

Categories of A- and B-Branes

Part 1

Topological Strings, B-Branes, and $D(\text{Coh}(X))$

ZMP Seminar, 29/04/21

• Outline

- 1 2d $N=2$ NLSM and twisting
 - 1.1 Twisting and topological field theory
 - 1.2 $N=2$ NLSM
 - 1.3 The A- and B-models
- 2 B-branes
 - 2.1 Boundary conditions for open strings
 - 2.2 Naive B-branes and vector bundles
 - 2.3 $D(\text{Coh}(X))$ as the category of B-branes

References:

- Aspinwall - D-Branes on Calabi-Yau Manifolds,
hep-th/0304166
- ABCDGKMSSW / Aspinwall et al. -
Dirichlet Branes and Mirror Symmetry
- Greene - String Theory on Calabi-Yau Manifolds,
hep-th/9702155

↙ Main reference

• 1 2d $N=2$ NLSM and twisting

• 1.1 Twisting and topological field theory

• In a generic QFT, physics depends on a **metric**

(features in the action ; $Z: \text{Bord}_d^g \rightarrow \mathbb{C}$)

• Stress-energy / energy-momentum tensor :

$$T^{\mu\nu} \sim \frac{\delta L}{\delta g_{\mu\nu}}, \text{ captures dependence on metric.}$$

$$H = \int_{\Sigma} T$$

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• Suppose, \exists operator Q s.t.

(1) $Q^2 = 0 \rightarrow$ can take its cohomology

(2) $T = \{Q, b\}$ is Q -exact \rightarrow Q -coho metric-indep.!

\Rightarrow Q -coho is TFT.

Supersymmetric QFTs:

- N lin. indep. supercharges Q_I , $I = 1, \dots, n$:

- hermitean operators

- $\{Q_I, Q_J\} = \delta_{IJ} H$

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E.g.: Suppose $\exists Q_1, Q_2$. Set $Q := Q_1 + iQ_2$

$\Rightarrow Q^2 = \frac{1}{2} \{Q, Q\} = H - H = 0$, Q defines a differential

$$H = \{Q, Q_1\} = \{Q, -iQ_2\}, \quad H \text{ is } Q\text{-exact}$$

But: \mathcal{T} generally not Q -exact or Q -closed!

Twisting:

Generates "R-symmetry"
↓

• Suppose \exists conserved bosonic $U(1)$ -current J_μ (with $\partial_\mu J^\mu = 0$)

$$\text{s.t. } \{Q, b_{\mu\nu}\} = T_{\mu\nu} - \frac{1}{4} \epsilon_{\mu\alpha} \partial^\alpha J_\mu - \frac{1}{4} \epsilon_{\nu\alpha} \partial^\alpha J_\mu$$

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- New, twisted theory has same Hamiltonian (but different Lorentz generators).

↳ Look for $N=2$ SQFTs with $U(1)$ -R-symmetry to obtain TFTs.

• 1.2 $N=(2,2)$ NLSM

- Σ a Riemann surface, X a Kähler manifold ($N=2$ Susy)
- Spin^c -structure on Σ : holomorph. line bundles L_1, L_2 and iso

$$L_1 \otimes L_2 \cong K \cong T^*\Sigma$$

↳ choose two Spin^c -strs $(L_1, L_2), (L_3, L_4)$ on Σ .

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Fields:

• Bosons: $\phi: \Sigma \rightarrow X$, in coordinates: $\phi^i, \phi^{\bar{i}}$

• Fermions:

$$\begin{aligned}\psi_+^i &\in \Gamma(L_1 \otimes \phi^* TX) \\ \psi_+^{\bar{i}} &\in \Gamma(L_2 \otimes \phi^* \bar{T}X) \\ \psi_-^i &\in \Gamma(\bar{L}_3 \otimes \phi^* TX) \\ \psi_-^{\bar{i}} &\in \Gamma(\bar{L}_4 \otimes \phi^* \bar{T}X)\end{aligned}$$

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$$\delta \phi^{\bar{j}} = i \tilde{\alpha}_- \psi_+^{\bar{j}} + i \tilde{\alpha}_+ \psi_-^{\bar{j}}$$

$$\delta \psi_+^j = -\tilde{\alpha}_- \partial \phi^j - i \alpha_+ \psi_-^k \Gamma_{kl}^j \psi_+^l$$

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• Action: $S = (\dots)$ involves spacetime metric g and B-field

• U(1) current: $J(z) = \frac{1}{4} g_{i\bar{j}} \psi_+^i \psi_-^{\bar{j}}$

• LaSring BRST operator Q with $Q^2 = 0$.

1.3 The A- and B-models

The A-model on Y

- Restrict SUSY trafo's to $\alpha := \alpha_- = \tilde{\alpha}_+$, $\alpha_+ = \tilde{\alpha}_- = 0$

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$$\delta \psi_{-}^{\bar{j}} = 0$$

$$\underline{\underline{Q^2 = 0.}}$$

• Spin^c - str : $L_1 = L_4$ trivial, $L_2 = L_3 = K$

• Action : $S = i \int_{\Sigma} \{Q, \mathcal{D}\} - 2\pi i \int_{\Sigma} \phi^*(B + i\omega)$
contains cplx str of Y Kähler form

\Rightarrow Depends on $B + i\omega$ only, on the level of \mathbb{Q} -coho.

• Local operators : $W[a] = a_{I_1 \dots I_p} \underbrace{x^{I_1} \dots x^{I_p}}_{\text{anticommuting / fermionic}}$,
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• Lemma : $\{Q, W[a]\} = -W[da]$, s.t. $H_{\mathbb{Q}}^i \cong H_{dR}^i(Y)$.

(as vector spaces; alg. str. more complicated)

The B-model on X

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- BRST sym from $\alpha := \tilde{\alpha}_+ = \tilde{\alpha}_-$, $\alpha_+ = \alpha_- = 0$.
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 - ↳ Variations in $g, B+i\omega$ induce \mathcal{Q} -exact terms \rightarrow dep on J only.

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Lemma: $H_{\mathcal{Q}}^i \cong H_{\bar{\partial}}^i(X; \Lambda^* TX)$.

Result: B-model has no instanton corrections
 \Rightarrow classically exact, computable.

• 2 B-branes

• 2.1 Boundary conditions for open strings

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• Invariance under B-field gauge transformations: $B \mapsto B + d\lambda$:

in S : $\int_{\Sigma} \phi^* B \rightarrow$ not invariant for open strings

\hookrightarrow add coupling of string ends to gauge field on N :

$\int_{\partial_{b_+} \Sigma} \phi^* A$, where $A \mapsto A - \lambda$ under B-transf.

-2.2 Naive B-branes

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↳ Above + $\mathbb{Q} = \bar{\partial}$ + the Chan-Paton degrees of freedom:

$$\begin{aligned} \mathbb{Q} &= \text{coho of open string vertex ops / states between } E_1, E_2 \\ &\cong H_{\bar{\partial}}^{0,q}(X, \text{Hom}(E_1, E_2)) \end{aligned}$$

• Non-space-filling B-branes:

$$\begin{array}{ccc} E & & \\ \downarrow & & \\ N & \xrightarrow{\mathcal{L}} & X \end{array}$$

↳ From an algebraic perspective, we can treat these on the same level as before:

• E is a (locally free) sheaf of \mathcal{O}_N -modules

$\mathcal{L}_* E \in \text{Coh}(X)$, where $\text{Coh}(X)$ is the Cat. of
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• These \mathcal{B} -branes ($E \rightarrow N \subset X$) are **not enough** to achieve mirror sym. with $\text{Fuk}(Y)$ (Part II).

↳ Generalise: Maybe \mathcal{B} -branes are objects in $\text{Coh}(X)$?

2.3 The category of B-branes

- Had seen: Open-string states between $E, F \in \text{Mod}_{\mathbb{C}_X}^{\text{loc. free}} \subset \text{Coh}(X)$
 $\cong H_{\mathbb{Z}}^{\text{odd}}(X, \text{Hom}(E, F)) \cong \text{Ext}^q(E, F) \cong \text{Hom}_{\mathcal{D}(\text{Coh}(X))}(E, F[q])$
 $\cong \text{Ext}^l(E[p], F[q+p-l])$
any gradings possible

- Allow for general grading / ghost number / shifting: $E = \bigoplus_{n \in \mathbb{Z}} E^n$,
string in $\text{Ext}^l(E^m, F^l)$ has *ghost number* $q + l - m$.

- Include deformations of B-branes: Bdry defs. \rightarrow ghost no. one
↳ Consider strings $\mathcal{E} \rightsquigarrow \mathcal{E}$, i.e. $\text{Ext}^k(\mathcal{E}^n, \mathcal{E}^{n-k+1})$

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$k=1$: $H_{\mathfrak{g}}^{0,1}(X, \text{End}(\mathcal{E}))$, deformations of \mathcal{O}_X -mod \mathcal{E} .

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• $k=0$: $\text{Ext}^0(\mathcal{E}^n, \mathcal{E}^{n+1}) \cong \text{Hom}(\mathcal{E}^n, \mathcal{E}^{n+1}) \ni d^n, \forall n$
(\mathcal{Q} -closed!)

\hookrightarrow Leads to $\mathcal{Q} \mapsto \mathcal{Q}' = \mathcal{Q} + d$.

$$\begin{aligned} \hookrightarrow \{\mathcal{Q}', \mathcal{Q}'\} = 0 &\Leftrightarrow \underbrace{\{\mathcal{Q}, \mathcal{Q}\}}_{\mathcal{Q}^2=0} + 2 \underbrace{\{\mathcal{Q}, d\}}_{d \text{ is } \mathcal{Q}\text{-cl.}} + \{d, d\} = 0 \\ &\Leftrightarrow \underline{\underline{d^2 = 0}} \end{aligned}$$

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$$\begin{aligned} \text{↳ } \{\mathcal{Q}', \mathcal{Q}'\} = 0 &\iff \underbrace{\{\mathcal{Q}, \mathcal{Q}\}}_{\mathcal{Q}^2=0} + 2 \underbrace{\{\mathcal{Q}, d\}}_{d \text{ is } \mathcal{Q}\text{-cl.}} + \{d, d\} = 0 \\ &\iff \underline{d^2 = 0} \end{aligned}$$

$\Rightarrow k=0$ -deformations of our "naive" B-branes are objs. in $\mathcal{U}(\text{Coh}(X))$.

- ↳ This encompasses all other deformations [Asp].

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↑ "homotopy cat"
- Strings $(\mathcal{E}^\bullet, d_{\mathcal{E}}) \rightsquigarrow F \in \text{Coh}(X)$, ghost no. q :
- $d_{\mathcal{E}}$ turned off: $F \rightarrow \mathcal{I}^\bullet$ inj. res.,

$$\text{Ext}^{q+n}(\mathcal{E}^n, F) = H^{q+n}(\text{Hom}(\mathcal{E}^n, \mathcal{I}^\bullet), d_{\mathcal{I}}) = \left(\mathcal{R}^{q+n} \text{Hom}(\mathcal{E}^n, -) \right)(F)$$
↓ Q_0

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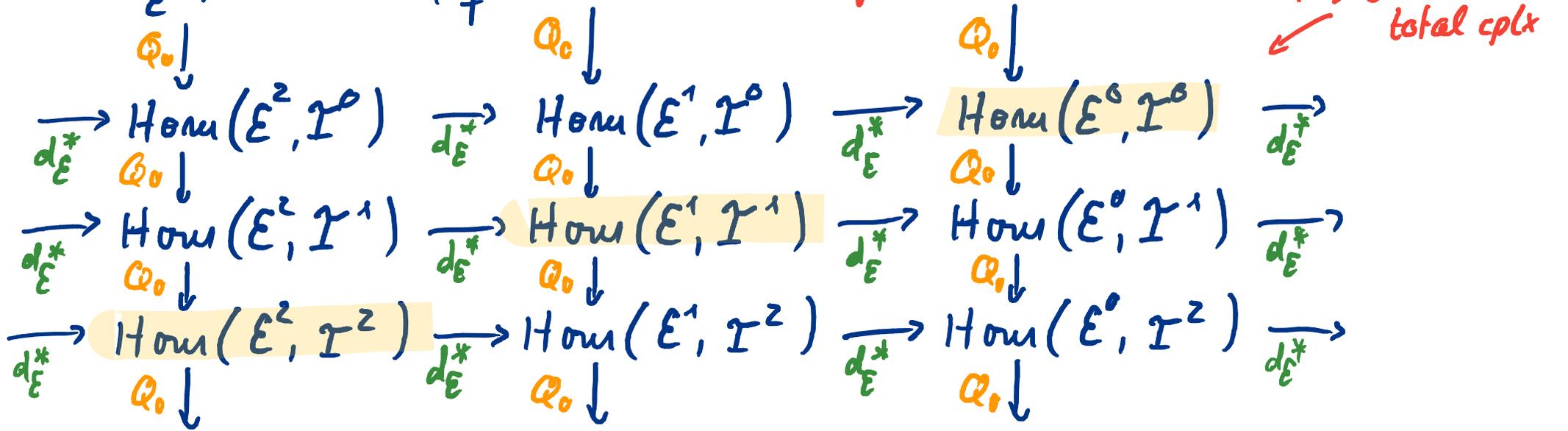
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 \uparrow "homotopy cat"

• Strings $(E^\bullet, d_E) \rightsquigarrow F \in \text{Coh}(X)$, ghost no. q :

• d_E turned off: $F \rightarrow \mathcal{I}^\bullet$ inj. res.,

$$\text{Ext}^{q+n}(E^n, F) = H^{q+n}(\text{Hom}(E^n, \mathcal{I}^\bullet), d_{\mathcal{I}}) = \left(\mathcal{R}^{q+n} \text{Hom}(E^n, -) \right)(F)$$

• d_E turned on, $q=0$: Need coho of $Q' = Q_0 + d_E^*$!

$$\begin{array}{ccccccc}
 & Q_0 \downarrow & & Q_0 \downarrow & & Q_0 \downarrow & \oplus_{r-s=0} \text{Hom}(E^s, \mathcal{I}^r) \\
 \xrightarrow{d_E^*} & \text{Hom}(E^2, \mathcal{I}^0) & \xrightarrow{d_E^*} & \text{Hom}(E^1, \mathcal{I}^0) & \xrightarrow{d_E^*} & \text{Hom}(E^0, \mathcal{I}^0) & \xrightarrow{d_E^*} \\
 & Q_0 \downarrow & & Q_0 \downarrow & & Q_0 \downarrow & \swarrow \text{total cplx} \\
 \xrightarrow{d_E^*} & \text{Hom}(E^2, \mathcal{I}^1) & \xrightarrow{d_E^*} & \text{Hom}(E^1, \mathcal{I}^1) & \xrightarrow{d_E^*} & \text{Hom}(E^0, \mathcal{I}^1) & \xrightarrow{d_E^*} \\
 & Q_0 \downarrow & & Q_0 \downarrow & & Q_0 \downarrow & \\
 \xrightarrow{d_E^*} & \text{Hom}(E^2, \mathcal{I}^2) & \xrightarrow{d_E^*} & \text{Hom}(E^1, \mathcal{I}^2) & \xrightarrow{d_E^*} & \text{Hom}(E^0, \mathcal{I}^2) & \xrightarrow{d_E^*} \\
 & Q_0 \downarrow & & Q_0 \downarrow & & Q_0 \downarrow &
 \end{array}$$

\Rightarrow Open \mathcal{B} -strings $E \rightarrow F$ of g.m. $q = \text{Hom}_{\mathcal{D}(\text{Coh}(X))}(E, F[q])$

• Proposal / Thm.: The cat of \mathcal{B} -branes on a CY mfd X is

$$\mathcal{B}\text{-Brane}(X) = \mathcal{D}^{(b)}(\text{Coh}(X)) = \mathcal{K}^{(b)}(\text{Coh}(X)) [q\text{-iso}^{-1}].$$

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- Physical interpretation of the triangulated structure:

- An SES $0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$ in $\text{Coh}(X)$ expresses F as a "sum" of E and $G \rightarrow F$ is a **bound state** of E, G .
- Last time: in $\mathcal{D}(\mathcal{A})$, SES generalise to dist. Δ / cones

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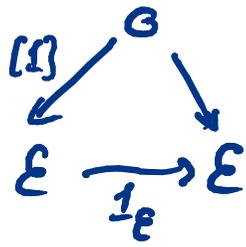
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• Remark: $\mathcal{K}(\text{Coh}(X))$ only keeps track of charges only, whereas dist. Δ in $\mathcal{D}(\text{Coh}(X))$ see the open strings f along which \mathcal{B} -branes bind \rightsquigarrow finer!

• $\forall E \in \mathcal{D}(X) \exists$



$\Leftrightarrow E$ binds with \emptyset to E .

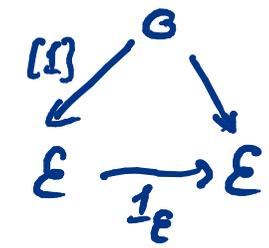
• $\forall E \in \mathcal{D}(X) \exists$

$$\begin{array}{ccc}
 & \mathcal{O} & \\
 \begin{array}{c} \{ \} \\ \downarrow \end{array} & & \downarrow \\
 E & \xrightarrow{1_E} & E
 \end{array}$$

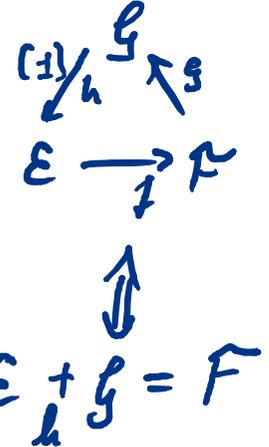
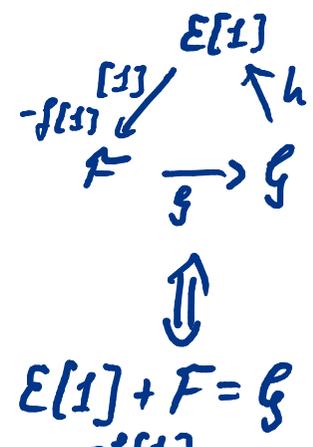
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\Leftrightarrow consistency, iso. objects rep. physically identical braes.

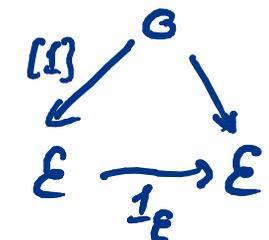
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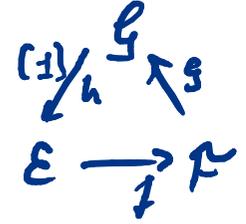
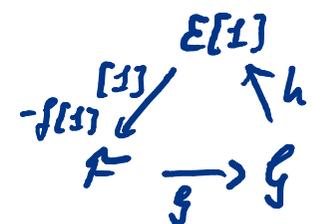
•  dist. iff 

"bind along h "

" $F - E = G$ " / decay

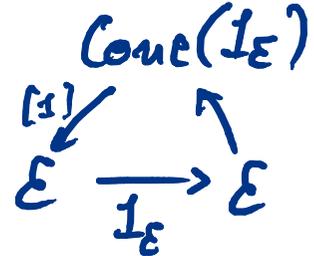
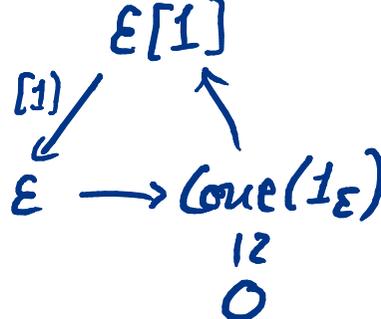
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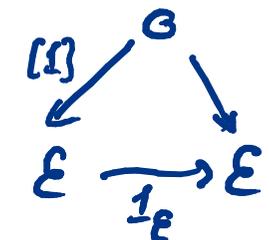
\updownarrow
 $E +_h G = F$
 "bind along h"

\updownarrow
 $E[1] + F = G$
 $-f[1]$
 "F - E = G" / decay

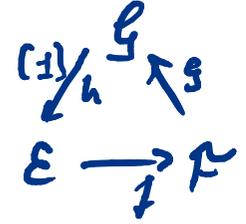
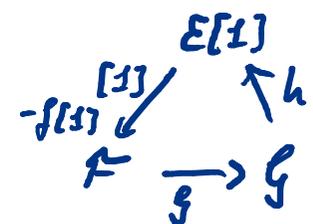
E.g.:  \Leftrightarrow 

$\Rightarrow E[1]$ is anti-brane for E ,
 under binding process along 1_E !

invisible in K-theory

• $\forall E \in \mathcal{D}(X) \exists$  $\Leftrightarrow E$ binds with \emptyset to E .

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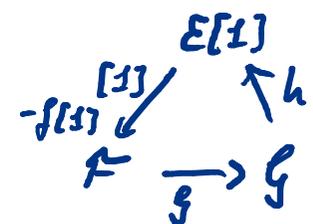
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$E \xrightarrow{[1]} F$ \xrightarrow{s} G \xleftarrow{h} E

$\Updownarrow h$

$E \xrightarrow{h} F = G$

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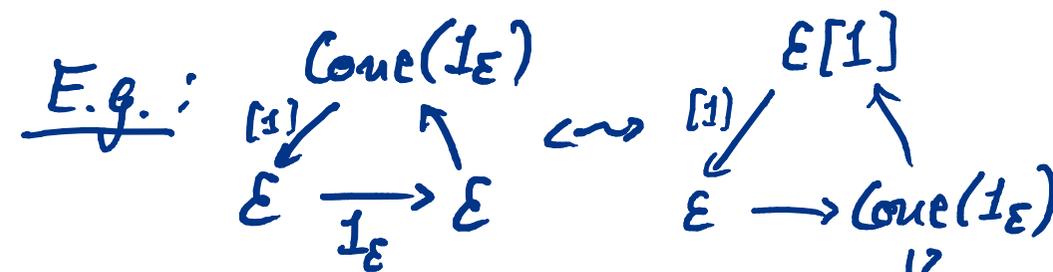
dist. iff 

$E \xrightarrow{[1]} G$ \xleftarrow{h} E \xrightarrow{s} G

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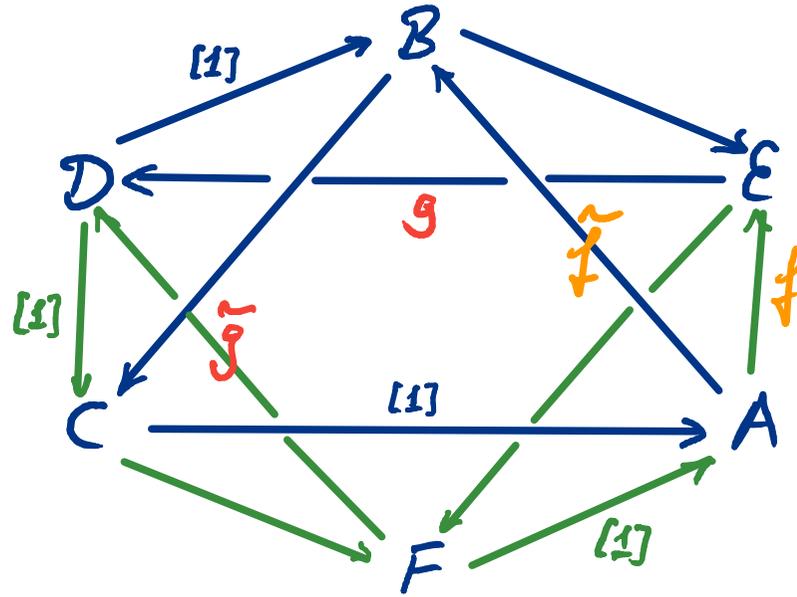
• Any cup. is part of a dist. $\Delta \Leftrightarrow$ branes can bind along any string between them.

$$\begin{array}{ccccccc}
 \bullet & \mathcal{E} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{E}[1] \\
 & \downarrow e & & \downarrow f & & \downarrow \exists g & & \downarrow e[1] \\
 & \mathcal{E}' & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{G}' & \longrightarrow & \mathcal{E}'[1]
 \end{array}$$

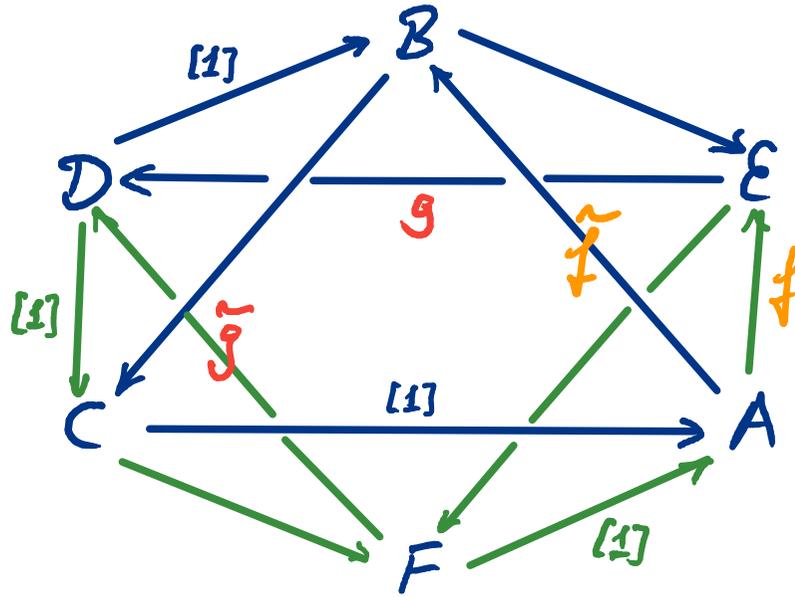
→ under rotation of triang. equiv to

$$\begin{array}{ccccccc}
 \mathcal{G} & \longrightarrow & \mathcal{E}[1] & \longrightarrow & \mathcal{F}[1] & \longrightarrow & \mathcal{G}[1] \\
 \downarrow g & & \downarrow e[1] & & \downarrow \exists f & & \downarrow g[1] \\
 \mathcal{G}' & \longrightarrow & \mathcal{E}'[1] & \longrightarrow & \mathcal{F}'[1] & \longrightarrow & \mathcal{G}'[1]
 \end{array} \iff$$

If there are strings constituents of two binding processes (compat. with the process), then there are strings between the bound-state branes.



Octahedral axiom \Leftrightarrow associativity of binding processes



Octahedral axiom \Leftrightarrow associativity of binding processes

E.g.: Bind A and D to form C (not simple composition)

$$\begin{aligned}
 C &= B \underset{\sim f}{+} A[1] \underset{R!}{=} (D[-1] \underset{g}{+} E) \underset{\sim f}{+} A[1] \\
 &= D[-1] \underset{\sim g}{+} (E \underset{f}{+} A[1]) = D[-1] \underset{\sim g}{+} F
 \end{aligned}$$

Thank you for your attention!

