

• FRS IV : Structure Constants and Correlation Functions

- Outline:
 - (1) Introduction and goal / where this fits into the programme
 - (2) Recollection of the ribbon graph construction
 - (3) Computing the fundamental structure constants
 - (4) Vertex operators and conformal blocks
 - (5) Computing the fundamental correlation functions

(1) Introduction and goal

- Broad goal: "obtain a universal, model independent, construction of CFT correlation functions." [p.3, FRS IV].
- Idea / claim: This can be done starting from relations between 3d TQFTs and rational CFTs (RCFTs).
- Last time: Fundamental construction on the TQFT side,
"ribbon graph construction" on "connecting manifold" (will be recalled in 53).
 ↳ Choose MTC \mathcal{C} and sym. special Frobenius alg. A in \mathcal{C}
- ↳ uses theory of MTCs + Reshetikhin-Turaev construction to associate to a surface X with field insertions Ψ a vector space $Z(X, \Psi)$ and a vector $v(X, \Psi) = Z(M_{X, \Psi}) \in Z(\hat{X}, \Psi)$ ↑ connecting w/ $M_{X, \Psi} : \emptyset \longrightarrow (X, \Psi)$ ← orientation double.
- Goal here: Understanding this further + get an idea of the relation to CFTs.
- Naming conventions (at least when relating this to CFTs), following [FRS IV]:
 - $Z(\hat{X}, \Psi)$ = (vector) space of conformal blocks (solv. to chiral
ward identities)
 - $v(X, \Psi) = Z(M_{X, \Psi}) \in Z(\hat{X}, \Psi)$ = correlator of (X, Ψ)

- Expanded $\underbrace{Z(X_i, \Psi)}_{\text{correlator}} = \sum_i \underbrace{c(X_i, \Psi)_i}_{\text{structure constants}} \cdot \underbrace{b(X_i, \Psi)_i}_{\text{basis of conformal blocks}}$

↳ This decomposition happens already purely in the 3d TQFT

- Relation to CFT:

- As a VOA, $C = \text{Rep}(A)$, A SSFA in $\text{Rep}(A)$.

- Tricky (incomplete!) part:

- relating $Z(\hat{X}, \Psi)$ to conf. blocks in the CFT / VOA sense (§4),
- analytic / convergence questions (neglected here)

- Upshot of a TQFT construction of CFT correlators:

TQFTs automatically satisfy sewing / cutting constraints.

- Rule: By factorisation and Ward identities, all correlators for oriented CFTs can be computed from the fundamental correlators:

- S^2 with three bulk fields,
 - D^2 with three boundary fields
 - D^2 with one bulk + one boundary field
- $\left. \begin{array}{l} \\ \\ \end{array} \right\}$ (oriented CFTs)

• 2 Recollection of the ribbon graph construction

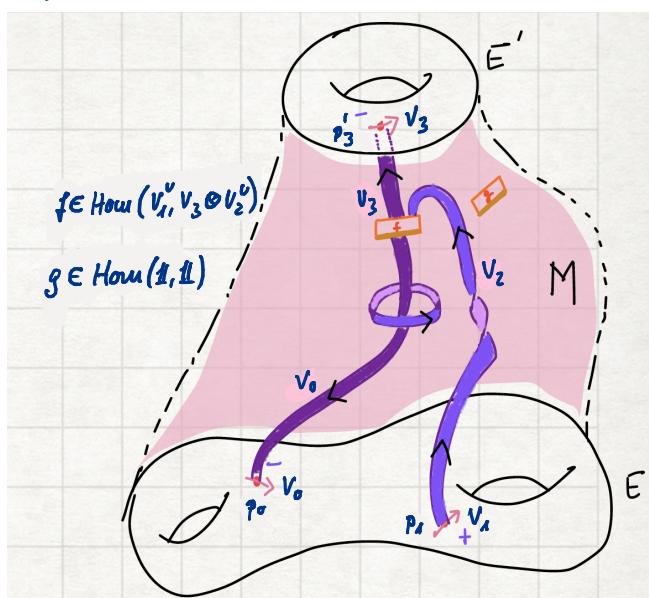
- Fix an MTC \mathcal{C} and a SSFA A in \mathcal{C} .
- Consider the following category $\text{Bord}_3^{\text{ext}}(\mathcal{C})$:
 - Objects: Extended surfaces , consisting of
 - Closed, oriented 2d surface E ($\partial E = \emptyset$),
 - Finite set of marked pts $(p_i, [\gamma_i], v_i, \epsilon_i)$,
 - $p_i \in E$. $p_i \neq p_j \quad \forall i \neq j$
 - $[\gamma_i]$ genus of arcs $\gamma_i : [-\delta, \delta] \longrightarrow E$, $\gamma_i(0) = p_i$
 - $v_i \in \mathcal{C}$.
 - $\epsilon_i \in \{-1, 1\}$,
 - $\lambda \subset H_1(E; \mathbb{R})$ Lagrangian subspace (not used here — but in RT?)

Morphisms: Extended cobordisms , consisting of

- An ordinary oriented cobordism $M : E \longrightarrow E'$,
(∂M oriented using inward-pointing normal vectors)
- Ribbon graph R in M , ending on

$$(-E, (p_i, [\gamma_i], v_i, -\epsilon_i)) \sqcup (E', (p'_j, [\gamma'_j], v'_j, \epsilon'_j)) ,$$

labelled by \mathcal{C} , ribbons pointing away from $\epsilon=1$ and towards $\epsilon=-1$,
inducing the genus $\epsilon_i[\gamma_i]$.



(image credit: Nivedita)

- From field insertions on surfaces to extended cobordisms

- FRS provide a recipe / construction

{surface X with field insertions (boundary, bulk, defects)}

↓ orientation double

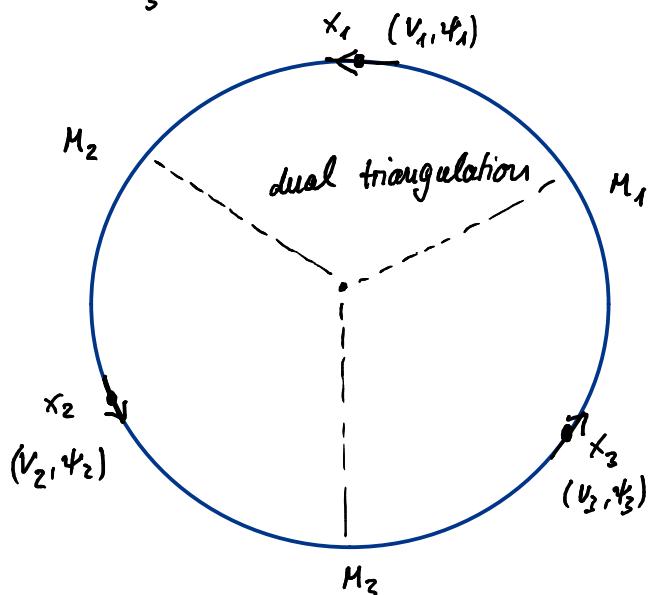
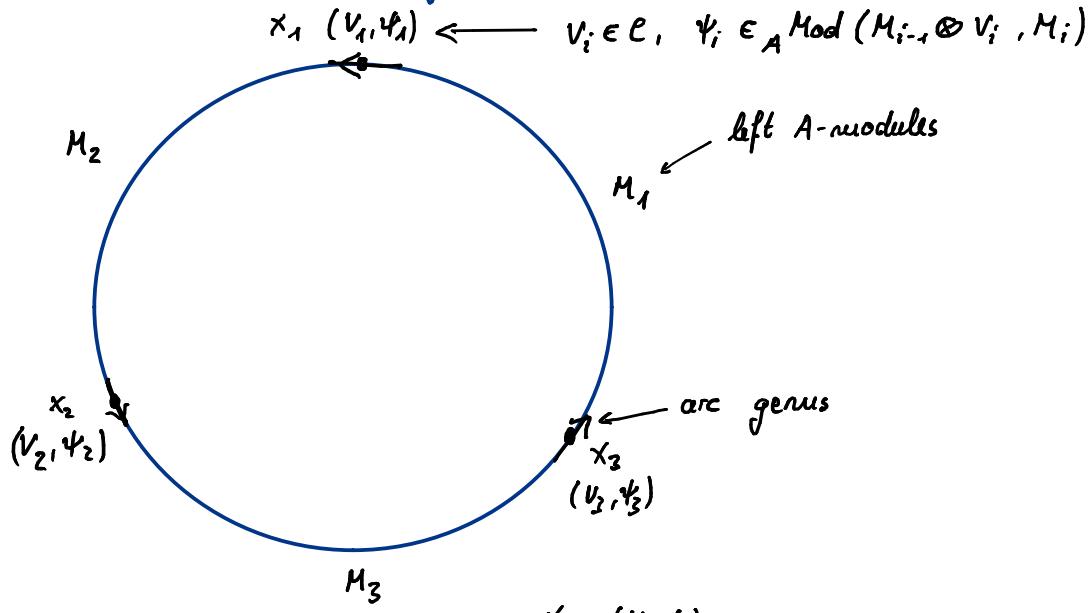
{extended surface \hat{X} / object in $\text{Bord}_3^{\text{ext}}(\mathcal{C})$ }

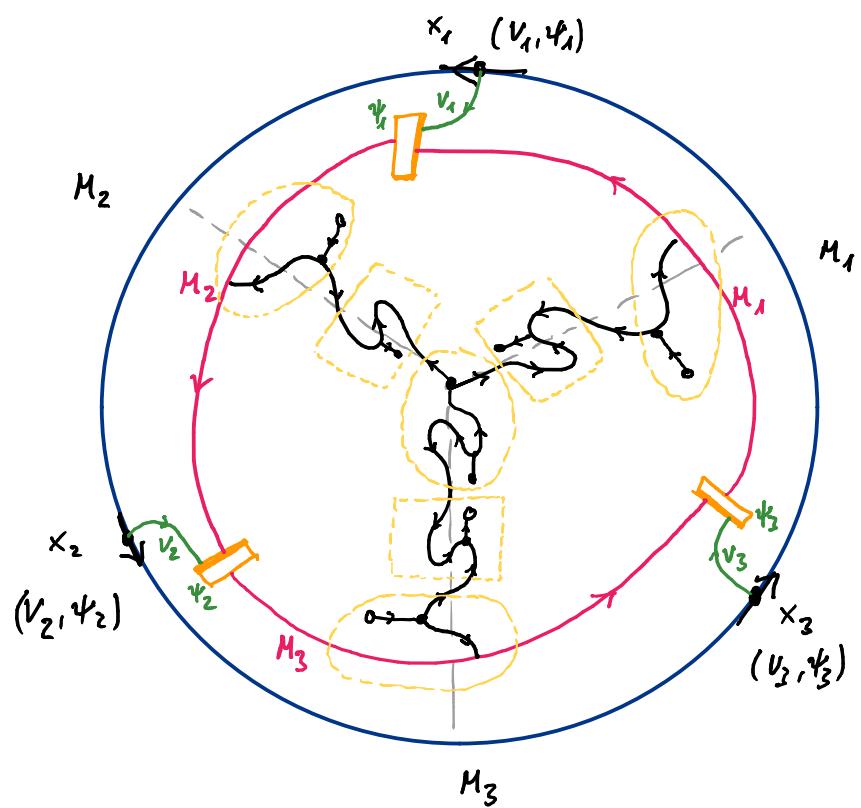
↓ connecting bordism

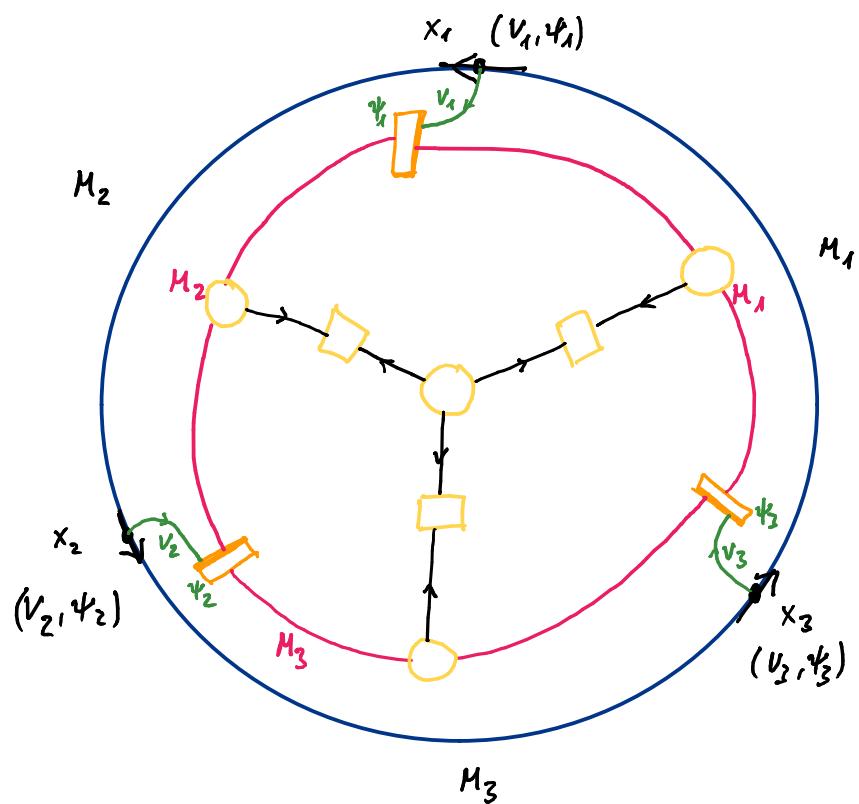
{Morphism $H_{\hat{X}} : \emptyset \longrightarrow \hat{X}$ in $\text{Bord}_3^{\text{ext}}(\mathcal{C})$ }

↳ Idea: $\underbrace{z(H_{X,\Psi})}_{\substack{\text{correlator assigned} \\ \text{to } (X,\Psi)}} \in \underbrace{Z(\hat{X},\bar{\Psi})}_{\substack{\text{Space of conformal blocks on } (X,\bar{\Psi})}}$

- Example: Three boundary fields on the disc







- Key to computations / ribbon graph manipulations:
- Lemma: Since $A \in \mathcal{C}$ is SSFA object, we have the identities

$$\textcircled{1} \quad \begin{array}{c} \text{Diagram showing two configurations of a dashed loop with arrows A and A} \\ \text{Diagram 1: } \text{Diagram 2: } \end{array} = \quad \text{and } \textcircled{2} \quad \begin{array}{c} \text{Diagram showing a dashed loop with arrows A and A} \\ \text{Diagram 1: } \text{Diagram 2: } \end{array} = \quad (5.11) \quad [\text{FRS I}]$$

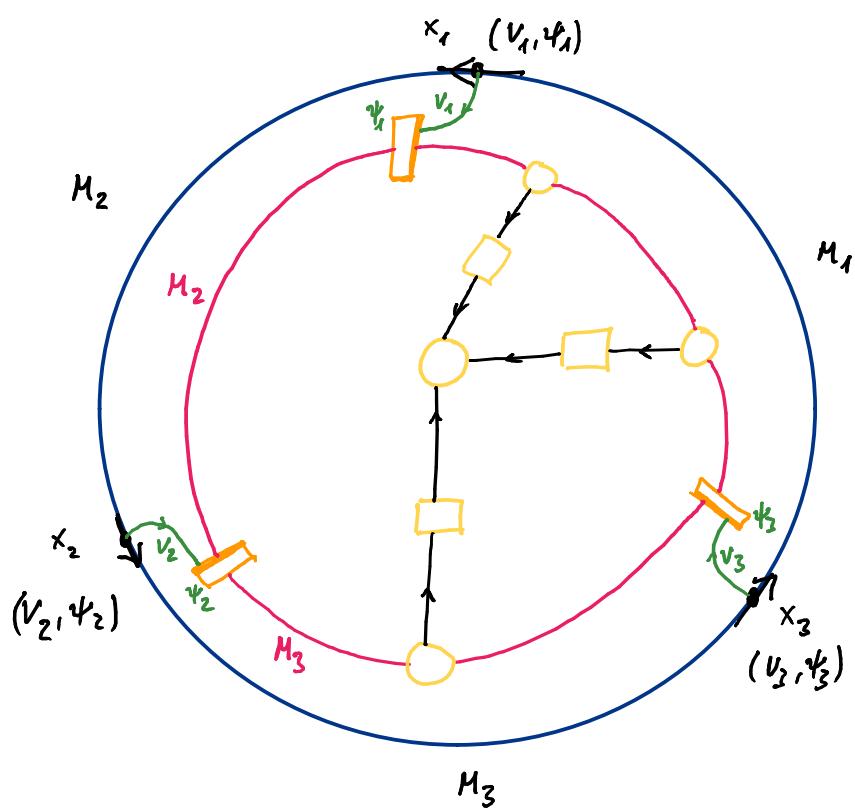
and on the boundary,

$$\textcircled{3} \quad \begin{array}{c} \text{Diagram showing a vertical dashed line with arrows A and M} \\ \text{Diagram 1: } \text{Diagram 2: } \end{array} = \quad \text{and } \textcircled{4} \quad \begin{array}{c} \text{Diagram showing a vertical dashed line with arrows A and M} \\ \text{Diagram 1: } \text{Diagram 2: } \end{array} = \quad (5.12) \quad [\text{FRS I}]$$

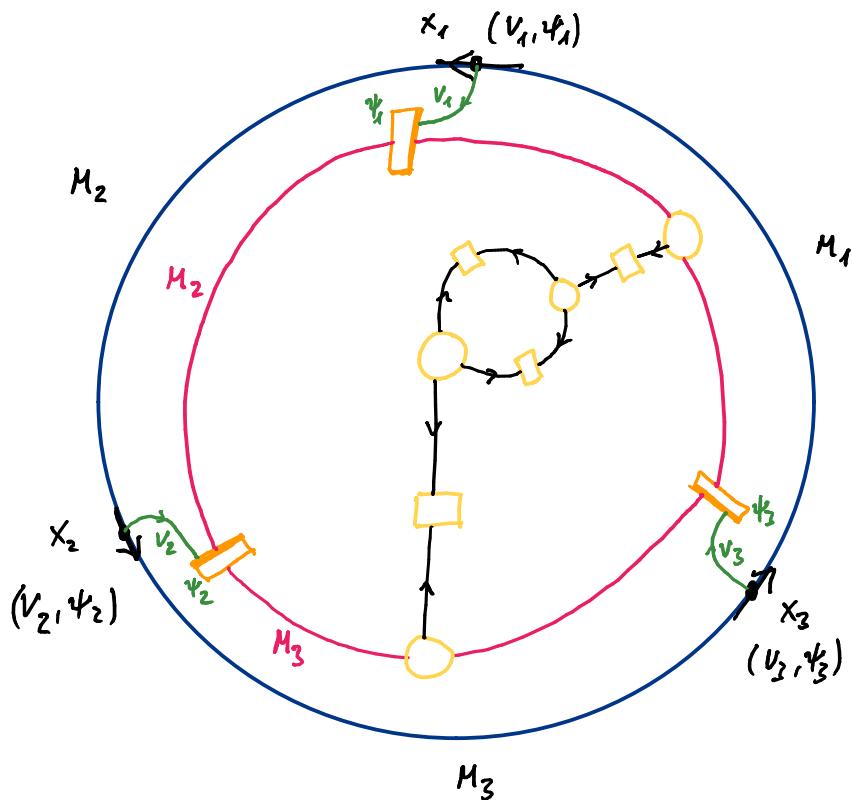
and

$$\textcircled{5} \quad \begin{array}{c} \text{Diagram showing a horizontal dashed line with arrows M and N, passing through a yellow box labeled psi} \\ \text{Diagram 1: } \text{Diagram 2: } \end{array} = \quad (3.15) \quad [\text{FRS IV}]$$

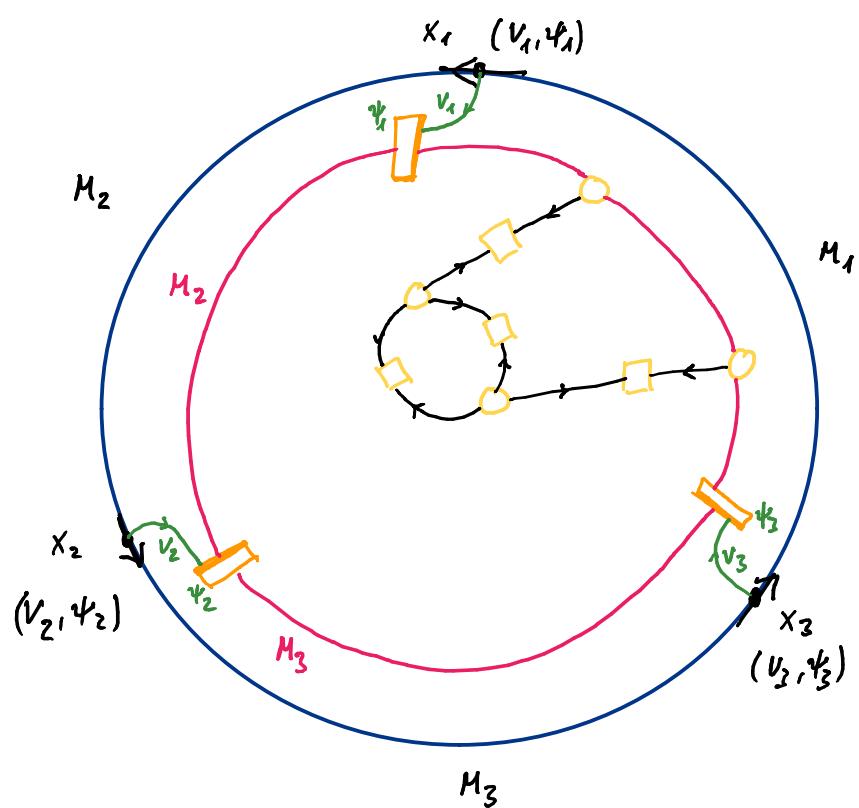
using ⑤:



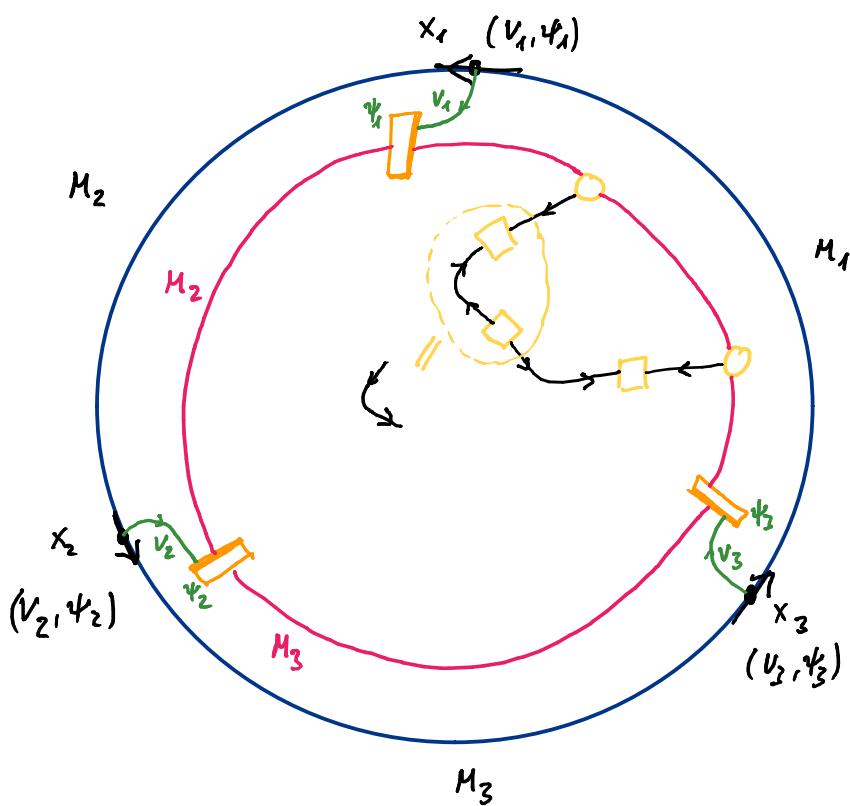
use ③:



use ⑤:

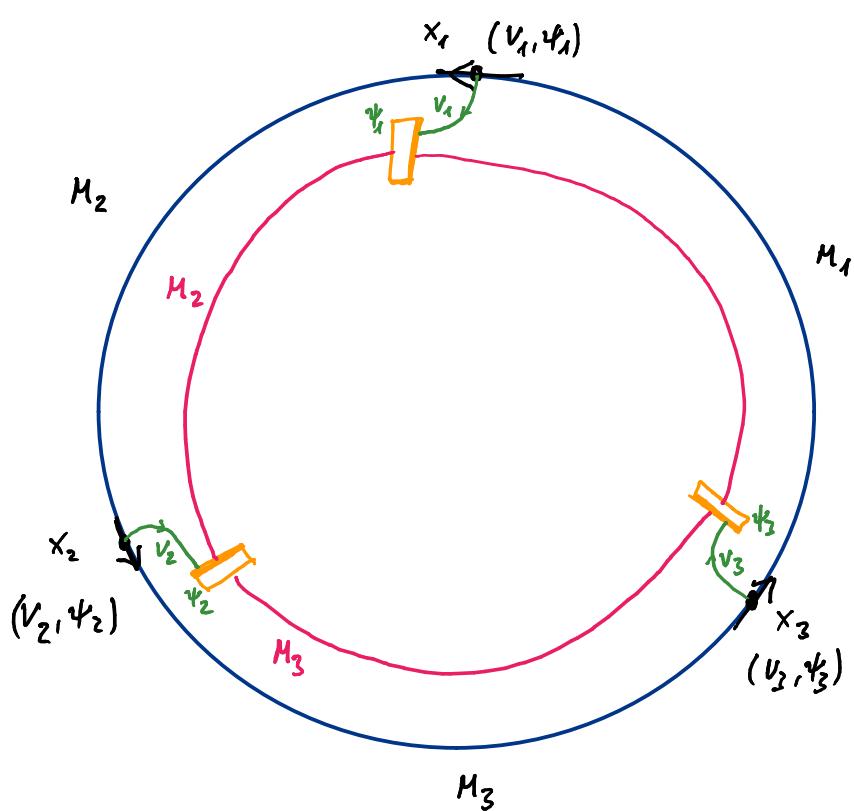


use ②:

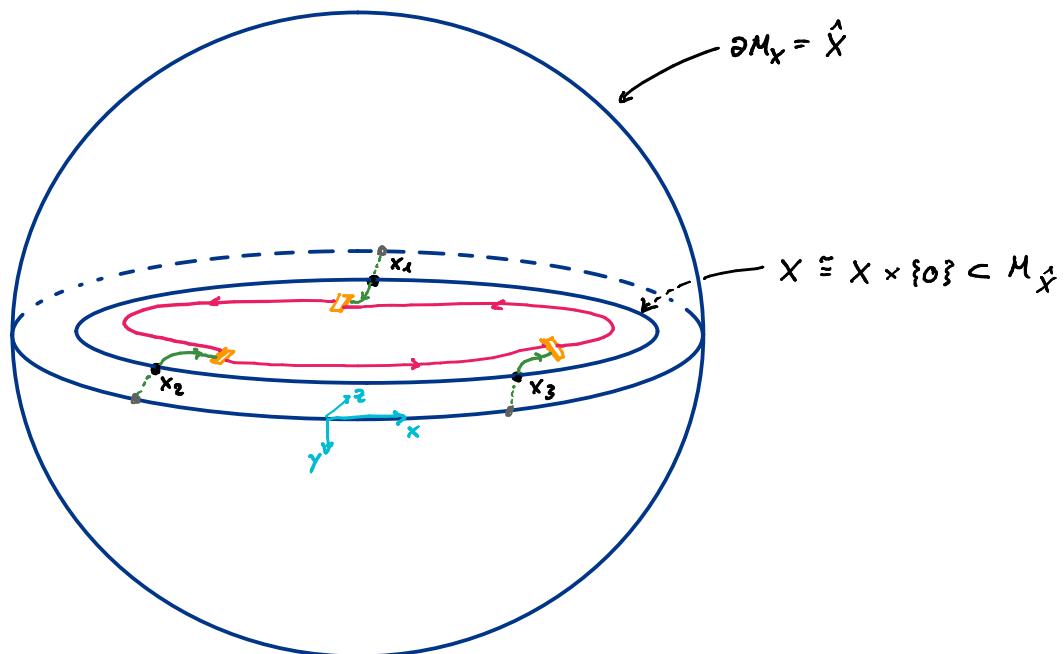


$\rightarrow \square \leftarrow$
 $A \rightarrow A^u$ is the
 iso induced by the
 non-diag. pairing
 $A \xrightarrow{\quad} A \circ 1$

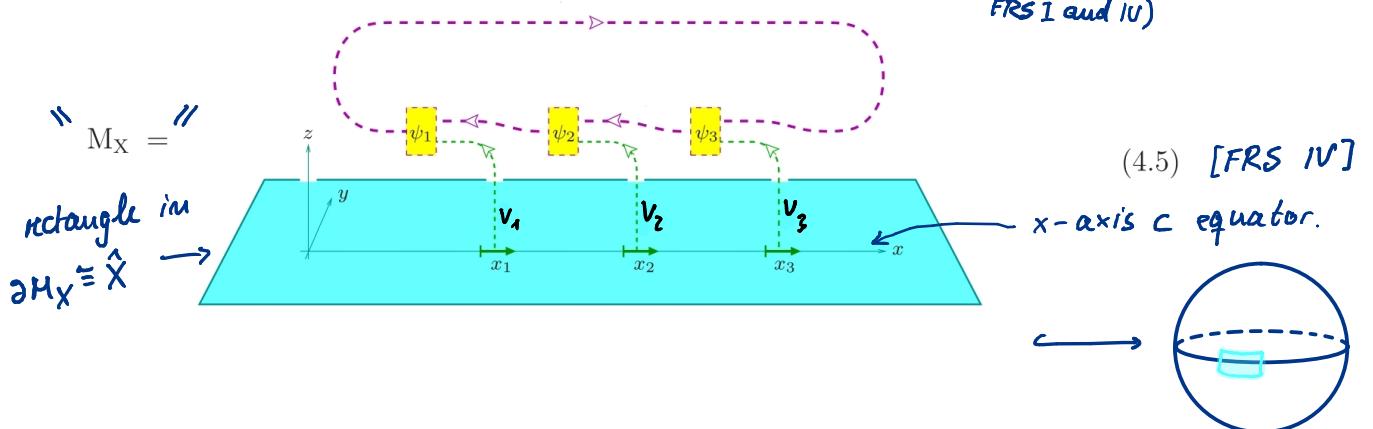
use ④:



- No bulk or defect fields \Rightarrow This is already the ribbon graph for $M_{\hat{X}}$.



- Moving insertions close to each other, FRS draw the case with opposite orientation as (convention change between FRS I and IV)



• 3 Computing fundamental structure constants

- Recall that $\underline{Z(H_x, \Psi)} = \sum_s \underline{c(x, \Psi)_s} \cdot \underline{b(x, \Psi)_s}$
 correlator structure constants basis of conformal blocks
 in $Z(\hat{x}, \Psi) \in \text{Vect}$.

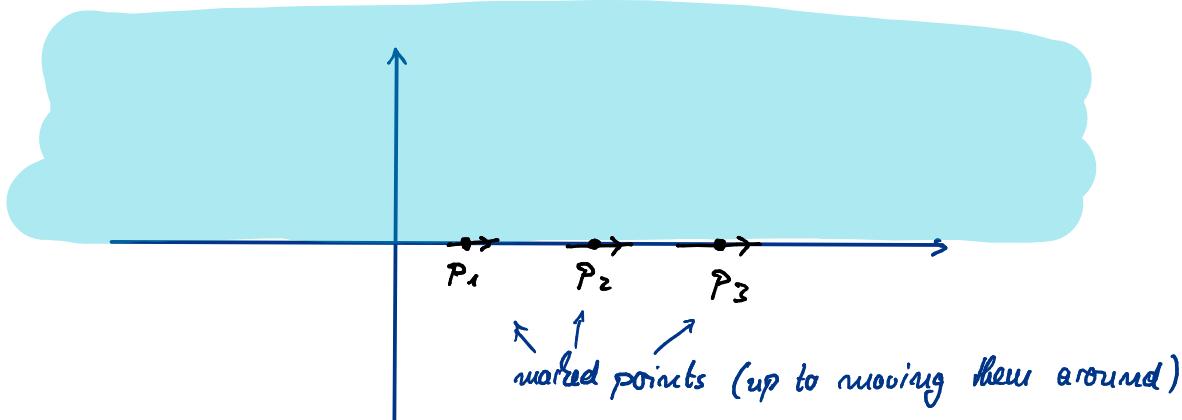
\Rightarrow We need to

- (1) Provide a "standard" basis for the vector space $Z(\hat{x}, \Psi)$,
- (2) Decompose the vector $Z(H_x, \Psi)$ in that basis.

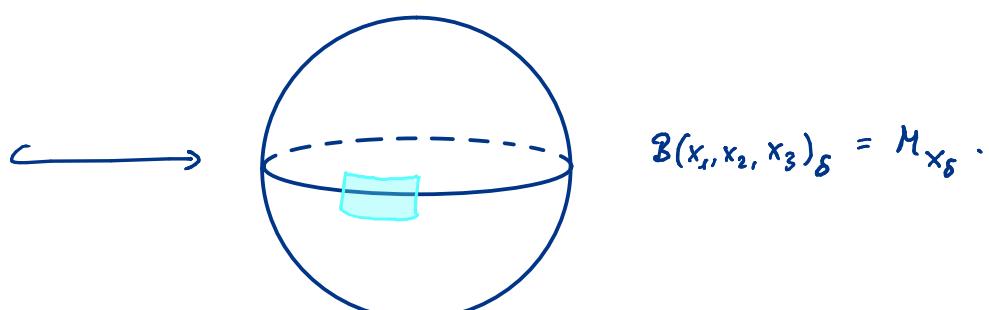
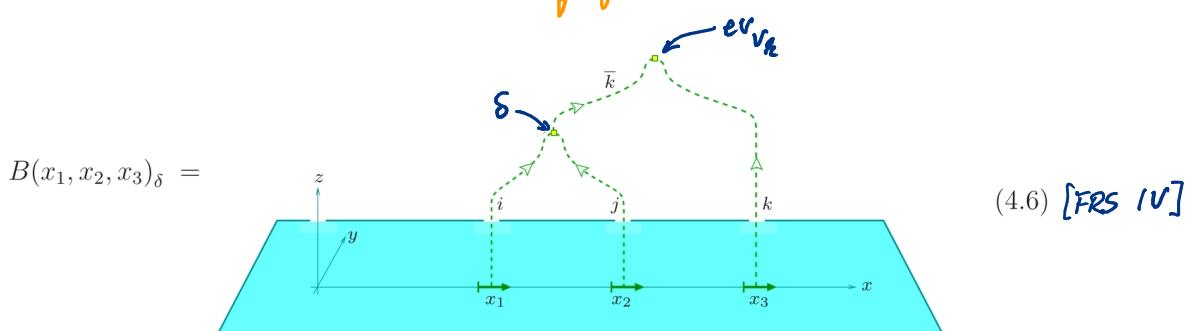
- [FRS IV] gives recipes for doing this for the fundamental correlators

\hookrightarrow Here: Disc with three boundary fields

- (1): • Represent D^2 as $H \cup \{\infty\} = \{z \in \mathbb{C} \cup \{\infty\} \mid \operatorname{Im}(z) \geq 0\}$
 • Consider D^2 with three marked points:



\hookrightarrow Orientation double: S^2 with "basis of field insertions"



\Rightarrow TFT construction turns this into a vector

$$b_\delta = Z(B(x_1, x_2, x_3)_\delta) \in Z(S^2, (v_e, \psi_e)_{e=1,2,3}).$$

- Prop.: Let $\{\delta\}$ be a basis of $\text{Hom}_e(V_i \otimes V_j, V_k)$.

Then, $\{b_\delta\}$ is a basis of $Z\left(\begin{array}{c} \uparrow \\ \xrightarrow{x_1} \bullet \xrightarrow{x_2} \bullet \xrightarrow{x_3} \\ v_i \quad v_j \quad v_k \end{array}\right)$.

\Rightarrow Let $(X, \Psi) =$ Then,

$$Z(M_X, \Psi) = \sum_{\delta} c(M\Psi_1, N\Psi_2, K\Psi_3) \cdot Z(B(x_1, x_2, x_3)_\delta).$$

- This is computable : postcompose by $\overline{B(x_1, x_2, x_3)_\delta}$ (\rightsquigarrow produces dual basis)
- to obtain the identity

$$c(M\Psi_1, N\Psi_2, K\Psi_3) =$$
(4.8) [F2S IV]

\hookrightarrow Now it remains to evaluate this in \mathcal{C} .

- Everything so far works for any MTC \mathcal{C} and SSFA A in \mathcal{C} .

The relation to (2d) CFT arises from choosing a vertex operator algebra (VOA) A and setting $\mathcal{C} = \text{Rep}(A)$.

• 4 Vertex operators and conformal blocks

conformal weight

- A VOA A consists, in particular, of an N_0 -graded vsp. $\mathcal{R}_{\mathcal{S}_2}$.

The space of states, with various additional structure, including:

- State-field correspondence: $\mathcal{R}_{\mathcal{S}_2} \longrightarrow \text{End}(\mathcal{R}_{\mathcal{S}_2})[[z^{\pm 1}]]$
 $u \longmapsto u(z) = \gamma(u, z)$.

- Distinguished vacuum state: $0 \neq v_{\mathcal{S}_2} \in \mathcal{R}_{\mathcal{S}_2}$, s.t. $\mathcal{R}_{\mathcal{S}_2(0)} = \text{span}_{\mathbb{C}} \{v_{\mathcal{S}_2}\}$.

$$v_{\mathcal{S}_2}(z) = \text{id}_{\mathcal{R}_{\mathcal{S}_2}}, \quad u(z)v_{\mathcal{S}_2} = u + O(z) \quad \forall u \in \mathcal{R}_{\mathcal{S}_2}, \dots$$

- A module over a VOA A of lowest conf. weight Δ consists, in particular, of a $(\Delta + N_0)$ -graded fin. dim. vsp. \mathcal{R} and an action map

$$s: \mathcal{R}_{\mathcal{S}_2} \longrightarrow \text{End}(\mathcal{R})[[z^{\pm 1}]] \quad + \text{conditions.}$$

(fcts. $\mathbb{C}^* \longrightarrow \text{Hom}(\mathcal{R}, \overline{\mathcal{R}})$, where $\overline{\mathcal{R}} = \prod_u \mathcal{R}_{\text{cusp}}$)

- An intertwining operator between $(\mathcal{R}_i, \Delta_i)$, $(\mathcal{R}_j, \Delta_j)$ and $(\mathcal{R}_k, \Delta_k)$ is a lin. map

$$V_{jk}^i(-, z): \mathcal{R}_j \longrightarrow z^{\Delta_i - \Delta_j - \Delta_k} \text{Hom}(\mathcal{R}_k, \mathcal{R}_i)[[z^{\pm 1}]] \quad + \text{conditions.}$$

Remark: We focus on the ideas here and therefore disregard convergence problems.

↳ There are various assumptions in [FRS IV] regarding these.

• Bundles of conformal blocks:

- A (chiral) CFT assigns vector spaces $H^c(E^c)$ not to surfaces, but to Riemann surfaces

↳ An extended Riemann surface E^c consists of:

- cpt. Riemann surface E^c , $\partial E^c = \emptyset$,
- finite ordered set of marked points $(p_i, [\varphi_i], R_i)$,
 $p_i \in E^c$, $[\varphi_i]$ genus of inv. holom. map $\sigma \in \mathbb{D}_E^{\times} \rightarrow E^c$ ($\sigma \mapsto p_i$),
 $R_i \in \text{Rep}(A)$,
- Lagrangian submodule $\lambda \subset H_*(E^c; \mathbb{Z})$.

- Let $\hat{\mathcal{M}}_{g,n}(R_1, \dots, R_n)$ be the moduli space of ext. Riemann surfaces,
with p_i decorated by R_i , $i=1, \dots, n$.

Then, $(E^c; p_i, [\varphi_i], R_i) \longmapsto H^c(E^c)$ produces a vector bundle

$$\begin{array}{ccc} \hat{\mathcal{B}}_{g,n}(R_1, \dots, R_n) & \longrightarrow & \hat{\mathcal{M}}_{g,n}(R_1, \dots, R_n) \times (R_1 \otimes \dots \otimes R_n)^* \\ \text{Bundle of conformal blocks} \nearrow & \searrow & \downarrow \\ & & \hat{\mathcal{M}}_{g,n}(R_1, \dots, R_n) \end{array}$$

(defined by a certain compatibility condition for the VOA action.)

- $\hat{\mathcal{B}}_{g,n}(R_1, \dots, R_n)$ carries a canonical projectively flat connection,
the Knizhnik - Zamolodchikov connection.

• Assumptions for rationality:

(1) $\text{Rep}(A)$ is an HTC.

(2) The spaces of conformal blocks associated to A define the same monodromy and factorisation data as the 3d TQFT constructed from $\text{Rep}(A)$. (see below)

↪ A with these properties is called **rational**.

• Idea/conjecture: (proven in [FRS IV] for $g=0$, mention other results, but generally open)

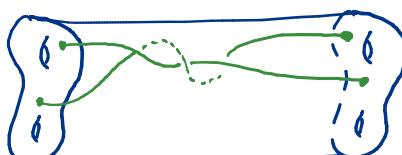
There exists a diagram of vector bundles

$$\begin{array}{ccccc}
 & & \text{connection-preserving iso (at least projectively)} & & \\
 & \searrow & & \downarrow & \\
 \hat{\mathcal{B}}_{g,n}(R_1, \dots, R_m) & \xrightarrow[N]{\cong} & \mathcal{F}^* V_{g,n}(R_1, \dots, R_m) & \longrightarrow & V_{g,n}(R_1, \dots, R_m) \\
 & \swarrow & & & \\
 \hat{\mathcal{M}}_{g,n}(R_1, \dots, R_m) & \xrightarrow[\mathcal{F}]{} & S_{g,n}^{\text{ext}}(R_1, \dots, R_m) & & \text{Mod.sp. of extended} \\
 & & & & \text{surfaces, as in TQFT} \\
 & & \uparrow \text{canonical forgetful map} & & \\
 & & (\text{path genus: } g_i = \varphi_i|_{D_E^2 \cap R}) & &
 \end{array}$$

where $V_{g,n}(R_1, \dots, R_m)|_E = Z(E) \in \text{Vect}$, Z the 3d TQFT obtained from $C = \text{Rep}(A)$.

↪ Connection on $V_{g,n}(R_1, \dots, R_m)$:

path γ in $S_{g,n}^{\text{ext}}$:



→ extended cobordism
 $M(\gamma)$

↪ parallel transport := $Z(M(\gamma))$.

Rank: $g=0$ case: $\hat{\mathcal{M}}_{0,n}$ is connected. \Rightarrow Enough to produce an iso of fibres at any single point, and then show that monodromy/holonomy agrees around each generator of π_1 . [FRS IV] sketch this.

• 5 Computing correlation functions

- Fix a rational VOA A , repr. of 160 classes of simple obj's $\{S_i \mid i \in \mathcal{I}\} \subset \text{obj}(\text{Rep}(A))$, and bases $\{V_{jk}^{i,\delta}(-, z) \mid \delta = 1, \dots, N_{jk}^i\} \subset \text{Hom}_{\text{Rep}(A)}(S_j \otimes S_k, S_i)$ (+ properties)
 - A **Riemannian world sheet** X^g is a cpt. possibly non-orientable Riemannian 2-mfd, possibly with $\partial X^g \neq \emptyset$, boundary, bulk and defect fields.
 - The **correlation function** for X^g is a linear map $C(X^g) : \bigotimes_{k=1}^m S_k \otimes \bigotimes_{l=1}^n S_l \otimes S_{j,k} \longrightarrow \mathbb{C}$
- $\begin{matrix} \uparrow \\ \text{bdry fields} \end{matrix} \quad \begin{matrix} \uparrow \\ \text{bulk + defect fields.} \end{matrix}$
- Holomorphic factorisation $\Rightarrow C(X^g) \in H^c(\hat{X}^g) = \mathcal{B}_{g,m+2n}(\bar{s})|_{(X^g, \bar{\rho}, [\bar{\varphi}])}$
- Idea: $Z(M_X) \in Z(\hat{X})$

- It suffices to give the fundamental correlators (sewing + Ward)
- Rule: Generally, $C(X^g)$ depends on g , not just on its conformal class, if $c \neq 0$. \leadsto Compute $C(X^g)/C(\dot{X}^g)$, $\dot{X}^g = X^g$ without insertions.

- E.g.: Three bdry fields on D^2 (with std. metric):

By our computation above,

$$C(X^g)(u, v, w) = \sum_{\delta=1}^{N_{ij}^k} c(M\Psi_1 N\Psi_2 K\Psi_3 M)_\delta \langle 0 | V_{kk}^0(w; x_3) V_{ji}^{\bar{k}; \delta}(v; x_2) V_{i0}^j(u; x_1) | 0 \rangle, \quad (6.15) \quad [\text{FRS IV}]$$

conf. invar: transform to $x_1=1, x_2=2, x_3=3$

$$\frac{C(X^g)(u, v, w)}{C(\dot{X}^g)} = \left(\sum_{\delta=1}^{N_{ij}^k} \frac{c(M\Psi_1 N\Psi_2 K\Psi_3 M)_\delta}{\dim(A)} B_{kji}^\delta(w, v, u) \right) (x_3 - x_2)^{\Delta_i(u) - \Delta_j(v) - \Delta_k(w)} (x_3 - x_1)^{\Delta_j(v) - \Delta_i(u) - \Delta_k(w)} (x_2 - x_1)^{\Delta_k(w) - \Delta_i(u) - \Delta_j(v)}. \quad (6.16) \quad [\text{FRS IV}]$$

$\cancel{c(\dot{X}^g)}$ $= \text{this term for } x_1=1, x_2=2, x_3=3$