Operads and their Algebras

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ZMP Seminar, Hamburg, 15/11/18

Contents

1 Introduction

1.1 Different realisations of algebraic structures

A unital algebra is a vector space $A \in \mathcal{V}\mathrm{ect}$, together with morphisms

$$\mathsf{e}_A \colon \mathbb{k} \to A\,, \qquad \mu_A \colon A \otimes A \to A\,,$$

specifying the unit element and the multiplication. The unitality and the associativity of these data are encoded in the commutativity of the diagrams

$$\begin{tikzcd}
\mathbb{k} \otimes A \arrow[r, "\mathsf{e}_A \otimes 1_A"] \arrow[dr] & A \otimes A \arrow[d, "\mu_A"] & A \otimes \mathbb{k} \arrow[l, "1_A \otimes \mathsf{e}_A"'] \arrow[dl] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mu_A \otimes 1_A"] \arrow[d, "1_A \otimes \mu_A"'] & A \otimes A \arrow[d, "\mu_A"] \\
A \otimes A \arrow[r, "\mu_A"'] & A
\end{tikzcd}$$

A unital topological monoid is a topological space X together with maps

$$\mathsf{e}_X \colon * \to X\,, \qquad \mu_X \colon X \times X \to X\,,$$

and the unitality and associativity conditions correspond to the exactly analogous diagrams as above, namely

$$\begin{tikzcd}
* \otimes X \arrow[r, "\mathsf{e}_X \otimes 1_X"] \arrow[dr] & X \otimes X \arrow[d, "\mu_X"] & X \otimes * \arrow[l, "1_X \otimes \mathsf{e}_X"'] \arrow[dl] \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \otimes X \otimes X \arrow[r, "\mu_X \otimes 1_X"] \arrow[d, "1_X \otimes \mu_X"'] & X \otimes X \arrow[d, "\mu_X"] \\
X \otimes X \arrow[r, "\mu_X"'] & X
\end{tikzcd}$$

with A replaced by X and $\Bbbk$ by $*$, as well as linear maps replaced by continuous maps. Similarly, a (unital) differential graded R-algebra is a chain complex $A_\bullet$ of R-modules together with chain maps and compatibilities of the above type, with $\Bbbk$ replaced by the complex $(R \to 0 \to 0 \to \ldots)$.

We can thus see already with these simple examples that the same algebraic structures implemented on different objects can yield structures of very different flavours. For example, a unital associative monoid in the category of sets is what is usually just called a 'unital associative monoid', but a unital associative monoid in the category of groups (or monoids) is an abelian group (monoid) by the Eckmann-Hilton argument!

Many categories have a notion of monoid internal to them. While the explicit forms of these monoids may be very different, their algebraic structure is yet the same – see the diagrams above! Thus, we should abstract the notion of an algebraic structure from its concrete realisations; this leads to the theory of operads.

1.2 Algebraic structures up to homotopy

Operads provide a canonical framework to manipulate algebraic structures abstractly. In particular, in environments where there is a notion of homotopy, or weak equivalence, the theory of operads provides means of replacing an algebraic structure, such as an associative product, by one that satisfies its axioms not strictly, but only up to coherent equivalence.

We will not discuss this formalism in general, but we will see the above example explicitly below: we will encounter operads encoding a strictly associative unital (binary) product, and we will encounter operads that encode a (binary) product that is unital and associative up to *coherent* homotopy.

1.3 Relations of different algebraic structures

One of the most important features of operads (certainly so for this seminar) is that they elegantly allow us to relate different algebraic structures, even ones in different categories (see next subsection). Some beautiful examples come from algebra and topology: we will see that if a topological space X carries an algebraic structure, taking the graded vector space $H_\bullet(X)$ of singular homology groups yields a graded vector space with certain other algebraic structure. The picture becomes even richer when we instead take the singular chain complex $C_\bullet(X)$ of X – here structures up to coherent homotopy will appear frequently.

1.4 Physics

In physics [BBZB+18], the crucial concept is that products of local operators in (topological) field theories vary in a certain way under varying the insertion points/cycles of the operators. In two dimensions, for instance there are continuous families of insertions that yield non-trivial 'monodromy'. This already yields a non-trivial family of products of two operators, which will of course induce even more non-trivial families of products of n operators.

The goal of the paper is to figure out how the *topological* means of combining insertion points manifest themselves on the *algebraic* side of the fields and observable algebras. We want to understand out how algebraic operations on configurations of (insertion) points, dictated by topology, yield algebraic operations on spaces/chain complexes of fields. We will show the basic of such a transfer in Theorem 3.3.

The following talks will work further towards understanding the conceptual ideas behind our guiding reference [BBZB+18]. In this talk, we will gather the mathematical tools to understand algebraic structures and their relations abstractly: we will introduce operads and their algebras.

2 Topological operads

We will start by considering operads in the category $\mathcal{T}\mathrm{op}_*$ of pointed topological (CGH) spaces, since this is where we have been lead to considering them in Walker's talk. We point out right here that operads make sense in any symmetric monoidal category $(\mathcal{V}, \otimes, \mathsf{u})$. For now, we fix

$$(\mathcal{V}, \otimes, \mathsf{u}) = (\mathcal{T}\mathrm{op}_*, \times, *)\,.$$

After seeing the definitions and the first examples, we will the use different choices for $\mathcal{V}$, such as vector spaces $\mathcal{V}\mathrm{ect}$, graded vector spaces $\mathcal{V}\mathrm{ect}_{\mathbb{Z}}$, or (co)chain complexes $\mathcal{C}\mathrm{h}_\bullet$ or $\mathcal{C}\mathrm{h}^\bullet$.

2.1 The definition of operads in $\mathcal{T}\mathrm{op}$

Definition 2.1 ([CG17, Definition A.3.1], [May72, Defintion 1.1]) *A* (symmetric) operad in $\mathcal{T}\mathrm{op}$ *consists of the following data:*

(1) a family $\{\mathcal{O}_n\}_{n\in\mathbb{N}_0}$ *of topological spaces of* n-ary operations*, with* $\mathcal{O}_0 = *$*,*

(2) a Σ_n*-action on* $\mathcal{O}_n$*,* permuting the n arguments of n-ary operations*, which we write as* $(\mu, \sigma) \mapsto \mu * \sigma$*,*

(3) for every $n \in \mathbb{N}$ *and* $j_1, \dots, j_n \in \mathbb{N}$ *a* multi-composition morphism

$$\mathsf{c}_{n;j_1,\dots,j_n} \colon \mathcal{O}_n \times \big(\mathcal{O}_{j_1} \times \cdots \times \mathcal{O}_{j_n}\big) \to \mathcal{O}_j\,,$$

where $j \coloneqq \sum_{i=1}^n j_i$*,*

(4) a unit element $\mathsf{e} \colon \mathcal{O}_0 \to \mathcal{O}_1$.

These data are subject to the following conditions:

(1) the composition morphisms are equivariant:

$$\begin{tikzcd}
\mathcal{O}_n \times \big(\mathcal{O}_{j_1} \times \cdots \times \mathcal{O}_{j_n}\big) \arrow[r, "(-*\sigma)\times\sigma_*^{-1}"] \arrow[d, "\mathsf{c}_{n;j_1,\dots,j_n}"'] & \mathcal{O}_n \times \big(\mathcal{O}_{j_{\sigma(1)}} \times \cdots \times \mathcal{O}_{j_{\sigma(n)}}\big) \arrow[d, "\mathsf{c}_{n;j_{\sigma(1)},\dots,j_{\sigma(n)}}"] \\
\mathcal{O}_j \arrow[r, "-*\sigma_{(j_1,\dots,j_n)}"'] & \mathcal{O}_j
\end{tikzcd}$$

Here, $\sigma_{(j_1,\dots,j_n)}$ *permutes blocks of lengths* $j_1, \dots, j_n$*. Moreover,*

$$\begin{tikzcd}
\mathcal{O}_n \times \big(\mathcal{O}_{j_1} \times \cdots \times \mathcal{O}_{j_n}\big) \arrow[r, "1\times(-*\tau_1)\times\cdots\times(-*\tau_n)"] \arrow[d, "\mathsf{c}_{n;j_1,\dots,j_n}"'] & \mathcal{O}_n \times \big(\mathcal{O}_{j_1} \times \cdots \times \mathcal{O}_{j_n}\big) \arrow[d, "\mathsf{c}_{n;j_1,\dots,j_n}"] \\
\mathcal{O}_j \arrow[r, "-*(\tau_1\oplus\cdots\oplus\tau_n)"'] & \mathcal{O}_j
\end{tikzcd}$$

(2) Compositions are associative:

$$
\begin{tikzcd}[column sep=small]
\mathcal{O}_n \times \bigtimes_{a=1}^{n} \Big(\mathcal{O}_{j_a} \times \bigtimes_{b=1}^{j_a} \mathcal{O}_{i_{a,b}} \Big) \arrow[rr, "\cong"] \arrow[d, "1 \times \bigtimes_{a=1}^{n} \mathsf{c}_{j_a,(i_{a,1}),\dots,(i_{a,j_a})}"'] & & \Big(\mathcal{O}_n \times \bigtimes_{a=1}^{n} \mathcal{O}_{j_a} \Big) \times \bigtimes_{a=1}^{n} \bigtimes_{b=1}^{j_a} \mathcal{O}_{i_{a,b}} \arrow[d, "\mathsf{c}_{n;j_1,\dots,j_n} \times 1"] \\
\mathcal{O}_n \times \bigtimes_{a=1}^{n} \mathcal{O}_{i_b} \arrow[dr, "\mathsf{c}_{n;i_1,\dots,i_n}"'] & & \mathcal{O}_j \times \bigtimes_{a=1}^{n} \bigtimes_{b=1}^{j_a} \mathcal{O}_{i_{a,b}} \arrow[dl, "\mathsf{c}_{j;(i_{1,1}),\dots,(i_{n,j_n})}"] \\
& \mathcal{O}_i &
\end{tikzcd}
$$

for all $n \in \mathbb{N}$, $j_1, \dots, j_n \in \mathbb{N}$, $i_{1,1}, \dots i_{n,j_n} \in \mathbb{N}$, with $j \coloneqq \sum_{a=1}^{n} j_a$, with $i_b \coloneqq \sum_{a=1}^{j_a} i_{a,b}$, and with $i \coloneqq \sum_{b=1}^{n} i_b$.

(3) Unitality:

$$
\begin{tikzcd}
\mathcal{O}_n \times \mathcal{O}_0^n \arrow[r, "1 \times \mathsf{e}^n"] \arrow[dr, "\cong"'] & \mathcal{O}_n \times \mathcal{O}_1^n \arrow[d, "\mathsf{c}_{n;1,\dots 1}"] \\
& \mathcal{O}_n
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{O}_0 \times \mathcal{O}_n \arrow[r, "\mathsf{e} \times 1"] \arrow[dr, "\cong"'] & \mathcal{O}_1 \times \mathcal{O}_n \arrow[d, "\mathsf{c}_{1;n}"] \\
& \mathcal{O}_n
\end{tikzcd}
$$

Remark 2.2 The first equivariance condition in an example: for $n = 2$, $j_1 = 3$ and $j_2 = 2$, and the non-trivial permutation $\sigma \in \Sigma_2$, we have

$$
\begin{tikzcd}
\Big((a,b,c,d,e) \mapsto \mu_2\big(\mu_3(a,b,c), \mu_2(d,e)\big) \Big) \arrow[r, mapsto] \arrow[d, mapsto] & \Big((a,b,c,d,e) \mapsto \mu_2\big(\sigma\big(\mu_2(a,b), \mu_3(c,d,e)\big)\big) \Big) \arrow[d, equal] \\
\Big((a,b,c,d,e) \mapsto \mu_2\big(\mu_3(c,d,e), \mu_2(a,b)\big) \Big) \arrow[r, equal] & \Big((a,b,c,d,e) \mapsto \mu_2\big(\mu_3(c,d,e), \mu_2(a,b)\big) \Big)
\end{tikzcd}
$$

Further, the asymmetry in the unit diagrams arises from the fact that we can operate with n units on the inputs of an n-ary operation, but we can only operate with one unit on the *one* output of an n-ary operation. $\triangleleft$

Definition 2.3 *A* morphism $f \colon \mathcal{O} \to \mathcal{P}$ of operads in $\mathcal{T}\mathrm{op}$ *is a family* $\{f_n \colon \mathcal{O}_n \to \mathcal{P}_n\}_{n \in \mathbb{N}_0}$ *of morphisms in* $\mathcal{T}\mathrm{op}$ *which intertwine the compositions and units.*

Example 2.4 A very simple example of an operad is

$$\mathcal{C}om_n \coloneqq * \qquad \forall\, n \in \mathbb{N}_0$$

with the trivial Σ_n-action. The unit and all composition morphisms are trivial. It encodes a single n-ary operation in every degree. Thus, by the composition maps, *any n-ary operation in $\mathcal{C}om$ can be built in a unique way from composing the binary operation*! That is, $\mathcal{C}om$ encodes the algebraic structure of *a single commutative and associative product.* Though simple, $\mathcal{C}om$ is a very important example, called the *commutative operad.*[1] $\triangleleft$

[1] This is the terminal object in the category of topological operads, which implies that an algebra over $\mathcal{C}om$ is canonically an algebra over any other operad.

Example 2.5 Another, more involved, operad is

$$\mathcal{A}ss_n \coloneqq \Sigma_n \qquad \forall\, n \in \mathbb{N}_0$$

with the Σ_n-action given by right multiplication. The compositions are given by the block inclusions $\Sigma_{j_1} \times \cdots \times \Sigma_{j_n} \hookrightarrow \Sigma_n$. Thus, we have two different binary operations that are related to each other by the flip $\sigma \in \Sigma_2$. Furthermore, any n-ary operation can be obtained by reordering from any other n-ary operation, and any given n-ary operation can again be written as compositions of one of the binaries. As there is no reordering in compositions, all of these compositions must result in the same n-ary operation – there aren't any more! – so that the binary products are, in particular, associative. Therefore, we see that $\mathcal{A}ss$ encodes the algebraic structure of *a single associative multiplication.* Correspondingly, this operad is called the *associative operad.* $\triangleleft$

Example 2.6 ([May72]) We already know an example of an operad in $\mathcal{T}\mathrm{op}$ from Walker's talk: we have

$$\mathcal{A}_{\infty,n} = \mathcal{E}_{1,n} \coloneqq \underline{\mathrm{Emb}}\Big(\bigsqcup_{i=1}^{n} \mathbb{D}^1,\, \mathbb{D}^1 \Big)\,,$$

where the space on the right-hand side is the space of embeddings of n intervals into one interval such that the n intervals have disjoint images and such that their images do not intersect the boundary $\{0,1\} \subset [0,1]$. The Σ_n-action permutes the 'little intervals'. The composition maps correspond to composing embeddings of discs.

Note that all connected components of $\mathcal{E}_{1,n}$ are contractible. As Walker has shown last time, this $\mathcal{A}_\infty$- *or* $\mathcal{E}_1$*-operad* encodes families of binary operations that are related to each other by either permuting arguments, or by continuous paths. Here, there are non-identical ways of composing the binary operations, but the resulting n-ary operations are again connected by continuous paths.

Observe that

$$\pi_0(\mathcal{E}_1) = \mathcal{A}ss\,,$$

where we take the connected components levelwise. This is one instance of the fact that $\mathcal{E}_1$ resolves the algebraic structure encoded by $\mathcal{A}ss$ by homotopical information. $\triangleleft$

2.2 Algebras over operads in $\mathcal{T}\mathrm{op}_*$

Our motivation for introducing operads above has been the idea that these should be objects that encode algebraic structures. Thus, consider an operad $\mathcal{O}$ in $\mathcal{T}\mathrm{op}_*$, and let us work out what it means for a topological space $X \in \mathcal{T}\mathrm{op}_*$ to carry the algebraic structure defined by $\mathcal{O}$.

We would like to understand the elements $\mu_n \in \mathcal{O}_n$ as n-ary operations on X, i.e. as maps $\chi_n(\mu_n)\colon X^n \to X$. Here, χ_n is tells us how the *abstract* datum of the n-ary operation $\mu_n \in \mathcal{O}_n$ is *realised on* X. In order to make this precise, we first consider the following important example:

Example 2.7 We would like to consider the collection of *all possible* n-ary operations on $X \in \mathcal{T}\mathrm{op}_*$. This is the space(!)

$$\mathcal{E}nd_n(X) \coloneqq \underline{\mathcal{T}\mathrm{op}_*}(X^n, X) \quad \forall\, n \geq 0 \qquad \in \mathcal{T}\mathrm{op}_*\,.$$

It carries a continuous right action of Σ_n by permuting the factors on the source side. The unit is given by $\mathsf{e}_X = 1_X$. Moreover, there are natural composition maps

$$\mathsf{c}_{n;j_1,\ldots,j_n}\colon \mathcal{E}nd_n(X) \times \Big(\bigtimes_{a=1}^{n} \mathcal{E}nd_{j_1}(X) \times \cdots \times \mathcal{E}nd_{j_n}(X) \Big) \longrightarrow \mathcal{E}nd_j(X)\,.$$

These data in fact define an operad, the so-called *endomorphism operad of X*. ◁

A realisation on X of the abstract operations encoded by $\mathcal{O}$ then just amounts to specifying which abstract operation should be realised as which actual operation on X:

Definition 2.8 *Let $\mathcal{O}$ be a topological operad. An* algebra over $\mathcal{O}$ *is a pair (X, χ) of a topological space $X \in \mathcal{T}\mathrm{op}_*$ and a morphism of topological operads*

$$\chi \colon \mathcal{O} \to \mathcal{E}nd(X)\,.$$

Remark 2.9 This procedure works whenever we have an operad defined in a symmetric monoidal category $\mathcal{V}$, and our algebra-to-be lives in a symmetric monoidal category $\mathcal{C}$ that is enriched over $\mathcal{V}$. Here we take $\mathcal{C} = \mathcal{V} = \mathcal{T}\mathrm{op}_*$, the category of compactly generated Hausdorff spaces. ◁

The information of $\{\chi_n \colon \mathcal{O}_n \to \underline{\mathcal{T}\mathrm{op}_*}(X^n X)\}_{n \in \mathbb{N}_0}$ is equivalent to specifying continuous maps

$$\chi_n^{\dashv} \colon \mathcal{O}_n \times X^n \to X\,, \qquad \chi_n^{\dashv}(\mu_n; x_1, \dots, x_n) \mapsto \chi_n(\mu_n)(x_1, \dots, x_n)\,.$$

The maps $\chi_n^{\dashv}$ then have to satisfy a translated version of the compatibility relations that the χ_n have to satisfy.

Example 2.10 Consider a morphism

$$\chi \colon \mathcal{A}ss \to \mathcal{E}nd(X)\,.$$

At level zero, we obtain nothing interesting:

$$* = \mathcal{A}ss_0 \xrightarrow{\chi_0} \underline{\mathcal{T}\mathrm{op}_*}(X^0 = *, X) \cong *\,.$$

At level one, we have an endomorphism

$$\mathcal{A}ss_1 = * \xrightarrow{\chi_1(*)} \underline{\mathcal{T}\mathrm{op}_*}(X, X)\,,$$

but the unitality *of the operad* implies that $\chi_1 = \chi_1 \circ \mathsf{e} = \mathsf{e}_X = 1_X$. This comes from the commutative diagram

$$\begin{tikzcd}
(\mathcal{O}_0 = *) \arrow[r, equal] \arrow[d, "\mathsf{e}"'] & (* = \mathcal{E}nd_0(X)) \arrow[d, "\mathsf{e}_X"] \\
\mathcal{O}_1 \arrow[r, "\chi_1"'] & \mathcal{E}nd_1(X)
\end{tikzcd}$$

that is part of the definition of a morphism of an operad.

At level two, we obtain *two* operations on X from

$$\chi_2 \colon \mathcal{A}ss_2 = \{(0,1), (1,0)\} \longrightarrow \underline{\mathcal{T}\mathrm{op}_*}(X^2, X)\,,$$

corresponding to products $(x_0, x_1) \mapsto \mu_X(x_0, x_1)$ and $(x_0, x_1) \mapsto \mu_X(x_1, x_0)$. From our arguments in Example 2.5 we know that this product will be associative and the only algebraic structure induced on X. Thus, $\mathcal{A}ss$-algebras in $\mathcal{T}\mathrm{op}_*$ are exactly unital topological monoids. ◁

Example 2.11 We have already seen another example of an algebra over an operad last time: the (based) loop space

$$\Omega X \coloneqq \underline{\mathcal{T}\mathrm{op}_*}(\mathbb{S}^1, X)$$

is an $\mathcal{A}_\infty$-algebra, or, equivalently, an $\mathcal{E}_1$-algebra. The maps

$$\chi_n^{\dashv}\colon \mathcal{E}_{1,n} \times (\Omega X)^n \longrightarrow \Omega X$$

takes an embedding of $([0,1]_1 \sqcup \ldots \sqcup [0,1]_n) \hookrightarrow [0,1]$ of n intervals with disjoint images (not hitting $\{0,1\} \subset [0,1]$), and an n-tuple of based loops $(\gamma_1, \ldots, \gamma_n)$ in X to the loop that is obtained by letting the image of $[0,1]_i$ parameterise γ_i, for $i = 1, \ldots, n$, and extending this to a map $\mathbb{S}^1 \to X$ by extending with the constant map to the basepoint (picture). Walker has shown us last time that $\mathcal{E}_1$ was built exactly such that this operation is associative up to *coherent(!)* homotopy. $\triangleleft$

3 Operads in other symmetric monoidal categories

We have already pointed out that the definition of an operad makes sense in *any* symmetric monoidal category $(\mathcal{V}, \otimes, \mathsf{u})$. A similar statement holds true for algebras over operads (cf. Remark 2.9). One replaces the product $\times$ by $\otimes$ and the point $*$ by the unit object u in Definition 2.1 and obtains the definition of an operad in a generic symmetric monoidal category $(\mathcal{V}, \otimes, \mathsf{u})$.

Remark 3.1 It is important that $\mathcal{V}$ is *symmetric* monoidal, because otherwise we would encounter things such as '$\mu_2(x_0, x_1) \neq \mu_2(\sigma^{-1} \circ \sigma(x_0, x_1))$'. $\triangleleft$

Similarly, if $\mathcal{C}$ is a symmetric monoidal category enriched in $\mathcal{V}$, then the (enriched) endomorphism operad of any object $X \in \mathcal{C}$ will be an operad in $\mathcal{V}$, and we can define $\mathcal{O}$-algebras in $\mathcal{C}$ for any operad $\mathcal{O}$ in $\mathcal{V}$: these simply consist of an object $X \in \mathcal{C}$ and a morphism

$$\chi\colon \mathcal{O} \to \mathcal{E}nd^{\mathcal{V}}(X)$$

of operads in $\mathcal{V}$.

This is possible, in particular, in the categories $s\mathcal{S}\mathrm{et}$ of simplicial sets, $\mathcal{V}\mathrm{ect}$ of vector spaces, $\mathcal{V}\mathrm{ect}_{\mathbb{Z}}$ of graded vector spaces, and $\mathcal{C}\mathrm{h}_\bullet$ or $\mathcal{C}\mathrm{h}^\bullet$ of (co)chain complexes (over any ring(?)). Note that these categories are all enriched over themselves.

3.1 Examples of operads and algebras

Example 3.2 We can straightforwardly define the associative and commutative operads in $\mathcal{V}\mathrm{ect}$, $\mathcal{V}\mathrm{ect}_{\mathbb{Z}}$, and $\mathcal{C}\mathrm{h}_\bullet$ or $\mathcal{C}\mathrm{h}^\bullet$ – we only need to remember that the monoidal unit is $\Bbbk$ in these cases (living in degree zero in the latter cases).

(1) The commutative operad in $\mathcal{V}\mathrm{ect}$ has

$$\mathcal{C}om_n^{\mathcal{V}\mathrm{ect}} := \Bbbk \qquad \forall\, n \in \mathbb{N}_0\,.$$

Compositions are given by multiplication.

(2) The associative operad in $\mathcal{V}\mathrm{ect}$ has

$$\mathcal{A}ss_n^{\mathcal{V}\mathrm{ect}} := \Bbbk[\Sigma_n] \qquad \forall\, n \in \mathbb{N}_0\,,$$

with the canonical Σ_n-action. Compositions are given by multilinear extension of the composition maps on $\mathcal{A}ss$.

(3) We can view these operads as operads in $\mathcal{V}\mathrm{ect}_{\mathbb{Z}}$ of $\mathcal{C}\mathrm{h}_\bullet$ or $\mathcal{C}\mathrm{h}^\bullet$ by putting the above spaces in degree zero, and letting all other levels of $\mathcal{O}_n$ be trivial (for $\mathcal{C}\mathrm{h}_\bullet$ or $\mathcal{C}\mathrm{h}^\bullet$, the differentials are trivial as well). $\triangleleft$

3.2 Relating structures: homology

The following principle will be crucial for the next (two) talks, and inspires much research in algebraic topology:

Theorem 3.3 ([Sin13, Proposition 6.2]) *The homology* $\mathrm{H}_\bullet(\mathcal{O};\Bbbk)$ *of any topological operad* $\mathcal{O}$ *is an operad in graded vector spaces (and hence also in* $\mathcal{C}\mathrm{h}_\bullet$ *or* $\mathcal{C}\mathrm{h}^\bullet$*). Further, for any algebra* X *over a topological operad* $\mathcal{O}$*, the homology* $\mathrm{H}_\bullet(X;\Bbbk)$ *is an algebra over the operad* $\mathrm{H}_\bullet(\mathcal{O};\Bbbk)$ *in* $\mathcal{V}\mathrm{ect}_{\mathbb{Z}}$*.*

Sketch of proof. This follows from applying the Künneth Theorem: it yields a *natural* morphism (in general not an isomorphism!) of graded vector spaces

$$\mathrm{H}_\bullet(X;\Bbbk)\otimes\mathrm{H}_\bullet(Y;\Bbbk)\to\mathrm{H}_\bullet(X\times Y;\Bbbk).$$

Applying this to the composition morphisms of an operad $\mathcal{O}$ in $\mathcal{T}\mathrm{op}_*$ then produces an operad structure on the graded vector spaces $\mathrm{H}_\bullet(\mathcal{O}_n;\Bbbk)$. The same strategy also yields the statement about algebras. □

Proposition 3.4 *(1) The homology (with coefficients in* $\Bbbk$*) of the topological commutative operad is the commutative operad:*

$$\mathrm{H}_\bullet(\mathcal{C}om;\Bbbk)=\mathcal{C}om^{\mathcal{V}\mathrm{ect}_{\mathbb{Z}}}\,.$$

(2) The homology of the topological associative operad is the associative operad:

$$\mathrm{H}_\bullet(\mathcal{A}ss;\Bbbk)=\mathcal{A}ss^{\mathcal{V}\mathrm{ect}_{\mathbb{Z}}}\,.$$

(3) The homology of the topological $\mathcal{A}_\infty=\mathcal{E}_1$ *operad is the associative operad*

$$\mathrm{H}_\bullet(\mathcal{E}_1;\Bbbk)=\mathcal{A}ss^{\mathcal{V}\mathrm{ect}_{\mathbb{Z}}}\,.$$

Sketch of proof. Claims (1) and (2) both follow from the fact that the level spaces of the respective operads are discrete. Statement (3) follows from the fact that $\mathcal{E}_{1,n}\cong\mathrm{Conf}_n([0,1])\simeq\Sigma_n$, so that $H_\bullet(\mathcal{E}_{1,n};\Bbbk)\cong\Bbbk[\Sigma_n]$ is concentrated in degree zero – here the Künneth morphisms from Theorem 3.3 are isomorphisms, and no mixing of degrees can possibly happen in the product maps. □

Remark 3.5 One can in fact take homology with arbitrary coefficients here, but $\Bbbk$ is the coefficient ring that will be of particular relevance in the following talks. ◁

These are just the easiest cases – much richer and more interesting examples will be presented to us next time by Vincentas. He will step by step add levels of commutativity to the operad $\mathcal{E}_1$ – the resulting operads are very important, and also encode the understanding of multiplying observables in [BBZB+18].

3.3 Generators and relations

In this section, we will outline how one can build operads in $\mathcal{V}\mathrm{ect}$ (and thus in $\mathcal{V}\mathrm{ect}_{\mathbb{Z}}$ and $\mathcal{C}\mathrm{h}_\bullet$ or $\mathcal{C}\mathrm{h}^\bullet$) by giving generators and imposing relations. In this way, one can start from algebraic structures and obtain operads that encode them. Here we closely follow [CV05, Section 2.1.5].

Remark 3.6 This does not just work in vector spaces – generalisations to symmetric monoidal categories that have certain colimits are possible. ◁

Definition 3.7 *Let $\mathcal{O}$ be an operad in $\mathcal{V}\mathrm{ect}$.*

(1) An ideal in $\mathcal{O}$ *is a family $\{\mathcal{I}_n \subset \mathcal{O}_n\}_{n\in\mathbb{N}_0}$ of Σ_n-invariant subspaces with the property that whenever at least on of the arguments of a composition $\mathsf{c}_{n;j_1,\ldots,j_n}(x_{1,1},\ldots,x_{n,j_n})$ is in $\mathcal{I}$, then so is the result, i.e. $\mathsf{c}_{n;j_1,\ldots,j_n}(x_{1,1},\ldots,x_{n,j_n}) \in \mathcal{I}_j$.*

(2) Given an ideal $\mathcal{I} \subset \mathcal{O}$, then the quotient operad *$\mathcal{O}/\mathcal{I}$ is the operad with n-th level $\mathcal{O}_n/\mathcal{I}_n$, an composition morphisms induced from those of $\mathcal{O}$.*

We first wish to define the *free operad* generated by a given collection of n-ary operations for any $n \in \mathbb{N}$. Note that, if we provide, for instance, a binary operation μ_2, an n-ary operation μ_n and an m-ary operation μ_m, we need to have an $(n{+}m)$-ary operation that should be $\mathsf{c}_{n+m;n,m}(\mu_2;\mu_n,\mu_m) = \mu_2 \circ (\mu_n,\mu_m)$. Moreover, in the absence of relations, for any n-ary operation, we obtain another, different, n-ary operation $\mu_n * \sigma$ for any permutation $\sigma \in \Sigma_n$.

Let $S \coloneqq \{S_n\}_{n\in\mathbb{N}_0}$ be a collection of sets, with $S_0 = *$. We need to extend this collection to an operad under the above modifications. For $n \in \mathbb{N}$, let Tree_n be the set of planar rooted (exactly one outgoing edge) trees with n leaves (incoming external edges) with a bijection of its leaves to $\underline{n} \coloneqq \{1,\ldots,n\}$. Note that all edges are directed towards the root, and that Tree_n carries a canonical Σ_n action that permutes the labels of the leaves.

For an (internal) vertex v of a tree T, let $in(v)$ denote the number of edges incoming to v. We think of a vertex v of a tree T as an $in(v)$-ary operation, and of any edge incoming to v as an insertion into that operation. Let $V(T)$ denote the set of all inner vertices of T; this is graded by the number $in(-)$ of incoming edges. Each such vertex v needs to be labelled by an $in(v)$-ary generator, i.e. an element from S_n. Thus, all possible n-ary operations generated by S form the set

$$\mathcal{F}_n(S) \coloneqq \coprod_{T\in\mathrm{Tree}_n} \mathcal{S}\mathrm{et}_{\mathbb{Z}}\big(V(T), S\big)\,,$$

where $\mathcal{S}\mathrm{et}_{\mathbb{Z}}$ denotes the category of graded sets. In order to obtain something in $\mathcal{V}\mathrm{ect}$, we set

$$\mathcal{F}_n^{\mathcal{V}\mathrm{ect}}(S) \coloneqq \bigoplus_{T\in\mathrm{Tree}_n} \Bbbk\big[\mathcal{S}\mathrm{et}_{\mathbb{Z}}\big(V(T), S\big)\big]\,.$$

Note that there always exists a distinguished planar rooted labelled tree $T_\emptyset$ with no vertex – it generates a copy of $\Bbbk$ in $\mathcal{F}_1^{\mathcal{V}\mathrm{ect}}(S)$.

Proposition 3.8 *Given a graded set S, the collection $\mathcal{F}^{\mathcal{V}\mathrm{ect}}(S)$, together with the element $T_\emptyset$ as the unit, the Σ_n-action of relabelling n labelled leaves, and grafting of trees as composition, defines an operad in $\mathcal{V}\mathrm{ect}$.*

Definition 3.9 *Let $\{R_n \subset S_n\}_{n\in\mathbb{N}}$ be a family of subsets. Let $\mathcal{I}(R) \subset \mathcal{F}^{\mathcal{V}\mathrm{ect}}(S)$ be the ideal generated by R under the Σ_n-actions and grafting. The operad $\mathcal{F}^{\mathcal{V}\mathrm{ect}}(S)/\mathcal{I}(R)$ is the* operad with generators S and relations R.

3.4 Further examples: $\mathcal{L}ie$, $\mathcal{P}ois_k$, $\mathcal{BV}$

We can now write down operads for loads of algebraic structures, basically by finding generators and relations for these structures. This may seem tautological at first, but in fact it is often desirable not to compare individual algebraic objects, but rather algebraic structures abstractly, i.e. on the level of operads – see for instance Theorem 3.3 and the upcoming talks.

Example 3.10 The associative operad $\mathcal{A}ss^{\mathcal{V}\mathrm{ect}}$ is generated by $S_2 = \{\mu_2\}$, $S_n = \emptyset$ for $n \neq 2, 0$, and has a single relation given by the associativity condition. (Note that we actually need associativity for all permutations of three arguments, but this is already guaranteed by dividing out the ideal generated by R). We can also view $\mathcal{A}ss^{\mathcal{V}\mathrm{ect}}$ as an operad in $\mathcal{V}\mathrm{ect}_{\mathbb{Z}}$, where its algebras are the graded $\Bbbk$-algebras, or we can view it in $\mathcal{C}\mathrm{h}_\bullet$ or $\mathcal{C}\mathrm{h}^\bullet$, where its algebras are the $\Bbbk$-dgas. Analogous statements hold true for $\mathcal{C}om^{\mathcal{V}\mathrm{ect}}$. ◁

Example 3.11 There is an operad $\mathcal{L}ie$ in $\mathcal{V}\mathrm{ect}$, whose algebras are exactly the Lie algebras. It is generated by a single binary operation $\mu_2 = [-,-]$ and the relations encoding antisymmetry and the Jacobi identity. Note that we can also view $\mathcal{L}ie$ as an operad in $\mathcal{C}\mathrm{h}_\bullet$ or $\mathcal{C}\mathrm{h}^\bullet$, where its algebras are dg Lie algebras. ◁

Example 3.12 Let $k \in \mathbb{N}$ (or even $\mathbb{Z}$, theoretically?). A *Poisson k-algebra* is a chain complex A, together with a commutative, associative product $- \cdot -\colon A \otimes A \to A$ of degree zero, and a Lie bracket $\{-,-\}\colon A \otimes A \to A$ of degree $1-k$; that is,

$$\{-,-\}_l \colon \bigoplus_{p+q=l} A_p \otimes A_q \to A_{l+1-k}$$

such that for any $a \in A$, the map $\{a,-\}$ is a derivation for $- \cdot -$.

These algebras are algebras over an operad in graded vector spaces (hence in $\mathcal{C}\mathrm{h}_\bullet$ or $\mathcal{C}\mathrm{h}^\bullet$) generated by $- \cdot -$ in $\mathcal{P}ois_{k,0}$ and $\{-,-\} \in \mathcal{P}ois_{k,1-k}$, under the relations indicated above.

Poisson 2-algebras are also called *Gerstenhaber algebras.* ◁

Example 3.13 (1) The smooth functions on a Poisson manifold, and in particular on any symplectic manifold, form a Poisson algebra concentrated in degree zero (which is, in this terminology, a Poisson 1-algebra).

(2) For any associative algebra A, its Hochschild cohomology $\mathrm{HH}^\bullet(A;A)$ with coefficients in itself, has the structure of a Gerstenhaber, or Poisson 2-algebra [LV12, Proposition 13.3.1].

◁

Theorem 3.14 (Hochschild-Kostant-Rosenberg-Theorem) *Let M be a manifold. There is an isomorphism of Gerstenhaber alegbras (or, equivalently, Poisson 2-algebras)*

$$\mathrm{HH}^\bullet\big(C^\infty(M,\mathbb{R}); C^\infty(M,\mathbb{R})\big) \cong \Gamma(M, \Lambda^\bullet TM)$$

of the Hochschild cohomology of the algebra of smooth functions on M and the Gerstenhaber algebra of smooth polyvector fields on M with the Schouten-Nijenhuis bracket. The latter is the extension of the Lie bracket to polyvector fields by imposing the Leibniz rule in each argument.

Note that the differential on the chain complexes on both sides are trivial here.

Example 3.15 A *BV algebra, or Batalin-Vilkovisky algebra* is a differential graded commutative algebra[2] A ($\mathcal{C}om^{\mathcal{V}\mathrm{ect}}$-algebra in $\mathcal{C}\mathrm{h}_\bullet$) together with a degree-1 morphism $\Delta\colon A \to A[1]$ satisfying

$$\begin{aligned} \Delta^2 &= 0\,, \\ \Delta(abc) &= \big(\Delta(ab)\big)c + (-1)^{|a|}\, a\,\Delta(bc) + (-1)^{(|a|-1)|b|}\, b\,\Delta(ac) \\ &\quad - (\Delta a)bc - (-1)^{|a|}\, a(\Delta b)c - (-1)^{|a|+|b|}\, ab(\Delta c) \qquad \forall\, a,b,c \in A\,. \end{aligned}$$

[2]There seems to be some ambiguity in the literature: sometimes one just uses a graded algebra, i.e. trivial differential.

Setting

$$[a,b] \coloneqq \Delta(ab) - (\Delta a)b - (-1)^{|a|}\, a(\Delta b) \qquad \forall\, a,b \in A\,,$$

we see that a BV algebra is equivalently Gerstenhaber algebra with a degree-1 derivation Δ for $[-,-]$ and where the bracket measures precisely the failure of Δ to be a derivation of the (graded!) commutative product. Note that a BV algebra is not the same as a dg Gerstenhaber algebra with differential Δ, because Δ is not a derivation for $- \cdot -$.

Again, this algebraic structure is described by an operad $\mathcal{BV}$ in $\mathcal{V}\mathrm{ect}_{\mathbb{Z}}$, obtained from either the operations $(-\cdot-, \Delta)$ and their relations, or the operations $(-\cdot-, \Delta, [-,-])$ and their relations. $\triangleleft$

Theorem 3.16 (e.g. [CG17, Section 4.1.1][3]) *Let M be an m-manifold with a volume form ω; this induces isomorphisms*

$$\iota_\omega \colon \Gamma(M, \Lambda^\bullet TM) \longrightarrow \Omega^{m-\bullet}(M)$$

of $C^\infty(M)$-modules. Then, the operation $\Delta \coloneqq \iota_\omega^{-1} \circ \mathrm{d} \circ \iota_\omega$ enhances the Poisson 2-algebra of polyvector fields on M to a BV algebra (with trivial differential).

Remark 3.17 In deformation quantization, Costello-Gwilliam use the Beilinson-Drinfeld operad $\mathcal{BD}$ which differs from the BV operad $\mathcal{BV}$. They view $\mathcal{BV}$ a quantum version of $\mathcal{BV}$ – it is an operad in dg modules over $\mathbb{R}[[\hbar]]$. For a brief outlook on the BV formalism in operad language and further literature in this direction, we refer to [LV12, CG17]. $\triangleleft$

Remark 3.18 The $\mathcal{BV}$ operad and its algebras appear frequently in theories related to two-dimensional conformal/topological field theory (string field theory, string topology, CFTs, TCFTs, ...). There, a topological operad of Riemann surfaces with n incoming punctures and one outgoing puncture plays a fundamental role. The reason that $\mathcal{BV}$ appears in such theories is that this Riemann surface operad is strongly related to a 2-dimensional version of the operad of little intervals, or 'one-disks', and this structure on the geometry induces a $\mathcal{BV}$ algebra structure on the fields, but I won't tell you how – that's the appetiser for Vincentas' talk in two weeks. $\triangleleft$

References

[BBZB+18] C. Beem, D. Ben-Zvi, M. Bullimore, T. Dimofte, and A. Neitzke. Secondary products in supersymmetric field theory. 2018. `arXiv:1809.00009`.

[CG17] K. Costello and O. Gwilliam. *Factorization algebras in quantum field theory. Vol. 1*, volume 31 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2017.

[CV05] R. L. Cohen and A. A. Voronov. Notes on string topology. In *String topology and cyclic homology. Proceedings, Summer School, Almeria, Spain, September, 16-20, 2003*, pages 1–95, 2005. `arXiv:math/0503625`.

[LV12] J.-L. Loday and B. Vallette. *Algebraic operads*, volume 346 of *Grundlehren der Mathematischen Wissenschaften*. Springer, Heidelberg, 2012.

[May72] J. P. May. *The geometry of iterated loop spaces*. Springer-Verlag, Berlin-New York, 1972. Lectures Notes in Mathematics, Vol. 271.

[Sin13] D. P. Sinha. The (non-equivariant) homology of the little disks operad. In *OPERADS 2009*, volume 26 of *Sémin. Congr.*, pages 253–279. Soc. Math. France, Paris, 2013. `arXiv:math/0610236`.

[3] This theorem was well-known long before that reference.