DIFFRACTIVE BEHAVIOR OF THE WAVE EQUATION IN PERIODIC MEDIA

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1. **Introduction and classical homogenization**
2. **WKB approximation**
3. **Diffractive behavior**
4. **Localization**

-I- INTRODUCTION

- ✍ We study wave propagation in a periodic medium.
- ✍ The small ratio between the period and a macroscopic characteristic lengthscale is denoted by ϵ.
- ✍ Many applications !
- ✍ We have in mind applications to photonic or phononic crystals.

The wave equation in a periodic medium

$$\begin{cases} \rho\left(x,\frac{x}{\epsilon}\right)\frac{\partial^2 u_\epsilon}{\partial t^2} - \operatorname{div}\left(A\left(x,\frac{x}{\epsilon}\right)\nabla u_\epsilon\right) = 0 & \text{in } \mathbb{R}^N \times \mathbb{R}^+ \\ \frac{\partial u_\epsilon}{\partial t}(t=0,x) = u^1_\epsilon(x) & \text{in } \mathbb{R}^N \\ u_\epsilon(t=0,x) = u^0_\epsilon(x) & \text{in } \mathbb{R}^N \end{cases}$$

- ☞ Scalar-valued unknown $u_\epsilon(t,x):\ \mathbb{R}^+ \times \mathbb{R}^N \ \rightarrow\ \mathbb{R}$
- ☞ $y \rightarrow A(x,y), \rho(x,y)$ $(0,1)^N$-periodic, real, measurable and bounded.
- ☞ $x \rightarrow A(x,y), \rho(x,y)$ smooth functions.
- ☞ A is a $N \times N$ symmetric, uniformly coercive, tensor, and $\rho(x,y) \geq \rho_0 > 0$.
- ☞ Initial data $u^0_\epsilon \in H^1(\mathbb{R}^N)$ and $u^1_\epsilon \in L^2(\mathbb{R}^N)$.

"Low frequency" homogenization

A "classical" result (see e.g. Brahim-Otsmane, Francfort and Murat 1992) that goes back to the beginning of homogenization (Babuska, Bakhvalov, Benssousan-Lions-Papanicolaou, Sanchez-Palencia...

Theorem. Denote by $\rho^*(x)$ the weak limit of $\rho\left(x, \frac{x}{\epsilon}\right)$.

If u_ϵ^0 converges weakly to u^0 in $H^1(\mathbb{R}^N)$ and $\rho_\epsilon u_\epsilon^1$ converges weakly to $\rho^* u^1$ in $L^2(\mathbb{R}^N)$, then u_ϵ converges weakly in the energy norm to the solution u of

$$\left\{\begin{array}{ll} \rho^* \dfrac{\partial^2 u}{\partial t^2} - \operatorname{div}\left(A^* \nabla u\right) = 0 & \text{in } \mathbb{R}^N \times \mathbb{R}^+ \\ \frac{\partial u}{\partial t}(t=0, x) = u^1(x) & \text{in } \mathbb{R}^N \\ u(t=0, x) = u^0(x) & \text{in } \mathbb{R}^N \end{array}\right.$$

where $A^*(x)$ is the "usual" elliptic homogenized matrix.

If the convergences of u_ϵ^0 and $\rho_\epsilon u_\epsilon^1$ are strong, so is that of the solution u_ϵ.

”Low frequency” homogenization (ctd.)

✗ Meaningfull result when the limits are non-zero (so-called low frequency initial data).

✗ True for example if

$$u_\epsilon^0(x) = u^0(x) \quad \text{and} \quad \rho\left(x, \frac{x}{\epsilon}\right) u_\epsilon^1(x) = \rho^*(x) u^1(x).$$

✗ What happens if, for example,

$$u_\epsilon^0(x) = \epsilon v^0\left(x, \frac{x}{\epsilon}\right) \text{ and } \rho\left(x, \frac{x}{\epsilon}\right) u_\epsilon^1(x) = v^1\left(x, \frac{x}{\epsilon}\right) \text{ with } \int v^1(x,y)\,dy = 0 \text{ ?}$$

In such a case, $u^0 = u^1 \equiv 0$ but

$$\lim_{\epsilon \to 0} E_\epsilon(t) \equiv \frac{1}{2} \int_{\mathbb{R}^N} \left(\rho^\epsilon \left(\frac{\partial u_\epsilon}{\partial t} \right)^2 + A^\epsilon \nabla u_\epsilon \cdot \nabla u_\epsilon \right) dx = \mathcal{O}(1) > 0.$$

Where does the (conserved) energy goes ?

-II- WKB APPROXIMATION

WKB method (Wentzel, Kramers, Brillouin) also known as geometric optics.

- ☞ WKB method works for smooth coefficients and short times.
- ☞ WKB method works also for Schrödinger equation (semi-classical limit).
- ☞ Huge literature: Bensoussan-Lions-Papanicolaou, Buslaev, Guillot-Ralston, Gérard-Martinez-Sjostrand, Gérard-Markowich-Mauser-Poupaud, Panati-Sohn-Teufel...
- ☞ Well prepared initial data:

$$u_\epsilon^0(x) = \epsilon\, e^{2i\pi \frac{S^0(x)}{\epsilon}} u^0\left(x, \frac{x}{\epsilon}\right) \quad \text{and} \quad u_\epsilon^1(x) = e^{2i\pi \frac{S^0(x)}{\epsilon}} u^1\left(x, \frac{x}{\epsilon}\right)$$

such that the energy is bounded.

- ☞ We still need to specify the oscillating behavior of u^0 and u^1: Bloch waves.

BLOCH WAVES

Bloch frequency (or quasi momentum) $\theta \in \mathbb{T}^N \equiv (0,1)^N$ (the unit torus).

The Bloch spectral problem is

$$-(\operatorname{div}_y + 2i\pi\theta)\Big(A(x,y)(\nabla_y + 2i\pi\theta)\psi\Big) = \lambda(x,\theta)\rho(x,y)\psi \quad \text{in } \mathbb{T}^N$$

A Bloch wave is $\phi(y) = \psi(\theta,y)e^{2i\pi\theta\cdot y}$ which satisfies

$$-\operatorname{div}_y\left(A(x,y)\nabla_y\phi\right) = \lambda(x,\theta)\rho(x,y)\phi \quad \text{in } \mathbb{R}^N.$$

Lemma. There exists an increasing sequence of real eigenvalues $(\lambda_n)_{n\geq 1}$ and normalized eigenfunctions $(\psi_n)_{n\geq 1}$ with $\int_{\mathbb{T}^N} \rho|\psi_n|^2 dy = 1$.

Assumption on the initial data

We choose n such that the eigenvalue $\lambda_n(x,\theta)$ is simple.

Replacing θ by ∇S^0, we consider Bloch wave packets

$$u_\epsilon^0(x) = \epsilon\, e^{2i\pi\frac{S^0(x)}{\epsilon}} \psi_n\left(x, \frac{x}{\epsilon}, \nabla S^0(x)\right) u^0(x)$$

and

$$u_\epsilon^1(x) = e^{2i\pi\frac{S^0(x)}{\epsilon}} \psi_n\left(x, \frac{x}{\epsilon}, \nabla S^0(x)\right) u^1(x)$$

where u^0 and u^1 are smooth envelope functions.

Notations. Define the operator

$$\mathcal{A}(x,\theta)\psi \equiv -(\operatorname{div}_y + 2i\pi\theta)\Big(A(x,y)(\nabla_y + 2i\pi\theta)\psi\Big)$$

Justification of using Bloch waves

Lemma. For any function $v(y) \in L^2(\mathbb{R}^N)$ there exist coefficients $\hat{v}_n(\theta)$ such that

$$v(y) = \sum_{n\geq 1} \int_{(0,1)^N} \hat{v}_n(\theta) e^{2i\pi\theta\cdot y} \psi_n(y,\theta)\, d\theta$$

and

$$\int_{\mathbb{R}^N} |v(y)|^2\, dy = \sum_{n\geq 1} \int_{(0,1)^N} |\hat{v}_n(\theta)|^2\, d\theta.$$

Remark. One can add $x \in \mathbb{R}^N$ as a parameter.

High frequency or WKB ansatz

We postulate

$$u_\epsilon(t,x) = \epsilon\, e^{2i\pi\frac{S(t,x)}{\epsilon}} \left(v\left(t,x,\frac{x}{\epsilon}\right) + \epsilon v_1\left(t,x,\frac{x}{\epsilon}\right) + \ldots \right)$$

The first derivatives are

$$\frac{\partial u_\epsilon}{\partial t} = e^{2i\pi\frac{S(t,x)}{\epsilon}} \left(2i\pi(v+\epsilon v_1)\frac{\partial S}{\partial t} + \epsilon\frac{\partial v}{\partial t} + \mathcal{O}(\epsilon^2) \right)$$

$$\nabla u_\epsilon = e^{2i\pi\frac{S(t,x)}{\epsilon}} \left(2i\pi(v+\epsilon v_1)\nabla S + \nabla_y v + \epsilon(\nabla_x v + \nabla_y v_1) + \mathcal{O}(\epsilon^2) \right)$$

Injecting them (as well as second order derivatives) in the wave equation we get a cascade of equations.

Cascade of equations

ϵ^{-1} order:

$$\mathcal{A}(x,\nabla S)v = 4\pi^2\rho\left(\frac{\partial S}{\partial t}\right)^2 v \quad \text{in} \quad \mathbb{T}^N.$$

This is an eigenvalue problem with respect to the y variable with fixed parameters (t,x). We deduce

$$4\pi^2\left(\frac{\partial S}{\partial t}\right)^2 = \lambda_n(x,\nabla S)$$

and by simplicity of the eigenvalue

$$v(t,x,y) = u(t,x)\psi_n(x,y,\nabla S(t,x))$$

We have thus obtained two eikonal equations (Hamilton-Jacobi) to determine the phase

$$2\pi\frac{\partial S}{\partial t} = \pm\sqrt{\lambda_n(x, \nabla S)} = \pm\omega_n(x, \nabla S)$$

with the initial data $S(0, x) = S^0(x)$.

By the method of characteristics we can solve the eikonal equation if we solve the Hamiltonian system

$$\begin{cases} \dot{x} = \nabla_\theta \omega_n(x, \theta) \\ \dot{\theta} = -\nabla_x \omega_n(x, \theta) \end{cases}$$

Unfortunately, the projection on the x-space of these bicharacteristics may intersect and produce caustics which means no smooth solutions for the eikonal equations. (This explains the restriction to small time.)

$\boxed{\epsilon^0 \text{ order:}}$

$$\mathcal{A}(x,\nabla S)v_1 = \lambda_n(x,\nabla S)\rho v_1 + f \quad \text{in} \quad \mathbb{T}^N,$$

with f depending on $u(t,x)$ (since $v = \psi_n u$)

$$f = -4i\pi\rho\frac{\partial S}{\partial t}\frac{\partial v}{\partial t} + \Big(\operatorname{div}_y + 2i\pi A\nabla S\Big)(A\nabla_x v) + \operatorname{div}_x\Big(A(\nabla_y + 2i\pi A\nabla S)v\Big).$$

A necessary condition for solving in v_1 is the Fredholm alternative

$$\int_{\mathbb{T}^N} f(t,x,y)\psi_n(y,\theta)\,dy = 0.$$

This yields an homogenized Liouville or transport equation for u.

Special case

We assume

- ☞ Purely periodic coefficients $y \to A(y), \rho(y)$ $(0,1)^N$-periodic.
- ☞ Monochromatic initial phase $S^0(x) = \theta \cdot x$.

The solution of the eikonal equation is global and linear

$$S^+(t,x) = \theta \cdot x + \sqrt{\lambda_n(\theta)}\, t \quad \text{and} \quad S^-(t,x) = \theta \cdot x - \sqrt{\lambda_n(\theta)}\, t$$

Furthermore, the homogenized transport equation is

$$\frac{\partial u^\pm}{\partial t} \pm \mathcal{V} \cdot \nabla_x u^\pm = 0$$

with the group velocity

$$\mathcal{V} = \frac{\nabla_\theta \lambda_n(\theta)}{4\pi\sqrt{\lambda_n(\theta)}} = \frac{1}{2\pi} \nabla_\theta \omega_n(\theta)$$

Conclusion

The phase is $S^{\pm}(t,x) = \theta \cdot x \pm \omega_n(\theta)$ and

$$\begin{aligned} u_\epsilon(t,x) \quad & \approx \epsilon\, e^{2i\pi \frac{S^+(t,x)}{\epsilon}} \psi_n\left(\frac{x}{\epsilon}, \nabla S^+(t,x)\right) u^+(t,x) \\ & + \epsilon\, e^{2i\pi \frac{S^-(t,x)}{\epsilon}} \psi_n\left(\frac{x}{\epsilon}, \nabla S^-(t,x)\right) u^-(t,x) \end{aligned}$$

with $u^{\pm}$ solutions of transport equations with group velocities $\pm\mathcal{V} = \pm\frac{1}{2\pi}\nabla_\theta \omega_n(\theta)$

$$u^{\pm}(t,x) = \frac{1}{2}\left(u^0 \pm \frac{1}{i\omega_n(\theta)} u^1\right)(x - \pm\mathcal{V}t)$$

We shall obtain a better asymptotic representation of $u^{\pm}$ for longer times of order ϵ^{-1}.

-III- DIFFRACTIVE BEHAVIOR

We change the time scale: the old time t is replaced by the new time t/ϵ.

We are thus looking at very long (old) time of order $1/\epsilon$.

New wave equation in a periodic medium:

$$\left\{\begin{array}{ll} \epsilon^2 \dfrac{\partial}{\partial t}\left(\rho_\epsilon \dfrac{\partial u_\epsilon}{\partial t}\right) - \operatorname{div}\left(A_\epsilon \nabla u_\epsilon\right) = 0 & \text{in } \mathbb{R}^N \times \mathbb{R}^+ \\ \frac{\partial u_\epsilon}{\partial t}(t=0,x) = u_\epsilon^1(x) & \text{in } \mathbb{R}^N \\ u_\epsilon(t=0,x) = u_\epsilon^0(x) & \text{in } \mathbb{R}^N \end{array}\right.$$

with weakly modulated coefficients

$$A_\epsilon(x) = A_0\left(\frac{x}{\epsilon}\right) + \epsilon^2 A_1\left(t, x, \frac{x}{\epsilon}\right), \quad \rho_\epsilon(x) = \rho_0\left(\frac{x}{\epsilon}\right) + \epsilon^2 \rho_1\left(t, x, \frac{x}{\epsilon}\right)$$

We consider a monochromatic wave packet as initial data

$$u_\epsilon^0(x) = \psi_n\left(\frac{x}{\epsilon}, \theta_0\right) e^{2i\pi\frac{\theta_0\cdot x}{\epsilon}} v_0(x) \quad \text{and} \quad u_\epsilon^1(x) = \frac{1}{\epsilon^2}\psi_n\left(\frac{x}{\epsilon}, \theta_0\right) e^{2i\pi\frac{\theta_0\cdot x}{\epsilon}} v_1(x)$$

with a Bloch wave $\psi_n(y, \theta_0)$ associated to a simple eigenvalue $\lambda_n(\theta_0)$.

Definitions.

$$\text{Frequency: } \omega_n(\theta_0) = \sqrt{\lambda_n(\theta_0)}.$$

$$\text{Group velocity: } \mathcal{V} = \frac{1}{2\pi}\nabla_\theta \omega_n(\theta_0) = \frac{1}{4\pi}\frac{1}{\sqrt{\lambda_n(\theta_0)}}\nabla_\theta \lambda_n(\theta_0)\,.$$

$$\text{Dispersion tensor: } A^* = \frac{1}{2\pi}\,\mathrm{div}_\theta \mathcal{V} = \frac{1}{4\pi^2}\nabla_\theta\nabla_\theta \omega_n(\theta_0)\,.$$

Assumptions

- the eigenvalue $\lambda_n(\theta_0)$ is simple (generic assumption)
- the modulated coefficients ρ_1 and A_1 are "invariant along group lines", i.e.,

$$\mathcal{V} \cdot \nabla_x \rho_1(t,x,y) = 0\,, \quad \mathcal{V} \cdot \nabla_x A_1(t,x,y) = 0.$$

(A weaker but more technical assumption is enough.)

Theorem (A.-Palombaro-Rauch). Under the above assumptions, the solution of the wave equation is given by

$$\begin{aligned} u_\epsilon(t,x) \;=\; e^{2i\pi\frac{\theta_0\cdot x}{\epsilon}}\psi_n\left(\frac{x}{\epsilon},\theta_0\right) & \left(e^{i\frac{\omega_n(\theta_0)t}{\epsilon^2}}v^+\left(t,x+\frac{\mathcal{V}}{\epsilon}t\right)\right. \\ & \left. +e^{-i\frac{\omega_n(\theta_0)t}{\epsilon^2}}v^-\left(t,x-\tfrac{\mathcal{V}}{\epsilon}t\right)\right)+r_\epsilon(t,x) \end{aligned}$$

with $\qquad \|r_\epsilon(t,x)\|_{L^\infty((0,T)\times\mathbb{R}^N))} \le C\epsilon\,,$

and $v^\pm \in C\left([0,T];L^2(\mathbb{R}^N)\right)$ is the solution of the homogenized problem

$$\begin{cases} \pm 2i\dfrac{\partial v^\pm}{\partial t} - \operatorname{div}\left(A^*\nabla v^\pm\right) + \gamma^* v^\pm = 0 & \text{in } \mathbb{R}^N\times(0,T), \\ v^\pm(t=0,x) = \dfrac{1}{2}\left(v_0(x) \pm \dfrac{1}{i\omega_n(\theta_0)}v_1(x)\right) & \text{in } \mathbb{R}^N, \end{cases}$$

with

$$\gamma^*(t,x) = \frac{1}{2\omega_n(\theta_0)}\int_{\mathbb{T}^N}\left(A_1(t,x,y)\nabla\psi_n(y)\cdot\nabla\overline{\psi}_n(y) - \lambda_n(\theta_0)\rho_1(t,x,y)|\psi_n(y)|^2\right)dy$$

REMARKS

- ☞ The fact that a wave equation yields a Schrödinger equation is well known (cf. paraxial approximation).
- ☞ In a constant homogeneous medium Bloch ≡ Fourier, the frequency is $\omega(\xi) = |\xi|$, the group velocity is $\mathcal{V} = \xi/|\xi|$ and the dispersion tensor is $A^* = (I - \mathcal{V} \otimes \mathcal{V})/|\xi|$ which is of rank $(N-1)$. In particular there is no dispersion in 1-d !
- ☞ On the contrary, A^* may have full rank N in a periodic medium.
- ☞ The homogenized coefficients do depend on the initial data ! It does not fit in the framework of G- or H-convergence.

Applications

- ☞ The group velocity may vanish: slow light !
- ☞ Dispersion effect could be important in photonic crystals and optic fibers.
- ☞ Some previous papers in physics: de Sterke and Sipe (Phys. Rev. A 1988), Sipe and Winful (Optics Let. 1988).

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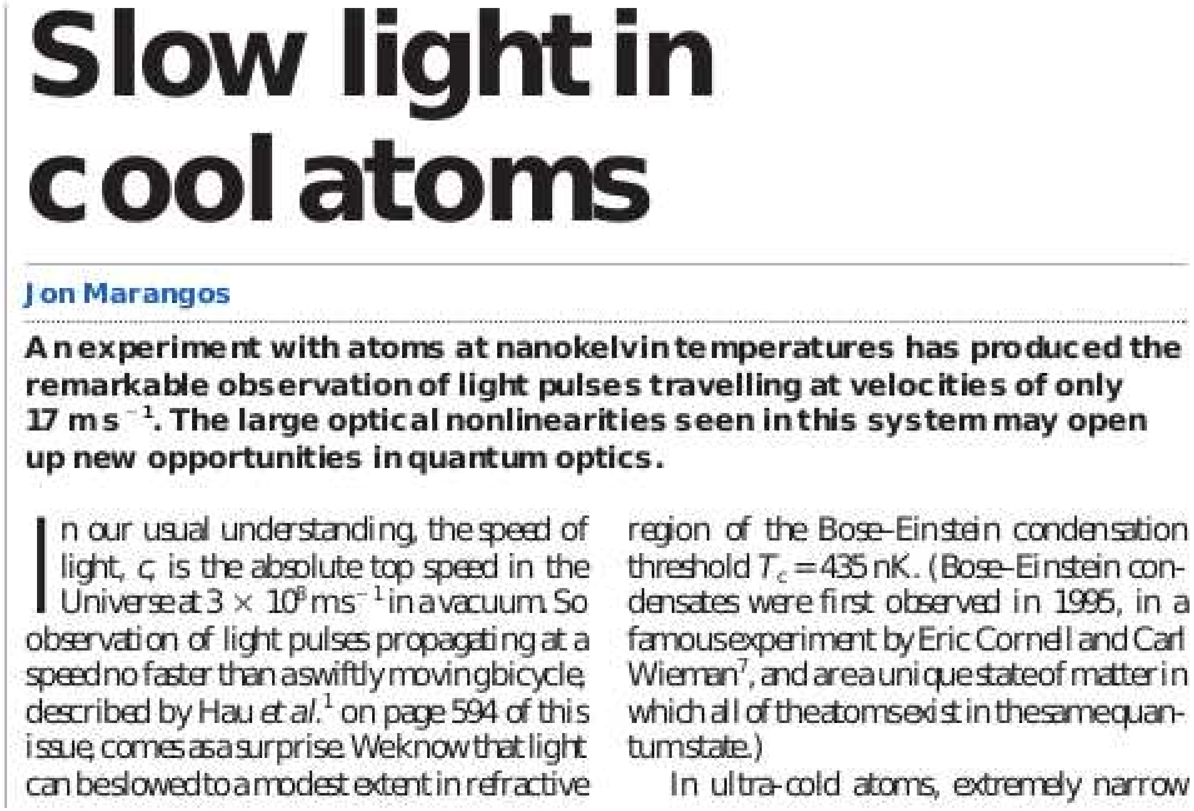

Slow light in cool atoms

Jon Marangos

An experiment with atoms at nanokelvin temperatures has produced the remarkable observation of light pulses travelling at velocities of only 17 m s^{-1}. The large optical nonlinearities seen in this system may open up new opportunities in quantum optics.

In our usual understanding, the speed of light, *c*, is the absolute top speed in the Universe at 3×10^8 m s^{-1} in a vacuum. So observation of light pulses propagating at a speed no faster than a swiftly moving bicycle, described by Hau *et al.*[1] on page 594 of this issue, comes as a surprise. We know that light can be slowed to a modest extent in refractive region of the Bose–Einstein condensation threshold $T_c = 435$ nK. (Bose–Einstein condensates were first observed in 1995, in a famous experiment by Eric Cornell and Carl Wieman[7], and are a unique state of matter in which all of the atoms exist in the same quantum state.)

In ultra-cold atoms, extremely narrow

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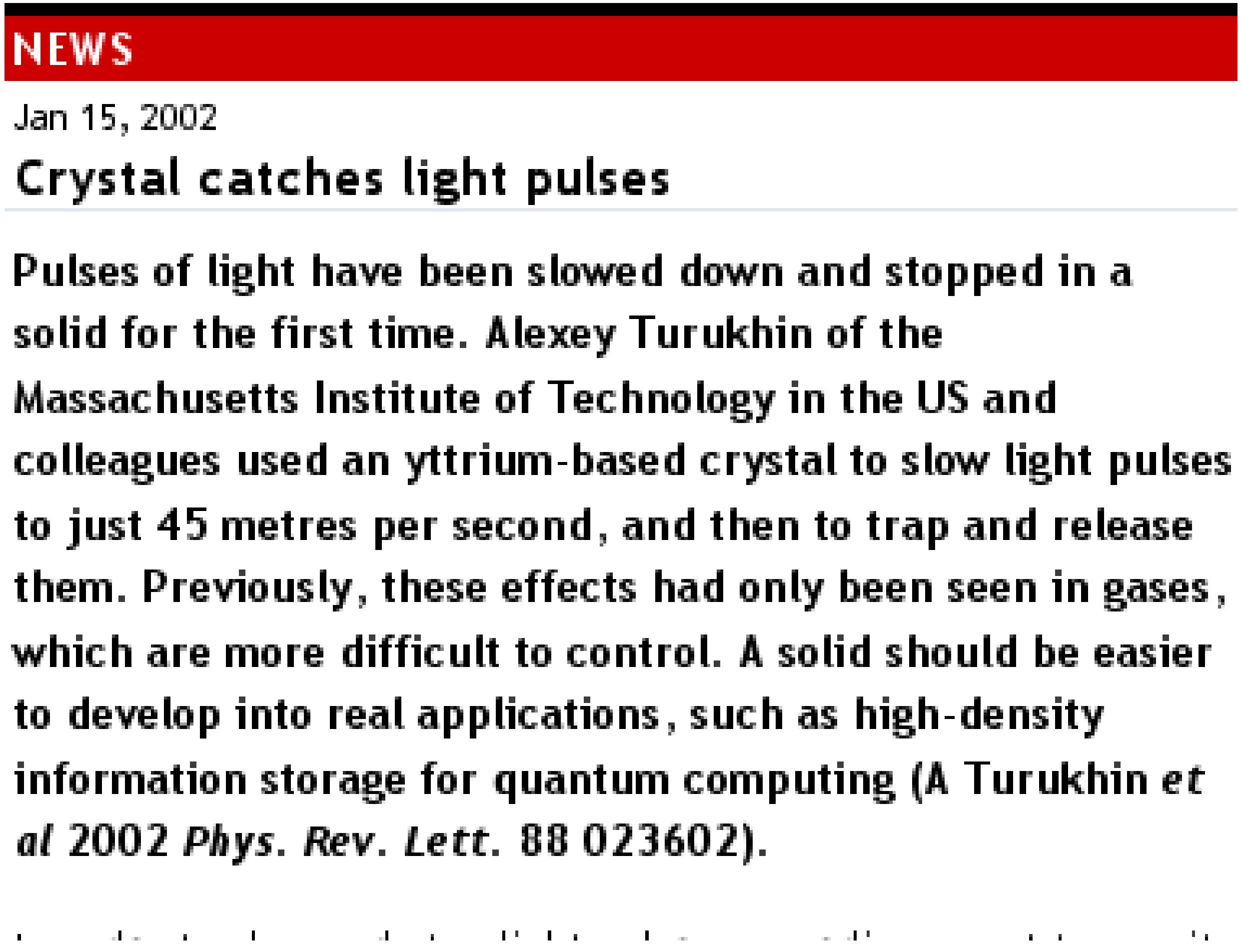
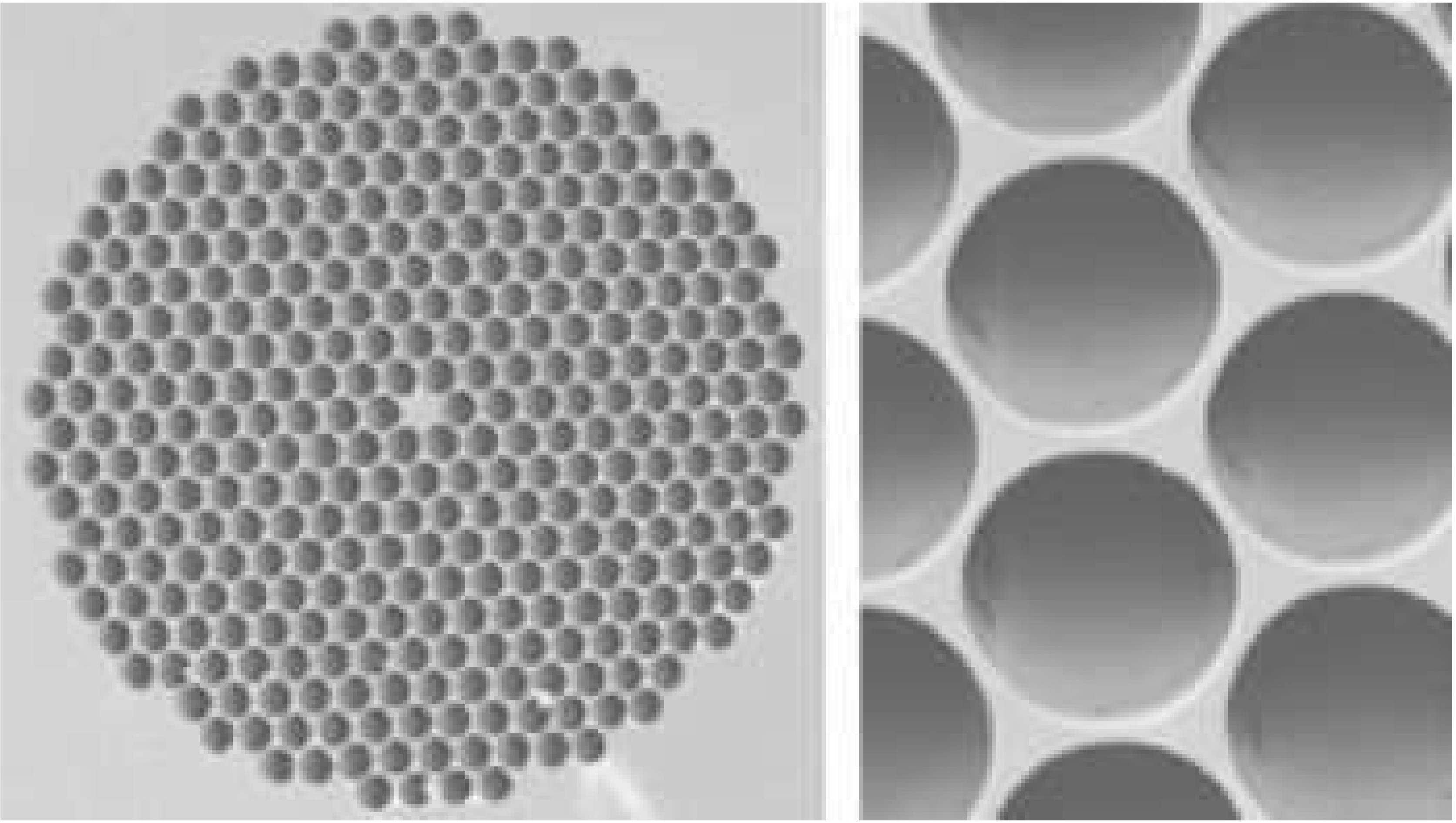


NEWS

Jan 15, 2002

Crystal catches light pulses

Pulses of light have been slowed down and stopped in a solid for the first time. Alexey Turukhin of the Massachusetts Institute of Technology in the US and colleagues used an yttrium-based crystal to slow light pulses to just 45 metres per second, and then to trap and release them. Previously, these effects had only been seen in gases, which are more difficult to control. A solid should be easier to develop into real applications, such as high-density information storage for quantum computing (A Turukhin *et al* 2002 *Phys. Rev. Lett.* 88 023602).



Constant coefficients case (Donnat, Rauch)

For simplicity consider the case of constant coefficients (Bloch waves become Fourier modes)

$$\left\{\begin{array}{ll} \epsilon^2 \dfrac{\partial^2 u_\epsilon}{\partial t^2} - c^2 \Delta u_\epsilon = 0 & \text{in } \mathbb{R}^N \times (0,T)\,, \\ u_\epsilon(0,x) = e^{i\frac{x\cdot\xi}{\epsilon}} v_0(x) & \text{in } \mathbb{R}^N\,, \\ \dfrac{\partial u_\epsilon}{\partial t}(0,x) = \epsilon^{-2} e^{i\frac{x\cdot\xi}{\epsilon}} v_1(x) & \text{in } \mathbb{R}^N\,, \end{array}\right.$$

where $c > 0$ is the speed of sound and $\xi \neq 0$ is a given frequency (we have dropped the 2π factor for simplicity).

We perform a WKB change of unknown and introduce a new function

$$u_\epsilon(t,x) = e^{i\left(\frac{x\cdot\xi}{\epsilon}+\frac{c|\xi|t}{\epsilon^2}\right)} v_\epsilon^+\left(t, x+\frac{c\xi t}{\epsilon|\xi|}\right).$$

We obtain a new form of the wave equation

$$2ic|\xi|\frac{\partial v_\epsilon^+}{\partial t} + \left(c\frac{\xi}{|\xi|}\cdot\nabla\right)^2 v_\epsilon^+ - c^2\Delta v_\epsilon^+ = -2c\epsilon\frac{\xi}{|\xi|}\cdot\nabla\frac{\partial v_\epsilon^+}{\partial t} - \epsilon^2\frac{\partial^2 v_\epsilon^+}{\partial t^2}.$$

Formally, passing to the limit yields a Schrödinger equation

$$2ic|\xi|\frac{\partial v^+}{\partial t} + \left(c\frac{\xi}{|\xi|}\cdot\nabla\right)^2 v^+ - c^2\Delta v^+ = 0.$$

Interpretation

Frequency $\omega(\xi) = c|\xi|$.

Group velocity $\mathcal{V} = \nabla_\xi \omega(\xi) = c\frac{\xi}{|\xi|}$.

Dispersion tensor $A^* = \nabla_\xi \nabla_\xi \omega(\xi) = \frac{c}{|\xi|}\left(I - \frac{\xi\otimes\xi}{|\xi|^2}\right)$.

Homogenized Schrödinger equation

$$\begin{cases} 2i\dfrac{\partial v^+}{\partial t} - \operatorname{div}\left(A^*\nabla v^+\right) = 0 & \text{in } \mathbb{R}^N \times (0,T)\,, \\ v^+(0,x) = \dfrac{1}{2}\left(v_0(x) + \dfrac{v_1(x)}{i\omega(\xi)}\right) & \text{in } \mathbb{R}^N\,. \end{cases}$$

In dimension $N = 1$, $A^* = 0$, which means no dispersion !

In general, no dispersion in the direction of propagation.

Another WKB change of unknowns is

$$u_\epsilon(t,x) = e^{i\left(\frac{x\cdot\xi}{\epsilon} - \frac{c|\xi|t}{\epsilon^2}\right)} v_\epsilon^- \left(t, x - \frac{c\xi t}{\epsilon|\xi|}\right),$$

which yields another limit Schrödinger equation

$$\begin{cases} -2ic|\xi|\dfrac{\partial v^-}{\partial t} + \left(c\dfrac{\xi}{|\xi|}\cdot\nabla\right)^2 v^- - c^2\Delta v^- = 0 & \text{in } \mathbb{R}^N \times (0,T)\,, \\ v^-(0,x) = \dfrac{1}{2}\left(v_0(x) - \dfrac{v_1(x)}{ic|\xi|}\right) & \text{in } \mathbb{R}^N\,. \end{cases}$$

Opposite frequency $\omega(\xi) = -c|\xi|$ and group velocity $\mathcal{V} = \nabla_\xi\omega(\xi) = -c\frac{\xi}{|\xi|}$, same dispersion tensor.

Theorem. (Donnat, Rauch)

Define an approximate solution

$$w_\epsilon(t,x) = e^{i\left(\frac{x\cdot\xi}{\epsilon}+\frac{c|\xi|t}{\epsilon^2}\right)} v^+\left(t, x+\frac{c\xi t}{\epsilon|\xi|}\right) + e^{i\left(\frac{x\cdot\xi}{\epsilon}-\frac{c|\xi|t}{\epsilon^2}\right)} v^-\left(t, x-\frac{c\xi t}{\epsilon|\xi|}\right).$$

It satisfies

$$\lim_{\epsilon\to 0} \|u_\epsilon - w_\epsilon\|_{L^\infty((0,T);L^2(\mathbb{R}^N))} = 0\,.$$

Back to the periodic case

Two methods of proof:

1. Infinite order asymptotic expansions.
2. Weak convergence analysis.

Weak convergence proof:

1. Prove a priori estimates.
2. Use oscillating test functions in the variational formulation.
3. Use a variant of two-scale convergence.
4. Deduce strong convergence from an energy convergence.

A priori estimates

The solution of

$$\begin{cases} \epsilon^2 \dfrac{\partial}{\partial t}\left(\rho_\epsilon \dfrac{\partial u_\epsilon}{\partial t}\right) - \operatorname{div}\left(A_\epsilon \nabla u_\epsilon\right) = 0 & \text{in } \mathbb{R}^N \times \mathbb{R}^+ \\ \frac{\partial u_\epsilon}{\partial t}(t=0,x) = \frac{1}{\epsilon^2}\psi_n\left(\frac{x}{\epsilon},\theta_0\right) e^{2i\pi\frac{\theta_0 \cdot x}{\epsilon}} v_1(x) & \text{in } \mathbb{R}^N \\ u_\epsilon(t=0,x) = \psi_n\left(\frac{x}{\epsilon},\theta_0\right) e^{2i\pi\frac{\theta_0 \cdot x}{\epsilon}} v_0(x) & \text{in } \mathbb{R}^N \end{cases}$$

satisfies

$$\epsilon \|\nabla u_\epsilon\|_{L^\infty((0,T);L^2(\mathbb{R}^N)^N)} + \epsilon^2 \left\| \frac{\partial u_\epsilon}{\partial t} \right\|_{L^\infty((0,T);L^2(\mathbb{R}^N))} \leq C\left(\|v^0\|_{H^1(\mathbb{R}^N)} + \|v^1\|_{L^2(\mathbb{R}^N)}\right),$$

and

$$\|u_\epsilon\|_{L^\infty((0,T);L^2(\mathbb{R}^N))} \leq C\left(\|v^0\|_{H^1(\mathbb{R}^N)} + \|v^1\|_{H^2(\mathbb{R}^N)}\right),$$

where $C(T) > 0$ is a constant which does not depend on ϵ.

Change of unknowns

$$v_\epsilon(t,x) := e^{-2i\pi\frac{\theta_0\cdot x}{\epsilon}} e^{-i\frac{\omega_n(\theta_0)t}{\epsilon^2}} u_\epsilon(t,x)$$

which satisfies

$$\left\{\begin{array}{ll} \epsilon^2\frac{\partial}{\partial t}\left(\rho_\epsilon\frac{\partial v_\epsilon}{\partial t}\right) + i\omega_n(\theta_0)\left(\rho_\epsilon\frac{\partial v_\epsilon}{\partial t} + \frac{\partial(\rho_\epsilon v_\epsilon)}{\partial t}\right) - \frac{\lambda_n(\theta_0)}{\epsilon^2}\rho_\epsilon v_\epsilon & \\ \quad - \left(\operatorname{div} + \frac{2i\pi\theta_0}{\epsilon}\right)\left(A_\epsilon\left(\nabla + \frac{2i\pi\theta_0}{\epsilon}\right)v_\epsilon\right) = 0 & \text{in } \mathbb{R}^N\times(0,T), \\ v_\epsilon(t=0,x) = \psi_n\left(\frac{x}{\epsilon},\theta_0\right)v_0(x) & \text{in } \mathbb{R}^N, \\ \frac{\partial v_\epsilon}{\partial t}(t=0,x) = \frac{1}{\epsilon^2}\psi_n\left(\frac{x}{\epsilon},\theta_0\right)\Big(v_1(x) - i\omega_n(\theta_0)v_0(x)\Big) & \text{in } \mathbb{R}^N. \end{array}\right.$$

Oscillating test function

To obtain the cell problem for ψ_n we use the test function

$$\epsilon^2 \phi \left(t, x + \frac{\mathcal{V}}{\epsilon} t, \frac{x}{\epsilon} \right)$$

To obtain the homogenized equation we use the test function

$$\Psi_\epsilon = \psi_n(\frac{x}{\epsilon}, \theta_0) \phi(t, x + \frac{\mathcal{V}}{\epsilon} t) + \frac{\epsilon}{2i\pi} \sum_{k=1}^{N} \frac{\partial \phi}{\partial x_k}(t, x + \frac{\mathcal{V}}{\epsilon} t) \frac{\partial \psi_n}{\partial \theta_k}(\frac{x}{\epsilon}, \theta_0)$$

Two-scale convergence with drift

Proposition (Marusic-Paloka, Piatnitski). Let $\mathcal{V} \in \mathbb{R}^N$ be a given drift velocity. Let u_ϵ be a bounded sequence in $L^2((0,T)\times\mathbb{R}^N)$. Up to a subsequence, there exist a limit $u_0(t,x,y) \in L^2((0,T)\times\mathbb{R}^N\times\mathbb{T}^N)$ such that u_ϵ *two-scale converges with drift* weakly to u_0 in the sense that

$$\lim_{\epsilon\to 0}\int_0^T\int_{\mathbb{R}^N} u_\epsilon(t,x)\phi\left(t, x+\frac{\mathcal{V}}{\epsilon}t, \frac{x}{\epsilon}\right)dt\,dx = \int_0^T\int_{\mathbb{R}^N}\int_{\mathbb{T}^N} u_0(t,x,y)\phi(t,x,y)\,dt\,dx\,dy$$

for all functions $\phi(t,x,y) \in L^2\left((0,T)\times\mathbb{R}^N; C(\mathbb{T}^N)\right)$.

-IV- LOCALIZATION (joint work with Luis Friz)

We come back to the semi-classical scaling and to **locally** periodic coefficients

$$\rho\left(x, \frac{x}{\epsilon}\right) \frac{\partial^2 u_\epsilon}{\partial t^2} - \operatorname{div}\left(A\left(x, \frac{x}{\epsilon}\right) \nabla u_\epsilon\right) = 0$$

Recall that the WKB method shows that the semi-classical limit is given by the dynamic of the effective Hamiltonian $(x, \theta) \in \mathbb{R}^N \times \mathbb{T}^N$

$$\begin{cases} \dot{x} = \nabla_\theta \omega_n(x, \theta) \\ \dot{\theta} = -\nabla_x \omega_n(x, \theta) \end{cases}$$

where $\lambda_n = \omega_n^2$ is an eigenvalue of the Bloch spectral problem

$$-(\operatorname{div}_y + 2i\pi\theta)\Big(A(x,y)(\nabla_y + 2i\pi\theta)\psi\Big) = \lambda(x,\theta)\rho(x,y)\psi \quad \text{in } \mathbb{T}^N$$

ASSUMPTIONS

We choose a point $(x^n, \theta^n) \in \mathbb{R}^N \times \mathbb{T}^N$ in the phase space such that

$$\lambda_n(x^n, \theta^n) \text{ is a simple eigenvalue, and } \nabla_\theta \lambda_n(x^n, \theta^n) = \nabla_x \lambda_n(x^n, \theta^n) = 0$$

We consider well-prepared initial data

$$u_\epsilon^0(x) = \psi_n\left(x^n, \frac{x}{\epsilon}, \theta^n\right) e^{2i\pi \frac{\theta^n \cdot x}{\epsilon}} v^0\left(\frac{x - x^n}{\sqrt{\epsilon}}\right)$$

$$u_\epsilon^1(x) = \frac{1}{\epsilon}\psi_n\left(x^n, \frac{x}{\epsilon}, \theta^n\right) e^{2i\pi \frac{\theta^n \cdot x}{\epsilon}} v^1\left(\frac{x - x^n}{\sqrt{\epsilon}}\right)$$

with $v^0 \in H^1(\mathbb{R}^N)$ and $v^1 \in H^2(\mathbb{R}^N)$ (degenerate case for WKB !)

Notations. New scale $z = \frac{x - x^n}{\sqrt{\epsilon}}$

Main result

Theorem (A.-Friz). Define the ansatz

$$w_\epsilon(t,x) = \psi_n\left(x^n, \frac{x}{\epsilon}, \theta^n\right) e^{2i\pi\frac{\theta^n\cdot x}{\epsilon}} \left(e^{i\frac{\omega_n t}{\epsilon}} v^+\left(t, \frac{x-x^n}{\sqrt{\epsilon}}\right) + e^{-i\frac{\omega_n t}{\epsilon}} v^-\left(t, \frac{x-x^n}{\sqrt{\epsilon}}\right)\right)$$

For any final time $T > 0$, it satisfies

$$\lim_{\epsilon\to 0} \frac{\|u_\epsilon(t,x) - w_\epsilon(t,x)\|_{L^2((0,T)\times\mathbb{R}^N)}}{\|w_\epsilon(t,x)\|_{L^2((0,T)\times R^N)}} = 0$$

when $v^\pm(t,z)$ are the solutions of the two homogenized Schrödinger equation

$$\left\{ \begin{array}{ll} \pm 2i\dfrac{\partial v^\pm}{\partial t} - \mathrm{div}(A^*\nabla v^\pm) + \mathrm{div}(v^\pm B^* z) + c^* v^\pm + v^\pm D^* z\cdot z = 0 & \text{in } \mathbb{R}^+\times\mathbb{R}^N \\ v^\pm(0,z) = \frac{1}{2}\Big(v^0(z) \pm \frac{1}{i\omega_n} v^1(z)\Big) & \text{in } \mathbb{R}^+ \end{array} \right.$$

The tensorial homogenized coefficients are the full Hessian of the time frequency

$$A^* = \frac{1}{8\pi^2}\nabla_\theta\nabla_\theta\omega_n(x^n,\theta^n),\ B^* = \frac{1}{2i\pi}\nabla_\theta\nabla_x\omega_n(x^n,\theta^n),\ D^* = \frac{1}{2}\nabla_x\nabla_x\omega_n(x^n,\theta^n),$$

Define

$$\nabla\nabla\omega_n = \left(\begin{array}{cc} \nabla_x\nabla_x\omega_n & \nabla_\theta\nabla_x\omega_n \\ \nabla_\theta\nabla_x\omega_n & \nabla_\theta\nabla_\theta\omega_n \end{array}\right)(x^n,\theta^n)\,.$$

Lemma. If the matrix $\nabla\nabla\omega_n$ is positive definite, then there exists an orthonormal basis $\{\varphi_k\}_{k\geq 1}$ of eigenfunctions of the two homogenized problems.

Moreover for each k there exists a real constant $\gamma_k > 0$ such that

$$e^{\gamma_k|z|}\varphi_k \, , \; e^{\gamma_k|z|}\nabla\varphi_k \in L^2(\mathbb{R}^N) \, .$$

This is localization ! (cf. Anderson in a stochastic framework)

Schrödinger case and references

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