# String structures and the Witten genus

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**Abstract**

We explore Witten's (heuristic) construction of what is nowadays called the Witten genus, i.e. his computation of the $\mathbb{S}^1$-equivariant index of the Dirac operator on the free loop space of a manifold. We then survey recent work by Berwick-Evans which provides a rigorous construction of the Witten genus as a derived global section of a sheaf of cdgas on the moduli space of elliptic curves. These are notes for a talk in the Topology Advanced Class on *supersymmetric field theories and the Stolz-Teichner programme* in Hilary term 2022 at the Mathematical Institute at the University of Oxford.

## Contents



## 1 Introduction and review of last time

The main goal in this seminar is to understand progress towards and motivation for the Stolz-Teichner Conjecture:

**Conjecture 1.1.** [ST04] *There is an isomorphism*

$$\pi_0\big(\mathrm{CFT}^{-n}_{2|1}(M)\big) \cong \mathrm{TMF}^n(M)\,.$$

Evidence for this is given by the identification of deformation classes of 0|1-theories with de Rham cohomology [HKST11] and of 1|1-EFTs with K-theory [HST10]. Last time, we saw a first relation between 2|1-sCFTs and (topological) modular forms:

**Theorem 1.2.** (proof incomplete, [ST04, Thm. 3.3.14]) *Evaluating a degree-$n$ two-dimensional CFT on the torus $E_\tau$ with the non-bounding spin structure and conformal structure labelled by $\tau \in \mathfrak{h}$, we obtain a function*

$$\mathfrak{h} \to \mathbb{C}\,, \qquad \tau \mapsto \mathcal{Z}(E_\tau)\,.$$

*This function is a* weak integral modular form *of weight $\frac{n}{2}$.*

In this and the upcoming talks we will give further evidence for Conjecture 1.1. In this talk, we will construct elements in $\mathrm{TMF}(M) \otimes \mathbb{C} = \mathrm{MF}(M) = \mathrm{H}(M;\mathrm{MF})$, the (de Rham) cohomology of $M$ with coefficients in the ring of weak integral modular forms. That is, we will not see any torsion phenomena related to genuine *topological* modular forms. That will happen in the next talk.

*Note that Theorem 1.2 is for $M = \mathrm{pt}$!* Our main goal in this talk will be to understand the following

**Question:** How does this generalise to/what do we obtain in the case of non-trivial $M$?

## 2 Witten's torus partition functions for non-trivial target space

One of the historical motivations to expect a relationship between supersymmetric (conformal) field theories comes from a series of works by Edward Witten. We start by reviewing these here.

In [Wit82], Witten showed that the partition function of supersymmetric quantum mechanics (1|1-EFTs) on a closed spin manifold $M$ computes the index $\mathrm{ind}(\not{D}_M)$ of the Dirac operator on $M$. In [Wit87], he provided physical arguments for what should happen for two-dimensional sCFTs on closed orientable manifolds $M$.

What do we get as the partition function of a 2|1-CFT on non-trivial $M$?

For a torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, the space of maps $\mathbb{T}^2 \to M$ is diffeomorphic to the space of maps $\mathbb{S}^1 \to LM$, where $LM = \mathrm{Map}(\mathbb{S}^1, M)$ is the *free loop space* of $M$. Generally, from every 2-dimensional theory on $M$ we can obtain a 1-dimensional theory on $LM$, via the identification

$$\{I_t \times \mathbb{S}^1 \longrightarrow M\} \cong \{I_t \longrightarrow LM\}$$

for each 1-dimensional bordism $I_t$ (possibly choosing a metric on $\mathbb{S}^1$.

The partition function of this theory should, by the above, be the index of a *hypothetical* Dirac operator $\not{D}_{LM}$ on $LM$.

**Remark 2.1.** Such an operator has indeed, at the time of writing, not been constructed yet. The infinite-dimensionality of $LM$ poses severe (but not necessarily unsurmountable) analytical difficulties here. $\triangleleft$

Nevertheless, it is worth taking this idea seriously. Let us assume that $LM$ is *spin* in some sense (which we will consider in more detail below), and that $\not{D}_{LM}$ does indeed exist. In the remainder of this section, we follow the presentation in [Sto96].

We write $\not{D}_X \otimes V$ for the Dirac operator $\not{D}_X$ on a closed spin manifold $X$, coupled to a vector bundle $V \to X$. Recall that

$$\mathrm{ind}(\not{D}_X \otimes V) = \dim\big(\ker(\not{D}_X \otimes V)\big) - \dim\big(\mathrm{coker}(\not{D}_X \otimes V)\big) \quad \in \mathbb{Z}\,.$$

**Theorem 2.2.** (Atiyah-Singer) *The index of $\not{D}_X \otimes V$ can be computed as*

$$\mathrm{ind}(\not{D}_X \otimes V) = \int_X [\hat{A}(X)] \cdot [\mathrm{ch}(V)]\,.$$

*Here, $[A(X)] \in \mathrm{H}^\bullet_{\mathrm{dR}}(M)$ is the* $\hat{A}$-class *of $X$, and* $\mathrm{ch}(V)$ *denotes the* Chern class *of a complex vector bundle. (The* $\hat{A}$-genus *of $X$ is $\int_X [\hat{A}(X)]$.)*

On $LM$, however, we have some more refined data: there is a canonical smooth $\mathbb{S}^1$-action on $LM$ given by

$$(z \triangleright \gamma)(z') = \gamma(zz')\,.$$

There is an enhanced version of index theory for situations of this kind. It takes compatibility with group actions into account as follows. If $X$ (as above) has a smooth $\mathbb{S}^1$-action which lifts to its spinor bundle and to $V$, then $\ker(\not{D}_X \otimes V)$ and $\mathrm{coker}(\not{D}_X \otimes V)$ are invariant under the $\mathbb{S}^1$-action, and thus furnish not just vector spaces but $\mathbb{S}^1$*-representations.* We let $\ker(\not{D}_X \otimes V)_m$ denote the subspace on which $z \in \mathbb{S}^1$ acts as multiplication by $z^m$, and similarly for $\mathrm{coker}(\not{D}_X \otimes V)_m$. Then, the refined, so-called *equivariant index* of $\not{D}_X \otimes V$ is

$$\mathrm{ind}^{\mathbb{S}^1}(\not{D}_X \otimes V) = \sum_{m \in \mathbb{Z}} \Big(\dim\big(\ker(\not{D}_X \otimes V)\big)_m - \dim\big(\mathrm{coker}(\not{D}_X \otimes V)\big)_m\Big) \cdot q^m \quad \in \mathbb{Z}[q, q^{-1}]\,.$$

(Only finitely many powers of $q$ appear non-trivially because both the kernel and the cokernel of $\not{D}_X \otimes V$ are still finite-dimensional, for $\not{D}_X \otimes V$ is an elliptic operator.)

In [Wit88], Witten heuristically computed the $\mathbb{S}^1$-equivariant index of $\not{D}_{LM}$, for a $d$-dimensional closed orientable manifold $M$:

$$\operatorname{ind}^{\mathbb{S}^1}(\not{D}_{LM}) = \eta(q)^{-d} \underbrace{\int_M \hat{A}(M) \wedge \operatorname{ch}\Big(\bigotimes_{m=1}^{\infty} S_{q^m}\big((TM - d) \otimes \mathbb{C}\big)\Big)}_{=:\phi_W(M)} .$$

Here, $S_t(V) = \mathbb{C} \oplus Vt \oplus S^2(V)t^2 \oplus \dots$ is the total symmetric power of a vector bundle $V$.

**Definition 2.3.** The power series $\phi_W(M) \in \mathbb{R}[[q]]$ is called the *Witten genus* of a closed orientable $d$-manifold $M$.

**Lemma 2.4.** *If $M$ is additionally spin, then $\phi_W(M) \in \mathbb{Z}[[q]]$.*

*Proof.* In this case, the Atiyah-Singer Index Theorem (Thm. 2.2) immediately yields

$$\begin{aligned}\phi_W(M) &= \int_M \hat{A}(M) \wedge \operatorname{ch}\Big(\bigotimes_{m=1}^{\infty} S_{q^m}\big((TM - d) \otimes \mathbb{C}\big)\Big) \\ &= \operatorname{ind}\Big(\not{D}_M \otimes \bigotimes_{m=1}^{\infty} S_{q^m}\big((TM - d) \otimes \mathbb{C}\big)\Big) .\end{aligned}$$

The integrality then follows because each analytical index is an integer. $\square$

We still have to specify what it means for $LM$ to have a spin structure. The manifold $M$ has a spin structure if its frame bundle can be reduced to the structure group $\mathrm{Spin}(d)$. Then, the frame bundle of $LM$ can be reduced to the structure group $L\mathrm{Spin}(d)$. The spinor bundle of $LM$ would then arise from a representation of $L\mathrm{Spin}(d)$. However, the relevant representation in this case is only *projective*, i.e. it is a representation of a central extension

$$1 \longrightarrow \mathbb{S}^1 \longrightarrow \widetilde{L\mathrm{Spin}(d)} \longrightarrow L\mathrm{Spin}(d) \longrightarrow 1$$

Thus, we can construct the spinor bundle of $LM$ as a genuine *vector* bundle—rather than merely as a *projective* vector bundle—if and only if the structure group of the frame bundle of $LM$ reduces to $\widetilde{L\mathrm{Spin}(d)}$.

**Definition 2.5.** $LM$ is *spin* if such a lift exists.

**Proposition 2.6.** [Kil87] *Let $M$ be a spin manifold. The following are equivalent:*

*(1) $LM$ is spin.*
*(2) The* fractional Pontryagin class $\frac{1}{2}p_1(M) \in \mathrm{H}^4(M;\mathbb{Z})$ *vanishes (this is a certain characteristic class whose square is the usual first Pontryagin class $p_1(M)$).*
*(3) The structure group of $M$ admits a further lift from* $\mathrm{Spin}(d)$ *to a group called* $\mathrm{String}(d)$ *(see below).*

**Definition 2.7.** We make the following definitions:

(1) The group $\mathrm{String}(d)$ is defined as the (or better 'a') topological group with a continuous group homomorphism $f\colon \mathrm{String}(d) \to \mathrm{Spin}(d)$ such that $\mathrm{String}(d)$ is 3-connected and $f$ induces an isomorphism on all homotopy groups in degree $i > 3$.
(2) A closed $d$-dimensional spin manifold $M$ is called *string* if it admits a *string structure*, i.e. a reduction of the structure group of its frame bundle from $\mathrm{Spin}(d)$ to $\mathrm{String}(d)$.

**Remark 2.8.** Since the homotopy groups $\pi_i \mathrm{Spin}(d)$ are trivial for $i = 4, 5, 6$ (at least for $d > 7$), $f$ is automatically a 7-connected covering in this case. ◁

The group String$(d)$ thus naturally sits in the *Whitehead tower* of $\mathsf{O}(n)$: each map in the sequence

$$\cdots \longrightarrow \mathrm{String}(d) \longrightarrow \mathrm{Spin}(d) \longrightarrow \mathsf{SO}(d) \longrightarrow \mathsf{O}(d)$$

is a covering which trivialises the respective highest non-trivial homotopy group of its target space.

**Theorem 2.9.** [Wit88, Zag88] *Let $M$ be a closed $d$-dimensional spin manifold, and let $\phi_W(M) \in \mathbb{Z}[[q]]$ be its Witten genus. If $p_1(M) = 0$, then the Witten genus $\phi_W(M)$ is a modular form, $\phi_W(M) \in \mathrm{MF}(\mathrm{pt})$.*

This applies, in particular, when $\frac{1}{2}p_1(M) = 0$, i.e. if $M$ is string, or equivalently $LM$ is spin. A trivialisation of $p_1(M)$ in $\mathrm{H}^4_{\mathrm{dR}}(M)$, rather than one of $\frac{1}{2}p_1(M)$ in $\mathrm{H}^4(M;\mathbb{Z})$, will be called a *rational string structure* on $M$; we will encounter these frequently below.

This provides another relation between sCFTs—this time on non-trivial $M$—and modular forms, and thus gives further evidence to Conjecture 1.1 (though historically, that conjecture was motivated by the results in this section).

# 3 The Witten class

From now on, we will follow Berwick-Evans paper [BE] to obtain a geometric, rigorous cocycle model for the Witten genus and relate it to 2|1-sCFTs. Note that an alternative geometric construction of the Witten genus has been given by Costello [Cos10, Cos] by means of factorisation algebras and his formalism of renormalisation for perturbative quantum field theory in this language.

For a closed spin manifold $M$, consider the diagram

$$\begin{tikzcd}
\mathrm{EFT}^\bullet_{1|1}(M) \arrow[r, "{[\mathrm{HST10}]}"] & K^\bullet(M) \arrow[r, "\mathrm{ch}"] \arrow[d, "\pi^{an}_!"'] & \mathrm{H}^\bullet_{\mathrm{dR}}(M) \arrow[d, "\int_M(-)\cdot[\hat{A}(M)]"] \\
& K^\bullet(\mathrm{pt}) \arrow[r, "\otimes\mathbb{C}"'] & \mathrm{H}^\bullet_{\mathrm{dR}}(\mathrm{pt})
\end{tikzcd}$$

The commutativity of the square is the *Local Index Theorem*, a version of Atiyah-Singer's Index Theorem. For $M$ a closed string manifold and TMF in place of K-theory, there is a diagram [AHS01, AHR]

$$\begin{tikzcd}
\mathrm{CFT}^\bullet_{2|1}(M) \arrow[r, dashed, "\text{Conj. 1.1}", "???"'] \arrow[rr, bend left, "W"] & \mathrm{TMF}^\bullet(M) \arrow[r, "\otimes\mathbb{C}"] \arrow[d, "\pi^{an}_!"'] & \mathrm{TMF}(M)\otimes\mathbb{C} \cong \mathrm{H}^\bullet(M;\mathrm{MF}) \arrow[d, "\int_M(-)\cdot[\mathrm{Wit}(M)]"] \\
& \mathrm{TMF}^\bullet(\mathrm{pt}) \arrow[r, "\otimes\mathbb{C}"'] & \mathrm{MF}^\bullet
\end{tikzcd}$$

where $[\mathrm{Wit}(M)]$ is the *Witten class* of $M$; it satisfies

$$\int_M [\mathrm{Wit}(M)] = \phi_W(M)\,.$$

We do not yet understand the dashed arrow in this diagram. As an approximation, here we will concentrate on describing the Witten class as well as the arrow $W$. The latter provides a *rational* version of the desired map from field theories to TMF and a better description of the partition function of 2|1-sCFT on a non-trivial manifold $M$.

# 4 Cocycle representatives for the Witten class and rational string structures

First, we need a model for $\mathrm{TMF}(M) \otimes \mathbb{C}$. Generally, the TMF-spectrum is constructed as the global sections of a sheaf of ring spectra over the moduli space of elliptic curves. Since here we are only interested in its rational version $\mathrm{TMF} \otimes \mathbb{C}$, we can give a model in terms of a sheaf of cdgas instead. We proceed as follows.

Consider the action of $\mathbb{Z}^2$ on $\mathbb{H} \times \mathbb{C}$ via

$$(\tau, z) \longmapsto (\tau, z + m + n\tau)\,.$$

The projection map

$$E = (\mathbb{H} \times \mathbb{C})/\mathbb{Z}^2 \longrightarrow \mathbb{H}\,, \quad [\tau, z] \longmapsto \tau$$

has as its fibre at $\tau \in \mathbb{H}$ the torus $E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$.

The space $E$ carries an action of $\mathrm{SL}(2;\mathbb{Z})$:

$$\tau \longmapsto \frac{a\tau + b}{c\tau + d}\,, \qquad z \longmapsto \frac{z}{c\tau + d}\,, \qquad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2;\mathbb{Z})\,.$$

We consider the holomorphic quotient orbifolds

$$\mathcal{M}_{\mathrm{ell}} = \mathbb{H}/\!\!/\mathrm{SL}(2;\mathbb{Z}) \quad \text{and} \quad \mathcal{E} = E/\!\!/\mathrm{SL}(2;\mathbb{Z})\,,$$

the *moduli stack of elliptic curves* and the *universal elliptic curve*, respectively.

There is a line bundle $\mathcal{L} \to \mathcal{M}_{\mathrm{ell}}$ whose fibre $\mathcal{L}_{|\tau}$ at $\tau \in \mathbb{H}$ is the space of holomorphic 1-forms on $E_\tau$. The form $\mathrm{d}z$ on $\mathbb{C}$ descends to a global section of $\mathcal{L}$; the transformation behaviour of $\mathrm{d}z$ under the $\mathrm{SL}(2;\mathbb{Z})$-action produces canonical isomorphisms

$$\Gamma(\mathcal{M}_{\mathrm{ell}}, \mathcal{L}^{\otimes k}) \cong \Big\{ f \in \mathcal{O}(\mathbb{H}) \,\Big|\, f\Big(\frac{a\tau + b}{c\tau + d}\Big) = (c\tau + d)^k f(\tau) \Big\} = \mathrm{MF}^{2k}\,.$$

Here, we demand that $f$ be meromorphic at $\mathrm{i}\infty$, i.e. that $f$ be a weak modular form, and we regard $\mathrm{MF}^\bullet$ as a graded ring with $\mathrm{MF}^{2k+1} = 0$.

**Definition 4.1.** Given a manifold $M$, we define a (strict) sheaf $(\mathrm{Ell}(M), \mathrm{d})$ of cdgas on $\mathbb{H}$ which assigns to $U \subset \mathbb{H}$ the complex of $\mathcal{O}(U)$-modules

$$\big(\mathrm{Ell}(M)(U), \mathrm{d}\big) = \big(\mathcal{O}(U; \Omega^\bullet(M)[\beta, \beta^{-1}]), \mathrm{d}\big) \cong \big(\Omega^\bullet(M; \mathcal{O}(U)[\beta, \beta^{-1}]), \mathrm{d}\big)\,,$$

where $\beta$ has degree $-2$ and the grading is the total grading by form degree and $\beta$. This sheaf is $\mathrm{SL}(2;\mathbb{Z})$-equivariant: the action first pulls back functions on $U$ and then acts via $\beta \mapsto \beta/(c\tau + d)$. Thus, $(\mathrm{Ell}(M), \mathrm{d})$ is a sheaf on $\mathcal{M}_{\mathrm{ell}} = \mathbb{H}/\!\!/\mathrm{SL}(2;\mathbb{Z})$.

**Remark 4.2.** We can view $(\mathrm{Ell}(M)(U), \mathrm{d})$ as the de Rham complex on $M$ with coefficients in (sections of) the Hodge bundle $\mathcal{L} \to \mathbb{H}/\!\!/\mathrm{SL}(2;\mathbb{Z})$; the element $\beta$ corresponds to the global section induced by $\mathrm{d}z$. That also justifies the transformation behaviour of $\beta$ under the $\mathrm{SL}(2;\mathbb{Z})$-action. ◁

**Proposition 4.3.** *The derived global sections of* $(\mathrm{Ell}(M), \mathrm{d})$ *over* $\mathbb{H}/\!\!/\mathrm{SL}(2;\mathbb{Z})$*, i.e. its hypercohomology, is isomorphic to* $\mathrm{H}^\bullet_{\mathrm{dR}}(M; \mathrm{MF})$.

(This can be expected from the observation that the map $\beta \mapsto \mathrm{d}z$ essentially induces an isomorphism from $(\mathrm{Ell}(M)(U), \mathrm{d})$ to $\Omega^\bullet(M, \Gamma(U; \mathcal{L}^\bullet))$. Nevertheless, the proof requires a lot of arguments about total complexes.)

Roughly speaking, *derived* global sections of degree $k$ consist of degree-$k$ sections $s_0$ over $\mathbb{H}$ together with coherence data $(s_1, s_2, \ldots)$ of sections $s_i$ of degree $k - i$ over $\mathbb{H} \times \mathrm{SL}(2;\mathbb{Z})^i$, satisfying $\delta s_i = \mathrm{d}s_{i+1}$

all the way down until we reach degree zero). Here, $\delta$ is the *equivariant*, or *Čech* differential for the $\mathrm{SL}(2;\mathbb{Z})$-action on $\mathbb{H}$.

Now that we have a model for $\mathrm{H}^\bullet_{\mathrm{dR}}(M;\mathrm{MF})$, we would like to find representatives for the Witten class in this model.

**Definition 4.4.** The *Witten class* of $M$ is the characteristic class

$$[\mathrm{Wit}(M)] = \exp\Bigg(\sum_{k=1}^{\infty}[ph_k(TM)]\beta^{2k}E_{2k}\Bigg) \quad \in \mathrm{H}_{\mathrm{dR}}\big(M;\mathcal{O}(\mathbb{H})[\beta,\beta^{-1}]\big)\,,$$

where $[ph_k(TM)] \in \mathrm{H}^{4k}_{\mathrm{dR}}(M)$ is the $4k$-th component of the Pontryagin character of $M$ and $E_{2k} \in \mathcal{O}(\mathbb{H})$ is the $2k$-th *Eisenstein series*

$$E_{2k}(\tau) = \sum_{n,m\in\mathbb{Z}^2_*} \frac{1}{(m\tau+n)^{2k}}\,, \quad \text{where} \quad \mathbb{Z}^2_* = \mathbb{Z}^2 \setminus \{(0,0)\}\,.$$

**Remark 4.5.** There is a subtlety in defining $E_2$ here; one can choose an ordering of the sum in its definition which makes $E_2$ a holomorphic function, which transforms under $\mathrm{SL}(2;\mathbb{Z})$ as

$$E_2\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^2E_2(\tau) - 2\pi\,\mathrm{i}\,c(c\tau+d)\,.$$

It is thus *not a modular form.* For $k>1$, the function $E_2$ *is a modular form* of weight $k$. We thus have

$$[\mathrm{Wit}(M)] \in \mathrm{H}_{\mathrm{dR}}(M;\mathrm{MF}) \quad \Longleftrightarrow \quad [ph_1(TM)] = [p_1(TM)] = 0\,.$$

Note that this is slightly weaker than $M$ admitting a string structure: this condition lives in *real*, or *de Rham* cohomology, whereas the condition $\frac{1}{2}p_1(TM)=0$ is a condition in *integer* cohomology. ◁

We use Chern-Weil theory to provide representatives for $ph_k(TM)$: given any connection on $TM$ (e.g. the Levi-Civita connection of a Riemannian metric on $M$) with curvature 2-form denoted by $R \in \Omega^2(M;\mathrm{End}(TM))$, we set

$$ph_k(TM) = \frac{\mathrm{tr}(R^{2k})}{2k(2\pi\,\mathrm{i})^{2k}}\,.$$

These represent the classes $[ph_k(TM)]$. In order to find a derived global section of $(\mathrm{Ell}(M),\mathrm{d})$ which represents $[\mathrm{Wit}(M)]$, we need to *choose* a trivialisation of the class $[ph_1(M)]$:

**Definition 4.6.** A *rational string structure* on a manifold $M$ is a pair $(g,H)$ of a Riemannian metric $g$ on $M$ and a 3-form $H\in\Omega^3(M)$ such that

$$\mathrm{d}H = \frac{\mathrm{tr}(R^2)}{2(2\pi\,\mathrm{i})^2} = -\frac{\mathrm{tr}(R^2)}{8\pi^2}\,.$$

**Proposition 4.7.** *Any choice of rational string structure $(g,H)$ on $M$ determines*

*(1) a lift of the class* $[\mathrm{Wit}(M)]$ *to a section*

$$\mathrm{Wit}(M) = \exp\Bigg(\sum_{k=1}^{\infty}ph_k(TM)\beta^{2k}E_{2k}\Bigg) \quad \in \Omega^\bullet\big(M;\mathcal{O}(\mathbb{H})[\beta,\beta^{-1}]\big)\,,$$

*(which is not a derived global section of* $(\mathrm{Ell}(M),\mathrm{d})$ *on* $\mathcal{M}_{\mathrm{ell}} = \mathbb{H}/\!\!/\mathrm{SL}(2;\mathbb{Z})$*) and*

*(2) a section* $A \in \Omega^\bullet(M;\mathcal{O}(\mathbb{H}\times\mathrm{SL}(2;\mathbb{Z}))[\beta,\beta^{-1}])$ *such that* $\delta\,\mathrm{Wit}(M) = \mathrm{d}A$*. The higher coherences are trivial to find (they are unobstructed and any two choices differ by exact data).*

*The section $A$ depends explicitly on the choice of $H$. Thus, the data* $\mathbb{W}\mathrm{it}(M) = (\mathrm{Wit}(X),A)$ *furnish a derived global section of* $(\mathrm{Ell}(M),\mathrm{d})$ *over* $\mathcal{M}_{\mathrm{ell}} = \mathbb{H}/\!\!/\mathrm{SL}(2;\mathbb{Z})$ *which represents* $[\mathrm{Wit}(M)]$ *under the isomorphism in Prop. 4.3.*

# 5 Relation to field theories and supersymmetric localisation

We would like to enhance this as follows: in a $2|1$-sCFT on $M$, the space of fields $\Phi_{\mathbb{T}^2,M}$ associated to $\mathbb{T}^2$ identifies with the *odd* tangent bundle of $\mathrm{Map}(\mathbb{T}^2, M)$. We can thus (non-canonically) identify the space of functions $C^\infty(\Phi_{\mathbb{T}^2,M})$—i.e. the observables of the field theory—with the cdga $\Omega^\bullet(\mathrm{Map}(\mathbb{T}^2, M))$. In the path integral formalism, the expectation value of an observable $\alpha \in C^\infty(\Phi_{\mathbb{T}^2,M}) \cong \Omega^\bullet(\mathrm{Map}(\mathbb{T}^2, M))$ would be computed as

$$\langle\alpha\rangle = \int_{\mathrm{Map}(\mathbb{T}^2,M)} \alpha$$

The path integral in quantum field theory computes quantum expectation values of observables by integrating them over the space of fields; here, that would amount to integrating forms $\alpha \in \Omega^\bullet(\mathrm{Map}(\mathbb{T}^2, M))$ over $\mathrm{Map}(\mathbb{T}^2, M)$. Due to the lack of integration techniques on the $\infty$-dimensional mapping space $\mathrm{Map}(\mathbb{T}^2, M)$, we restrict our attention to *equivariant* forms, where the situation is (slightly) better, due to the following localisation scheme.

For $\tau \in \mathbb{H}$, the group $E_\tau$ acts on $\mathbb{T}^2$, and thereby also on $\mathrm{Map}(\mathbb{T}^2, M) \cong LLM$. The fixed point set $X_0$ of this action consists precisely of the constant maps $x \colon \mathbb{T}^2 \to \mathrm{pt} \to M$, i.e. $X_0 \cong M$. Let $N \to M$ denote the ($\infty$-rank) $E_\tau$-equivariant normal bundle to the inclusion $M \hookrightarrow \mathrm{Map}(\mathbb{T}^2, M)$. Explicitly,

$$N = \left\{ v \in \Gamma(M \times \mathbb{T}^2; \mathrm{pr}_M^* TM) \,\middle|\, \int_{\mathbb{T}^2} v = 0 \right\}$$

Let $g$ be a Riemannian metric on $M$ with curvature 2-form $R$. This induces a 2-form $R \in \Omega^2(M; \mathrm{End}(N))$. Let $\xi_\tau$ be the (complexified) vector field on $\mathrm{Map}(\mathbb{T}^2, M)$ which generates the $E_\tau$-action.

*Pretending* for a moment that $\mathrm{Map}(\mathbb{T}^2, M)$ was finite-dimensional, we *would* be able to apply a localisation result by Berline and Vergne [BV85], which *would* read as

**Theorem 5.1.** *The following statements hold true:*

*(1) The inclusion $M \hookrightarrow \mathrm{Map}(\mathbb{T}^2, M)$ induces a quasi-isomorphism of cdgas*

$$\mathrm{res} \colon \big(\Omega^\bullet(\mathrm{Map}(\mathbb{T}^2, M))^{\mathbb{T}^2}, \mathrm{d} - \iota_{\xi_\tau}\big) \xrightarrow{\sim} (\Omega^\bullet(M), \mathrm{d})\,,$$

*where $\Omega^\bullet(\mathrm{Map}(\mathbb{T}^2, M))^{\mathbb{T}^2}$ denotes the cdga of* equivariant *forms on $\mathrm{Map}(\mathbb{T}^2, M)$.*

*(2) For $\alpha \in \Omega^\bullet(\mathrm{Map}(\mathbb{T}^2, M))^{E_\tau}$ with $(\mathrm{d} - \iota_{\xi_\tau})\alpha = 0$ we have*

$$\int_{\mathrm{Map}(\mathbb{T}^2,M)} \alpha = \int_M \frac{\mathrm{res}(\alpha)}{\det^{\frac{1}{2}}(\mathscr{L}_{\xi_\tau} + R)}\,.$$

Our last goal in this talk is to mimick this formula in the present context. Berwick-Evans first enhances the formalism to consider all $E_\tau$, $\tau \in \mathbb{H}$ at the same time: assembling the observables $C^\infty(\Phi_{\mathbb{T}^2,M})^{E_\tau} \cong \Omega^\bullet(\mathrm{Map}(\mathbb{T}^2, M))^{E_\tau}$ together produces a sheaf of cdgas $(\mathrm{Ell}(\mathrm{Map}(\mathbb{T}^2, M), Q)$ on $\mathbb{H}/\!/\mathrm{SL}(2;\mathbb{Z})$ (essentially an enhancement of the construction of $(\mathrm{Ell}(M), \mathrm{d})$). The precise version of the first part of the above localisation statement is

**Theorem 5.2.** *The inclusion $M \hookrightarrow \mathrm{Map}(\mathbb{T}^2, M)$ induces a quasi-isomorphism of sheaves of cdgas on $\mathbb{H}/\!/\mathrm{SL}(2;\mathbb{Z})$:*

$$\mathrm{res} \colon \big(\mathrm{Ell}(\mathrm{Map}(\mathbb{T}^2, M), Q\big) \xrightarrow{\sim} \big(\mathrm{Ell}(M), \mathrm{d}\big)\,.$$

*In particular,* $(\mathrm{Ell}(\mathrm{Map}(\mathbb{T}^2, M), Q)$ *also is a model for* $\mathrm{H}_{\mathrm{dR}}(M; \mathrm{MF})$.

Berwick-Evans then computes a suitably renormalised version of the Pfaffian and uses this to prove the following theorem:

**Theorem 5.3.** *The following statements hold true:*

*(1)* $\det_{\mathrm{ren}}^{-1/2}(\mathcal{L}_\xi+\beta R)$ *is a conditionally convergent product (the problem arising from a summand which is $E_2$).*

*(2) Any choice of rational string structure on $M$ and choice of ordering allow us to make* $\det_{\mathrm{ren}}^{-1/2}(\mathcal{L}_\xi+\beta R)$ *into a derived global section of* $(\mathrm{Ell}(M), \mathrm{d})$ *over* $\mathbb{H}/\!/\mathrm{SL}(2;\mathbb{Z})$*. This represents the Witten class* $[\mathrm{Wit}(M)]$*.*

*(3) We have*

$$\int_M \det{}_{\mathrm{ren}}^{-\frac{1}{2}}(\mathcal{L}_\xi+\beta R) = \phi_W(M) \quad \in \Gamma\big(\mathcal{M}_{\mathrm{ell}}, \mathrm{Ell}^{-d}(\mathrm{pt})\big) = \mathrm{MF}^d\ .$$

The computation of $\det_{\mathrm{ren}}^{-1/2}(\mathcal{L}_\xi+\beta R)$ essentially relies on Fourier decomposition of normal vectors to $M \subset \mathrm{Map}(\mathbb{T}^2, M)$. This leads to a block-decomposition of $\mathcal{L}_\xi + \beta R$ into infinitely many finite-dimensional operators, whose Pfaffians can be computed. The key observation is that the action of $\mathcal{L}_\xi$ produces exactly the terms which assemble into the Eisenstein series upon putting together the contributions of each Fourier mode.

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