

Regularity results for functionals with general growth

Lars Diening¹ Bianca Stroffolini² Anna Verde²

¹Universität Freiburg, Germany

²Università Federico II, Napoli

Oxford, PDE seminar, February 15, 2010

Outline

- 1 Introduction**
 - The Model
 - Main Theorem
- 2 Sketch of the proof**
 - Useful Ingredients
 - Uhlenbeck trick
 - Excess decay estimate
- 3 Further directions**
 - Asymptotically convex problems
 - Quasiconvex problems
 - Systems with critical growth

Our model case is:

$$I(u) = \int_{\Omega} \phi(|Du|) dx$$

- $\Omega \subset \mathbb{R}^n$ bounded open set, $u : \Omega \rightarrow \mathbb{R}^N$
- $\phi : [0, \infty) \rightarrow [0, \infty)$ convex function
- $\phi \in C^1([0, \infty) \cap C^2((0, \infty))$

A function $u \in W_{loc}^{1,\phi}(\Omega, \mathbb{R}^N)$ is a local minimizer for I if

$$I(u, \text{spt}\eta) \leq I(u + \eta, \text{spt}\eta) \quad \forall \eta \in C_0^1(\Omega, \mathbb{R}^N).$$

Orlicz-Sobolev

- $L^\phi : f \in L^\phi$ iff there exists $K > 0$ such that $\int \phi(\frac{|f|}{K}) dx < \infty$
- $W^{1,\phi} : f \in W^{1,\phi}$ iff $f, Df \in L^\phi$.

Framework

A fundamental problem in the Calculus of Variations is the one of identifying classes of functionals for which everywhere $C^{1,\alpha}$ -regularity or even just continuity of minimizers occurs

A well known result of [K. Uhlenbeck](#) (77) states that the $C^{1,\alpha}$ -regularity holds for local minimizers if the integrand function is of the type

$$f(|z|)$$

for a convex function f of p -growth, with $p \geq 2$.

Question

What are suitable assumptions on ϕ that guarantee everywhere $C^{1,\alpha}$ -regularity for local minimizers?

Known results

- Uhlenbeck '77
- Giaquinta-Modica '86
- Acerbi-Fusco '89 $1 < p < 2$
- Marcellini '89-'96 general growth
- Lieberman : scalar case '91 ; vectorial case '93
- Mingione-Siepe '99
- Esposito-Mingione '00 nearly linear growth
- Fuchs-Mingione '00
- Marcellini-Papi '06

Euler system

$$u \in W^{1,\infty}, A \in C^1 \implies u \in C^{1,\alpha}$$

without excess decay estimate!

ϕ N-function

- $\phi(0) = 0$
- ϕ' right continuous, non-decreasing
- $\phi'(0) = 0$, $\phi'(t) > 0$ for $t > 0$, and $\lim_{t \rightarrow \infty} \phi'(t) = \infty$.
- $\phi \in C^1([0, \infty)) \cap C^2((0, \infty))$
- H1. $\phi'(t) \sim t\phi''(t)$ uniformly in $t > 0$
- H2. Hölder continuity for ϕ''

$$|\phi''(s+t) - \phi''(t)| \leq c\phi''(t) \left(\frac{|s|}{t}\right)^\beta \quad \beta > 0$$

for all $t > 0$ and $s \in \mathbb{R}$ with $|s| < \frac{1}{2}t$.

- $H1. \implies \Delta_2(\phi, \phi^*) < \infty$ (ϕ^* is the conjugate)

Δ_2 condition

$$\phi \in \Delta_2 \Leftrightarrow \exists c_1 > 0 : \phi(2t) \leq c_1 \phi(t)$$

Examples

- $\phi(t) = t^p \quad \forall p > 1$
- $\phi(t) = t^p \log^\alpha(e + t)$
- $\phi(t) = t^p \log \log(e + t)$

Main Theorem

Let $u \in W_{loc}^{1,\phi}(\Omega, \mathbb{R}^n)$ local minimizer for

$$\int_{\Omega} \phi(|Du|) dx$$

with ϕ like before



“excess decay estimate”

$$\int_{B_{\rho}} |V(Du) - (V(Du))_{\rho}|^2 \leq c \left(\frac{\rho}{R}\right)^{\alpha} \int_{B_R} |V(Du) - (V(Du))_R|^2 \quad \forall \rho < R$$



Du locally Hölder continuous

$$A(Q) = \phi'(|Q|) \frac{Q}{|Q|} \quad V(Q) = \psi'(|Q|) \frac{Q}{|Q|} \quad \psi'(t) = \sqrt{\phi'(t)t}$$

The nonlinearity of the problem is inserted in V !

$$A(Q) \cdot Q \sim |V(Q)|^2 \sim \phi(|Q|)$$

$$(A(P) - A(Q)) \cdot (P - Q) \sim |V(P) - V(Q)|^2$$

uniformly in $P, Q \in \mathbb{R}^{N \times n}$.

Principal steps

- 1. **Poincaré** and **Caccioppoli** in the Orlicz-Sobolev setting;



Gehring – type result

- 2. **Bernstein-Uhlenbeck** trick: $\phi(|Du|)$ is a subsolution of a uniformly elliptic equation



weak Harnack inequality

- 3. **Excess decay estimate**



conclusion using the integral characterization of **Campanato** spaces.

Acerbi-Fusco type lemma

$$\int_0^1 \frac{\phi'(|P_\theta|)}{|P_\theta|} d\theta \sim \frac{\phi'(|P_0| + |P_1|)}{|P_0| + |P_1|} \quad |P_0| + |P_1| > 0$$

where $P_\theta = (1 - \theta)P_0 + \theta P_1$.

$$\begin{aligned} (A(P) - A(Q)) \cdot (P - Q) &\sim |V(P) - V(Q)|^2 \\ &\sim |P - Q|^2 \phi''(|P| + |Q|) \end{aligned}$$

A key ingredient is the **shifted function** ϕ_λ defined for $\lambda > 0$ by:

$$\phi'_\lambda(t) = \frac{\phi'(\lambda + t)t}{\lambda + t}$$



ϕ_λ inherits all properties of ϕ uniformly in λ

Higher integrability

Caccioppoli: $u \in W_{loc}^{1,\phi}(\Omega, \mathbb{R}^n)$ local minimizer then $\forall B$ with $2B \subset \Omega$, $R = \text{diam}(B)$

$$\int_B \phi(|Du|) dx \leq c \int_{2B} \phi\left(\frac{|u - u_{2B}|}{R}\right) dx$$

plus

Poincaré: $\exists 0 < \theta < 1$, $K = K(\Delta_2, R)$ such that

$$\int_B \phi\left(\frac{|v - v_{2B}|}{R}\right) dx \leq K \left(\int_B \phi(|Dv|)^\theta dx \right)^{\frac{1}{\theta}}$$

Reverse Hölder: $\exists q_1 > 1$ such that $\forall q \in [1, q_1]$

$$\left(\int_B |V(Du)|^{2q} dx \right)^{\frac{1}{2q}} \leq c \left(\int_{2B} |V(Du)|^2 dx \right)^{\frac{1}{2}}$$

and

Reverse Hölder for the oscillation:

$$\int_B |V(Du) - V(Q)|^2 dx \leq c \left(\int_{2B} |V(Du) - V(Q)|^{2\theta} dx \right)^{\frac{1}{2\theta}}$$

Using difference quotient technique: $V(Du) \in W^{1,2}$ and

$$\int_B |DV(Du)|^2 dx \leq \frac{c}{R^2} \int_{2B} |V(Du)|^2 dx$$

Marcellini and Papi proved the boundedness of the gradient of a minimizer of an integral functional of the type

$$\int_{\Omega} g(|Dv|) dx$$

where $g \in W^{2,\infty}[0, T]$, $\forall T > 0$, is a convex function s.t. $g(0) = g'(0) = 0$ and satisfies the following:

- H1'. Let $t_0, H > 0$ and $\beta \in (\frac{1}{n}, \frac{2}{n})$. For every $\alpha \in (1, \frac{n}{n-1}]$ there exists $K = K(\alpha)$ such that

$$Ht^{-2\beta} \left[\left(\frac{g'(t)}{t} \right)^{\frac{n-2}{n}} + \left(\frac{g'(t)}{t} \right) \right] \leq g''(t) \leq K \left[\left(\frac{g'(t)}{t} \right)^{\alpha} + \left(\frac{g'(t)}{t} \right) \right]$$

$\forall t \geq t_0$.

Remark

$$H1 \implies H1' \implies Du \in L_{loc}^{\infty}.$$

Uhlenbeck trick

We use the approximated functionals $F_\lambda = \int_\Omega \phi_\lambda(|Dv|) dx$

⇓

$\phi_\lambda(|Du_\lambda|)$ subsolution of a problem whose coefficients satisfy

$$c_0 |\xi|^2 \leq \sum_{kl} G_\lambda^{kl}(Q) \xi_k \xi_l \leq c_1 |\xi|^2$$

⇓

$$\sup_B \phi_\lambda(|Du_\lambda|) \leq c \int_{2B} \phi_\lambda(|Du_\lambda|) \quad (\text{DeGiorgi – Nash – Moser})$$

⇓

$$\sup_B \phi(|Du|) \leq c \int_{2B} \phi(|Du|) \quad \left((\lambda, Q) \rightarrow V_\lambda^{-1}(Q) \text{ is continuous} \right)$$

⇓

$$\int_{\frac{1}{2}B} |V(Du) - \langle V(Du) \rangle_{\frac{1}{2}B}|^2 \leq c \left(\sup_B \phi(|Du|) - \sup_{\frac{1}{2}B} \phi(|Du|) \right)$$

Excess decay estimate

Using the Hölder continuity of ϕ'' we get:

$\forall \tau \in (0, 1) \exists \varepsilon_0(\tau) \in (0, 1)$ such that

$$\Phi(u, R) \leq \varepsilon_0 \sup_{B_{R/2}} \phi(|Du|) \implies \Phi(u, \tau R) \leq c\tau^2 \Phi(u, R)$$

$$\Phi(u, R) = \int_{B_R} |V(Du) - (V(Du))_R|^2 dx$$

How to remove the “**smallness**” assumption?

We prove an alternative using the weak **Harnack** inequality.

- Using a standard **iteration** technique, we prove that
 $\exists \alpha > 0 : \forall B \subset \Omega \quad \Phi(u, \rho) \leq c\left(\frac{\rho}{R}\right)^\alpha \Phi(u, R) \quad \forall \rho < R$
- From **Campanato** characterization of Hölder continuous functions, we get $V(Du)$ locally Hölder continuous;
- Using that V^{-1} Hölder continuous, we conclude:
 Du locally Hölder continuous.

- Asymptotically convex problems
- Quasiconvex problems
- ϕ -systems with critical growth

Given $H(\xi) = (1 + |\xi|^2)^{\frac{p}{2}}$ we say that f is C^2 asymptotically convex if

$$\forall \epsilon > 0, \exists \gamma_\epsilon > 0 : \left| \frac{\partial^2 f}{\partial \xi^2}(\xi) - \frac{\partial^2 H}{\partial \xi^2}(\xi) \right| \leq \epsilon |\xi|^{p-2}$$

whenever $|\xi| > \gamma_\epsilon$.

Question

Which kind of regularity we can expect for local minimizers of

$$I(u) = \int_{\Omega} f(Du) dx?$$

Local Lipschitz regularity

- Chipot-Evans '86 $p = 2$
- Giaquinta-Modica '86 $p \geq 2$
- Leone-Passarelli di Napoli-V. '07 $1 < p < 2$
- Raymond '91, Kristensen-Taheri '03,
Dolzman-Kristensen '05,
- Dolzman-Kristensen-Zhang,
- Scheven-Schmidt '09
- Carozza-Passarelli-Schmidt-Verde '10

f smooth uniformly convex function with uniformly bounded second derivatives

Classical results: smooth minimizers

Morrey, De Giorgi, Nash

- $n = 2, N \geq 1$ or
- $n \geq 2, N = 1$

Counterexamples

$n > 2, N > 1 \implies$ non smooth minimizers, but Lipschitz continuous, Necas '77.

Recent results, Sverak-Yan (2002)

Using “null Lagrangian” approach, they construct counterexamples showing that in general for $n \geq 3$ Lipschitz continuity cannot be expected.

For $n \geq 5$ even locally unbounded minimizers.

Main Theorem

$$I(u) = \int_{\Omega} f(Du) dx$$

- $f \in C^2(\mathbb{R}^{nN})$
- $|D^2 f(\xi)| \leq c\phi''(|\xi|), \forall \xi \in \mathbb{R}^{nN} \setminus \{0\}$
- $\lim_{|\xi| \rightarrow \infty} \frac{|D^2 f(\xi) - D^2 \phi(|\xi|)|}{\phi''(|\xi|)} = 0$

If $u \in W_{loc}^{1,\phi}(\Omega, \mathbb{R}^N)$ is a local minimizer for I , then Du is locally bounded.

$$\sup_B \phi(|Du|) \leq c(1 + \int_{2B} \phi(|Du|) dx)$$

The proof is achieved comparing our minimizer with the minimizer of the model functional for which we have the decay estimate!

What about C^1 asymptotically convex problems?
 Counterexample of Dolzmann, Kristensen,
 Zhang: $n = N = 2, p = 2$

$$\int_B \text{dist}^2(\nabla u, SO(2)),$$

$B \subset \mathbb{R}^2$ unit disc. The quasiconvex envelope F is C^1 asymptotically convex; $F'(\xi) = 2(\xi - c(\xi))$, with c bounded Lipschitz. There exists a minimizer in $W_0^{1,2}(B, \mathbb{R}^2)$ with unbounded gradient near 0 (also minimizer for F): it is $\frac{1}{4} \bar{z} \log |z|^2 \in W^{1,BMO}$!

Scheven-Schmidt result:

f locally bounded Borel integrand + asymptotically regular



Ω can be decomposed into three disjoint sets:

- H is open u is $C^{1,\alpha}, \forall \alpha$;
- $B_L x \in B_L$ is a Lebesgue point and $|Du| \leq L$;
- Σ is negligible set.

H and the interior of B_L are contained in the regular set, that is dense in Ω . But $S(u) \cap \partial B_L$ can have positive measure!

Quasiconvex problems

What about **Quasiconvex functionals**?

A function $F : \mathbb{R}^{nN} \rightarrow \mathbb{R}$ is quasiconvex iff

$$\int_Q [F(z_0 + D\eta) - F(z_0)] dx \geq 0$$

for every $z_0 \in \mathbb{R}^{nN}$ and every η with compact support in Q .

Partial regularity for quasiconvex functionals was initiated by Evans who considered:

$$\int_{B_1} f(A + D\eta) - f(A) \geq \nu \int_{B_1} (\mu + |A| + |D\eta|)^{p-2} |D\eta|^2$$

for every $A \in \mathbb{R}^{nN}$ and every $\eta \in C^\infty(B_1, \mathbb{R}^N)$.

uniform strict quasiconvexity

$$\int_{B_1} f(A + D\eta) - f(A) \geq \nu \int_{B_1} \phi''(\mu + |A| + |D\eta|) |D\eta|^2$$

for every $A \in \mathbb{R}^{nN}$ and every $\eta \in C^\infty(B_1, \mathbb{R}^N)$ with compact support.

- $\mu > 0$ Non degenerate case
- $\mu = 0$ Degenerate case

Non degenerate case

Partial regularity: The basic idea is to linearize the problem near the gradient average.

We have different methods to implement a local linearization scheme:

- **Indirect method** via blow up techniques: Morrey, Evans, Acerbi-Fusco, Carozza-Passarelli di Napoli, Carozza-Fusco-Mingione,...
- **A-harmonic approximation method:** De Giorgi (minimal surfaces), Duzaar-Steffen (geometric measure theory), Duzaar-Mingione, Duzaar-Gastel-Grotowski, Duzaar-Grotowski-Kronz,...

Degenerate case

Partial regularity: When we linearize near the gradient average, it may happen that $(Du)_{x,r}$ is near the origin or even 0 so that the linearized problem loses the ellipticity!

Idea for partial regularity:

- when Du is far from 0, then one can linearize as before;
- when Du is near 0, then one directly compares u with minimizers of the model case functional $\int_{\Omega} |Dv|^p dx$ via "p-harmonic approximation" due to Duzaar-Mingione (thanks to additional structure conditions).

In this context of more general growth we prove a ϕ -harmonic approximation that also in case of powers give a new approximation in terms of the gradients

ϕ -harmonic approximation

$\phi \in C^1([0, \infty)) \cap C^2((0, \infty))$ s.t. $\phi'(t) \sim t\phi''(t)$ uniformly in $t > 0$. For every $\epsilon > 0$ and $\theta \in (0, 1)$, $\exists \delta = \delta(\epsilon, \theta, \phi) > 0$ s.t. if $u \in W^{1, \phi}(B, \mathbb{R}^N)$ is *almost ϕ -harmonic* i.e. $\forall \xi \in C_0^\infty(B, \mathbb{R}^N)$

$$\left| \int_B \phi'(|\nabla u|) \frac{\nabla u}{|\nabla u|} \nabla \xi \, dx \right| \leq \delta \left(\int_B \phi(|\nabla u|) \, dx + \phi(\|\nabla \xi\|_\infty) \right),$$

then the unique ϕ -harmonic map h with $h = u$ on ∂B satisfies

$$\left(\int_B |V(\nabla u) - V(\nabla h)|^{2\theta} \, dx \right)^{\frac{1}{\theta}} < \epsilon \int_B \phi(|\nabla u|) \, dx.$$

The proof is based on a generalization of the Lipschitz approximation Lemma in the context of Orlicz spaces.

Lipschitz approximation Lemma

ϕ N -function s.t. $\Delta_2(\phi, \phi^*) < \infty$, Ω bounded domain and $u^n \in W_0^{1,\phi}(\Omega, \mathbb{R}^N)$ be such that $u^n \rightharpoonup 0$ in $W_0^{1,\phi}(\Omega, \mathbb{R}^N)$. Set $\gamma_n := \|u^n\|_\phi \rightarrow 0$. Let $\theta_n > 0$ s.t. $\theta_n \rightarrow 0$, $\frac{\gamma_n}{\theta_n} \rightarrow 0$ and $\mu_j := 2^{2^j}$



$\exists \lambda_{n,j} > 0$ s.t. $\mu_j \leq \lambda_{n,j} \leq \mu_{j+1}$ and $u^{n,j} \in W_0^{1,\infty}(\Omega, \mathbb{R}^N)$ s.t.

- $\|u^{n,j}\|_\infty \leq \theta_n \rightarrow 0$, $\|\nabla u^{n,j}\|_\infty \leq c \lambda_{n,j} \leq c \mu_{j+1}$.
- $\{u^{n,j} \neq u^n\} \subset \Omega \cap (\{Mu^n > \theta_n\} \cup \{M(\nabla u^n) > 2\lambda_{n,j}\})$
- when $n \rightarrow \infty$ $u^{n,j} \rightarrow 0$ in $L^s(\Omega, \mathbb{R}^N) \forall s \in [1, \infty]$,
- $u^{n,j} \rightharpoonup 0$ in $W_0^{1,s}(\Omega, \mathbb{R}^N) \forall s \in [1, \infty)$, $\nabla u^{n,j} \rightharpoonup^* 0$ in L^∞
- $\limsup_{n \rightarrow \infty} \int \phi(\nabla u^{n,j} \chi_{\{u^{n,j} \neq u^n\}}) dx \leq c 2^{-j}$,
- $\exists \epsilon_j (\rightarrow 0)$ s.t. $\|\nabla u^{n,j} \chi_{\{u^{n,j} \neq u^n\}}\|_\phi \leq c \frac{\gamma_n}{\theta_n} \mu_{j+1} + \epsilon_j$.

Remark

Acerbi-Fusco Lemma in the power case $p \implies$

$$\|\nabla u^{n,j} \chi_{\{u^{n,j} \neq u^n\}}\|_p \leq \lambda_{n,j} |\{u^{n,j} \neq u^n\}|^{\frac{1}{p}} \leq c \|u^n\|_{1,p} \leq K.$$

So just boundedness of the above term!

Sketch of the ϕ -harmonic approximation:

- Take h solution of the homogeneous problem in a ball B with $h = u$ on ∂B ;
- Let $\gamma > 0$ s.t. $\phi(\gamma) = \int_B \phi(|\nabla u|) dx$ and $\lambda \in [\gamma, 2^{m_0} \gamma]$ for suitable m_0 . Take $w = h - u$ and w_λ s.t. $\|w_\lambda\|_\infty \leq c\lambda$ and $\int_B \phi(\lambda \chi_{\{w_\lambda \neq w\}}) dx \leq \frac{\phi(\gamma)}{m_0}$. We consider w_λ as test function in both problems (*almost* ϕ -harmonic estimate and ϕ -harmonic system);
- Monotonicity of the operator, Young's inequality and useful properties of w_λ .

Let us remark the main differences with respect to the result of Duzaar and Mingione.

Remark

- We use a direct approach without a contradiction argument. This allows us to show that the constants involved in the approximation only depend on ϕ .
- We are able to preserve the boundary values of our original function. In particular, $u = h$ on ∂B .
- We show that h and u are close with respect to the gradients rather than just the functions.

Idea for partial regularity

- harmonic approximation
- interpolation inequality
- excess decay estimate for the harmonic map h



- $C^{1,\alpha}$ -partial regularity for minimizers of quasiconvex functionals of this type: $\mu \geq 0, \phi'' \nearrow$

$$\int_{B_1} f(A + D\eta) - f(A) \geq \nu \int_{B_1} \phi''(\mu + |A| + |D\eta|) |D\eta|^2.$$

ϕ -systems with critical growth

Hölder regularity

Suppose $c \geq 1$ is given. Then there exists $\delta(n, N, \phi, c) > 0$ such that if $u \in W^{1,\phi}(\Omega, \mathbb{R}^N)$ satisfies the system :

$$\int_{\Omega} \frac{\phi'(|\nabla u|)}{|\nabla u|} \nabla u \nabla \eta \, dx = \int_{\Omega} G \eta \, dx \quad (1)$$

for all $\eta \in C_0^\infty(\Omega)$ where $G \in L^1(\Omega)$ satisfies for a.e. $x \in \Omega$

$$|G(x)| \leq c\phi(|\nabla u|) \quad (2)$$

and if u satisfies a Caccioppoli inequality and

$$\phi^{-1} \left(\int_{B_R} \phi(|\nabla u|) \, dx \right) \leq \frac{\delta}{R}$$

on some ball $B_R \subset \Omega$, then $V(\nabla u)$ is Hölder continuous on $B_{\frac{R}{2}}$ with exponent μ , for suitable μ depending on δ, ϕ, n, N, c .

Principal steps

- smallness + Caccioppoli assumption $\implies u$ is almost ϕ -harmonic;
- ϕ -harmonic approximation+ excess decay estimate for the ϕ -harmonic map $h \implies$ Morrey-type estimate for the gradient
- convex-hull property for the functional $\implies u - h$ is continuous;
- subtracting the equations with $u - h$ as test function, we conclude by using a classical iteration argument.