} \in \{0,1\}^n, \underbrace{\vec{i} \rightarrow \vec{k}}$ inert,
the diag.

defines an $m = \sum_a i_a$ -tuple of
single cospan in dir. a , $i_a = 1$

$(\vec{i}^* p)_{A,\mathcal{E}}: \tilde{Sp}^{\times u} \rightarrow \text{Mod}_{A,\mathcal{E}}$ is a limit diag. ("oriented m -uplespan")

- Prop.: For each $(H, \bar{I}, f) \in \text{Bord}_{d,d+u}^{\text{or}}$ w. $j_i \in \{0,1\}$, $m := \sum_{i=1}^m j_i$,
 $\text{circut}_{d,d+u}(H, \bar{I}, f): \tilde{Sp}^{\times u} \rightarrow \text{PreOr}^{g,d}$ is oriented.

\Rightarrow Cor.: $\text{circut}_{d,d+u}$ factors through $\text{Or}_m^{g,d} \subset \text{Cospan}_m(\text{PreOr}^{g,d})$. Gives Step 3 ✓

- Proof of Prop.: Let $A \in \text{CAlg}_K, \mathcal{E} \in \text{Mod}_A$.

Show: $\hat{p}_{A,\mathcal{E}}: \tilde{Sp}^{\times u} \rightarrow \text{Mod}_A$ is limit diag.

- By Lemma 2.88, equiv to showing that

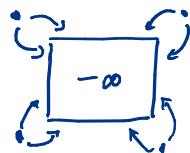
purely ~~plat~~,
about $\frac{5}{4}$ -shaped
diags in stable ~~plat~~

$$\begin{array}{ccc}
 \text{N} C^*(H; E^\vee) & & "Sp^{x_{\infty}} \setminus \{\infty\}", \quad Sp^{x_{\infty}} = (Sp^{x_{\infty}, 0})^\Delta \\
 \widehat{P}_{A, E}(-\infty) \longrightarrow \lim_{J \in Sp^{x_{\infty}, 0}} \widehat{P}_{A, E}(J) & \downarrow & \simeq C^*(\partial H; E^\vee|_{\partial H}) \\
 \downarrow & & \\
 0 \longrightarrow \underbrace{\widehat{P}_{A, E}(\infty)}_{\simeq C_*(H; E)[1 - \mu] - d} & &
 \end{array}$$

$$\bullet \quad \hat{p}_{A,E}(J) = C(M_J; \mathcal{E}^{\nu}_{|M_J})$$

$$\Rightarrow \lim_{j \in Sp^{\times w_{i,0}}} \hat{P}_{A,E}(j) \simeq \lim_j C^*(M_j; E|_{M_j})$$

$$\begin{array}{c} \vdots \\ \cong \end{array} \lim_{J \in Sp^{xw, 0}} \quad \lim_{x \in H_J} E_{1x}$$



$$\begin{array}{c}
 \text{w-loc} \\
 \Rightarrow \text{fin + cofin}
 \end{array}
 \xrightarrow{\quad \int M(-) \quad}
 \begin{array}{c}
 \parallel \int M(-) \parallel = \underset{J \in \text{Sp}^{\text{tw}, 0}}{\text{colim}} M_J \simeq \mathcal{M} \\
 \xleftarrow{\quad E^v(-) \quad} \xleftarrow{\quad \text{Mod}_A \quad}
 \end{array}
 \begin{array}{c}
 \text{$E^v(-)$ factorises uniquely} \\
 (\text{it sends all maps to equisur})
 \end{array}$$

$$Sp^{u, v} \xrightarrow{H_{(C)}} g^{\text{op}}$$

Σ linear $x \in \partial M$ $E^v|_X$

$$= C^*(\partial M; \mathcal{E}^\vee|_{\partial M}) \quad (\text{modulo smoothing of } M, \partial M).$$

Now the claim follows by Alexander - Lefschetz duality.

\Rightarrow Thm. [CHS]: To each s -symplectic derived stack (X, ω) and each $n \in \mathbb{N}_0$, there is a fully extended, oriented, n -dim TQFT given as the corp.

$$Z_{(X,\omega)} : \text{Bord}_{\partial, n}^{\text{or}} \xrightarrow{\text{out}_n} \mathcal{O}_{\Gamma_n}^{g, 0} \xrightarrow{(-)_B \times S} \mathcal{O}_{\Gamma_n}^{\text{cpt}, X\text{-good}, S, 0} \xrightarrow{\text{AKSZ}_{n, (X,\omega)}^{S, 0}} \text{Lag}_n^{S, 0},$$

with $Z_{(X,\omega)}(*_+) = (X, \omega)$.

- Bonus: Thomas's talk: $\text{Lag}_n = \text{Lag}_{n,\text{fd}}$ \rightsquigarrow should expect only framed TQFTs!
 Nivedita's talk: $\text{Span}_n(\mathcal{C}) = \text{Span}_{n,\text{fd}}(\mathcal{C})$ $\left| \begin{array}{l} \hookrightarrow \text{Difference: hofpts for} \\ \forall \mathcal{C} \in \text{Cat}_\infty \text{ w.r.t. fd classes.} \end{array} \right. \left| \begin{array}{l} \text{SO}(n) - \text{O}(n) - \text{action, resp.} \end{array} \right.$

\rightsquigarrow Why should we expect the AKSZ TQFTs to lift to oriented bordisms?

\hookrightarrow Resolution: $\text{Span}_n(\mathcal{C})_{\text{fd}}^{\simeq} \xrightarrow{[\text{Hau}]} \mathcal{C}^{\simeq}$ \rightsquigarrow inherits $\text{O}(n)$ -action from Cob Hyp.

- Thm [CHS]: (1) The above $\text{O}(n)$ -action on $\mathcal{C}^{\simeq} = \text{Span}_n(\mathcal{C})_{\text{fd}}^{\simeq}$ is trivial.
 (2) The $\text{SO}(n)$ -action on $\text{Lag}_n^{S, \times}$ is trivial.
 (3) The $\text{O}(n)$ -action on $\text{Lag}_n^{S, \times}$ factors through $\mathbb{Z}_2 = \pi_0 \text{O}(n)$, via $(X, \omega) \mapsto (X, -\omega)$.

Lemma: $X \in \mathcal{F}$, $G \in \text{Grp}(\mathcal{G})$, $G \curvearrowright X$, s.t. $X^{hG} \xrightarrow{\text{forg}} X$ admits a section. Then $G \curvearrowright X$ is the trivial action.

Pf. of Thm. (1): ((2) analogous w.r.t. $\text{ord}\text{ctbo}, u$, $\mathcal{I}_{\text{fin}} \mapsto \text{PreOr}^{J_{\text{fin}}, \circ}$, $\text{Span}_n \mapsto \text{Lag}_n$)

- Each $x \in \mathcal{C}$ defines an unor. TQFT

$$\text{Bord}_{0,n} \xrightarrow{\text{cut}_{0,n}} \text{Cospan}_n(\mathcal{I}_{\text{fin}}) \xrightarrow{x^{(-)}} \text{Span}_n(\mathcal{C})$$

$$\begin{array}{ccccc}
 & \overset{x}{\nearrow} & & \overset{x^{(-)}}{\searrow} & \\
 & \text{I} & \longrightarrow & & \\
 \Rightarrow & \mathcal{C}^{\simeq} & \longrightarrow & \text{Fun}^{\otimes}_{\text{rect}}(\mathcal{I}_{\text{fin}}^{\text{op}}, \mathcal{C})^{\simeq} & \longrightarrow \text{Fun}^{\otimes}(\text{Cospan}_n(\mathcal{I}_{\text{fin}}), \text{Span}_n(\mathcal{C}))^{\simeq} \\
 & \parallel & \text{section} & & \\
 & \mathcal{C}^{\simeq} & \xrightarrow{\text{forg}} & (\mathcal{C}^{\simeq})^{h\text{O}(n)} & \xleftarrow{\text{Gbhyp} \simeq} \text{Fun}^{\otimes}(\text{Bord}_{0,n}, \text{Span}_n(\mathcal{C}))^{(\simeq)} \\
 & & & \downarrow \psi & \\
 & z(*) & \longleftarrow & z & \xleftarrow{\text{cut}_{0,n}^*}
 \end{array}$$

- By the above Lemma, this completes the proof. \square

- Thanks all for taking part so actively!