Obstacle type problems
An overview and some recent results

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The good old obstacle problem

Original form

Find the smallest superharmonic function over a given obstacle.

This in terms of Hamilton-Jacobi equation can be written as

$$\min(-\Delta u, u - \psi) = 0,$$

where ψ is a give obstacle; vi ignore boundary values.

The good old obstacle problem

Example 2. *In this example* We Choose $\Omega = [-2,2] \times [-2,2]$ *and*

$$\psi(x,y) = \begin{cases} \sqrt{1-x^2-y^2}, & \text{for } x^2+y^2 \le 1, \\ -1, & \text{elsewhere.} \end{cases}$$

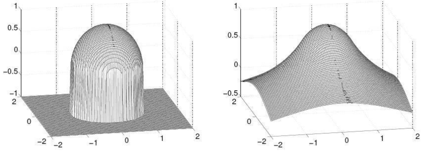

FIGURE 2. The obstacle (left) and the analytical solution (right)

Figure: Equilibrium state of the membrane over the obstacle (Curtsy of Farid Bozorgnia)

The good old obstacle problem

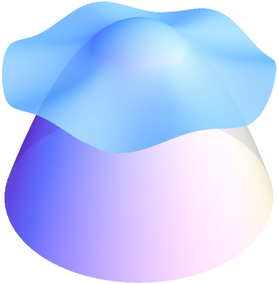

Figure: Equilibrium state of the membrane over the obstacle,

from a forthcoming book by Petrosyan, Uraltseva, Shahgholian

The good old obstacle problem (Applications)

Various processes in engineering sciences

Dam problem or filtration through porous medium. Usually uses Darcy's law and results to an obstacle problem for the accumulated pressure.

Cavitation in hydrodynamic lubrication. Cavities are the coincident sets.

Mathematical Finance: Parabolic versions of such problems represents the pricing of American type derivatives/options.

The good old obstacle problem (Applications)

Smash sum of P. Diaconis and W. Fulton (Lattice sets)

The smash sum C of two sets A, B is a certain random set defined as follows. Let $A \cap B = \{x_1, \cdots, x_k\}$. Each point in $A \cap B$, is sent out through random walks.

Once the point hits *first time* any point y_j outside the region $A \cup B$ it stops and adds y_j to the union:

$$C_j = A \cup B \cup \{y_j\}.$$

The process continuous until all points x_j are sent out. The resulting region $C := C_k$ is the Smash sum.

The good old obstacle problem (Applications)

Smash sum of P. Diaconis and W. Fulton (Lattice sets)

The key observation is that the law does not depend on the ordering of the points x_j.
As the lattice spacing goes to zero, the smash sum has a deterministic scaling limit, which is an obstacle problem with potential function of the set $A \cap B$, as the obstacle.

Quadrature Domains: Ramifications

The related topic *Quadrature Domains* has found interesting applications in Quantum Hall theory, Laplacian growth, and Random Matrices. This is a growing research area.

Reformulation

The reformulated form of the obstacle problem

$$\min(-\Delta v, v - \psi) = 0$$

when ψ is smooth is by replacing v with

$$u = v - \psi,$$

which leads to

$$\Delta u = (-\Delta \psi)\chi_{\{u>0\}}, \qquad u \geq 0, \qquad \text{in } B_1.$$

or more generally

$$\Delta u = f(x)\chi_{\{u>0\}}, \qquad u \geq 0, \qquad \text{in } B_1,$$

for given $f > 0$ (smooth/continuous).

Two examples

Potential Theory: harmonic continuation of potentials

$$\Delta u = \chi_\Omega, \quad \text{in } B_1, \qquad u = \nabla u = 0, \qquad \text{in } B_1 \setminus \Omega,$$

for some unknownwn Ω.

Superconductivity

$$\Delta u = \chi_{\{\nabla u \neq 0\}}, \quad \text{in } B_1,$$

More general form

A semilinear problem

Consider a semilinear equation of the form

$$\Delta u = F(u), \qquad \text{(more generally} \quad F(x,u))$$

where $F(.,t)$ has discontinuity at $t = 0$, i.e. across the zero level sets of u.

What can we say about the optimal regularity of the solution u?

What can we say about the optimal regularity of the set $\partial\{u > 0\}$?

More general form

Free boundary points of interest are $\nabla u = 0$, otherwise implicit function theorem gives regularity of the level sets.

Invariant Scaling

Let $u(x) = |\nabla u(x)| = 0$ and set

$$u_r(x) := \frac{u(rx + x^0)}{r^2}.$$

Then

$$\Delta u_r(x) = F(u(rx + x^0)) = F(r^2 u_r)$$

and we retain the problem.

So we need u_r to be uniformly bounded in r?

$C^{1,1}$ regularity

Conditions on F

The condition

$$F_t'(x,t) \geq -C, \qquad |\nabla_x F(x,t)| \leq C$$

guarantees a $C^{1,1}$ regularity for u.

Indeed, for $u_e = D_e u$ we get

$$\Delta(u_e)^{\pm}(x) = F_e + F_t'(u_e)^{\pm} \geq -C'$$

and a monotonicity formula of CJK can be applied to obtain bounds on ∇u_e.

Failure of $C^{1,1}$ regularity

Conditions on F

If F' produces a NEGATIVE Dirac type measures, i.e.

$$F'_t \not\geq -C$$

then $C^{1,1}$ regularity may fail.

Good Examples of F: Obstacle problem

Obstacle problem

For the obstacle problem (with smooth obstacle) one has

$$F(x,u) = f(x)\chi_{\{u>0\}}, \qquad u \geq 0 \qquad f > 0$$

$$\Delta u_r(x) = f(rx)\chi_{\{u_r>0\}},$$

and

$$\Delta (u_e)^{\pm} \geq f' \chi_{\{u>0\}} + f(u_e)^{\pm}\delta_{\partial\{u>0\}} \geq -C$$

In this case, not much is left for study!
The case when $f \geq 0$ has zeros is untouched!

Good Examples of F: Two-phase case

Two phase problems

$$F(x,u) = \lambda_+(x)\chi_{\{u>0\}} - \lambda_-(x)\chi_{\{u<0\}},$$

$$\Delta u_r(x) = \lambda_+(rx)\chi_{\{u_r>0\}} - \lambda_-(rx)\chi_{\{u_r<0\}}$$

with $\lambda_+ + \lambda_- \geq 0$ and $C^{0,1}$.
and

$$\Delta(u_e)^{\pm} \geq D_e\lambda_+(x)\chi_{\{u>0\}} - D_e\lambda_-(x)\chi_{\{u<0\}} +$$

$$(\lambda_+ + \lambda_-) \text{ pos. measure } \geq -C$$

Here too, not much is left for study! The case when $\lambda_\pm$ can take zero value or one of them is identically zero are not studied.

Less Good Examples of F: Unstable

Unstable problems

$$F(x,u) = \lambda_+(x)\chi_{\{u>0\}} - \lambda_-(x)\chi_{\{u<0\}},$$

$$\lambda_+(x) + \lambda_-(x) < 0$$

A scaling does not necessarily converge.

Examples of F: Failure of $C^{1,1}$ regularity

A simple case

$$\Delta u = -\chi_{u>0},$$

related to traveling wave solutions in solid combustion with ignition temperature.

J. Andersson, G. Weiss

There exist solutions that are not $C^{1,1}$.

Examples of F: Failure of $C^{1,1}$ regularity

J. Andersson, G. Weiss

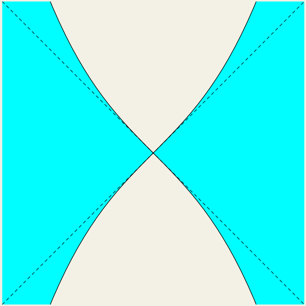

Figure: **NOT** $C^{1,1}$. In A-W example level curves are cross-shaped.

from a forthcoming book by Petrosyan, Uraltseva, Shahgholian

Applications: Unstable case

Composite membrane

Build a body of a prescribed shape out of given materials of varying densities, in such a way that the body has a prescribed mass and with the property that the fundamental frequency of the resulting membrane (with fixed boundary) is lowest possible.

Composite membrane

Given a fix body, of two different materials. Rearrange the materials so that we get the lowest resonance/vibration.

This problem results in an equations of the type above with

$$\lambda_+ + \lambda_- < 0.$$

Ref. S.J. Cox, J.R. Mclaughlin.

Applications: Unstable case

Population dynamics

Other applications are that of Population Dynamics, where the optimal arrangement of favorable and unfavorable regions for species' survival is in consideration. This again is a an eigenvalue problem

$$\Delta u = -\lambda m(x) u$$

with

$$m(x) = m_1 \chi_{u>t} - m_2 \chi_{u<t}$$

($t > 0$, $u > 0$, $\lambda > 0$).

Ref. Y. Lou, E. Yanagida.

Systems: An example

Stable (+), and Unstable (-) cases

If we consider systems, then the following problem can be considered a generalization of the the obstacle type problems

$$\Delta u_i = \frac{\pm u_i}{|\mathbf{u}|}, \quad i = 1, 2, ...$$

and it introduces some challenge. $+$ gives the stable case, and $-$ the unstable one.

Systems: An example

Unstable case

The real and imaginary parts of the function

$$S(z) = z^2 \log |z|$$

satisfy the unstable equation (up to a multiplicative constant) and they have singularities at the origin.

Hence optimal $C^{1,1}$ regularity is lost!

Main tools in regularity theory

Quadratic growth

Standard in such problems is that u is $C^{1,1}$ and we have the possibility of scaling the solution

$$u_r(x) := \frac{u(rx + x^0)}{r^2},$$

in order to analyze local properties of the solution and the free boundary.

Observe that u_r is then a solution in $B_{1/r}$ and as $r \longrightarrow 0$ then $u_r \longrightarrow u_0$ in the entire space $\mathbb{R}^n$, and we have

$$\Delta u_0 = \lambda_+(0)\chi_{\{u_0>0\}} - \lambda_-(0)\chi_{\{u_0<0\}}, \qquad \text{in } \mathbb{R}^n .$$

Main tools in regularity theory

Non-degeneracy

In blowing up a solution, one needs to show that the limit solution is not identically zero (i.e it doesn't flatten out). Therefore one needs the so-called non-degeneracy:

$$\sup_{B_r(x^0)} u \geq cr^2, \qquad \inf_{B_r(x^0)} u \leq -cr^2.$$

Unstable case: Troubles

Failure!

Quadratic growth and non-degeneracy may fail

$$C_0 r^2 \not\leq \sup_{B_r(x^0)} u \not\leq C_1 r^2.$$

Standard scaling is not available, anymore.

So how do we study the regularity of the free boundary?

Unstable case: Troubleshooting

Consider a solution to the equation

$$\Delta u = -\chi_{\{u>0\}} \text{ in } B_1,$$

and define

$$S(u) = \{X;\ u(X) = \nabla u(X) = 0, u \notin C^{1,1}(B_r(X)) \quad \forall r > 0\}.$$

Can $S(u)$ be embedded in a smooth lower dimensional manifold?

Unstable case: $\mathbb{R}^2$

Andersson, Sh., Weiss:

For x^0 a singular point we have
(i) there exists a polynomial $p^{x^0,u} = p$ such that

$$\left\| \frac{u(x^0 + sx)}{\sup_{B_s(x^0)} |u|} - p \right\|_{C^{1,\beta}(B_1)} \leq C_{\alpha,\beta,n,M} \left(\frac{\delta}{1 + \delta \log(r/s)} \right)^{\alpha}$$

(ii) the set $\{u = 0\} \cap B_r(x^0)$ consists of two C^1 curves intersecting each other at right angle at x^0.

Here $1/\delta = \sup_{B_r} |u|/r^2$, is large enough, and $s < r$, $0 < \beta, \alpha < 1/2$ are arbitrary, M is supnorm of u.

Unstable case: 3-space dimension

Singular points $S = S(u)$ are points where u is not $C^{1,1}$.

Homogenous scaling

For singular points X^0 we have

$$\Delta \left(\frac{u(r_j X + X^0)}{\sup_{B_{r_j}(X^0)} |u|} \right) = - \frac{r^2}{\sup_{B_{r_j}(X^0)} |u|} \chi_{\{u(r_j X + X^0) > 0\}} \quad \to \quad 0$$

Hence

$$\lim_{r_j \to 0} \frac{u(r_j X + X^0)}{\sup_{B_{r_j}(X^0)} |u|} = p(X)$$

where p is a second order homogeneous harmonic polynomial.

Unstable case: $\mathbb{R}^3$

Questions

- Does p depend on the sequence $r_j \to 0$?
- Are there some further restrictions on p?

Unstable case: $\mathbb{R}^3$

Limiting harmonic polynomials in $\mathbb{R}^3$

If for some sequence r_j

$$\lim_{r_j \to 0} \frac{u(r_j X + X^0)}{\sup_{B_{r_j}} |u|} = p$$

then the limit exists for all sequences r_j and it is p with

$$p = x^2 - z^2 \qquad \text{or} \qquad p = \pm\left((x^2 + y^2)/2 - z^2\right)$$

after suitable rotation.

Unstable case: $\mathbb{R}^3$

Limiting harmonic polynomials in $\mathbb{R}^3$

Observe that this is a very strong statement, since there are a range of other possible quadratic harmonic polynomials, that we exclude

$$ax^2 + by^2 - z^2, \qquad a + b = 1.$$

Unstable case: $\mathbb{R}^3$

Theorem; Andersson, Sh., Weiss

In $\mathbb{R}^3$, the singular set S is divided into two parts

$$S_1 = \{x \in S;\ \lim_{r\to 0} u(x + r\cdot)/\sup_{B_r} |u| = \pm(x^2 + y^2 - 2z^2)\}$$

and

$$S_2 = \{x \in S;\ \lim_{r\to 0} u(x + r\cdot)/\sup_{B_r} |u| = xy\}$$

where S_1 consists only of isolated points and S_2 is contained in a C^1 manifold.

Unstable case: Ideas of the proof

A non-standard blow-up

The above blow-up does not provide us with enough information about the solution, since the non-linearity of $\chi_{\{u>0\}}$ disappears in the limit.

Instead we suggest a blow-up of the following kind

$$\lim_{r_j \to 0} \frac{u(r_j X + X^0)}{r_j^2} - \Pi(u, r_j, X^0),$$

where $\Pi(u, r_j, X^0)$ is the projection of $u(r_j X + X^0)/r_j^2$ in B_1 into the homogeneous harmonic second order polynomials.

Unstable case: Ideas of the proof

Preserving Non-linearity in the limit

One can show that

$$\lim_{r_j \to 0} \frac{u(r_j X + X^0)}{r_j^2} - \Pi(u, r_j, X^0) = Z_p$$

where

$$\Delta Z_p = -\chi_{\{p>0\}}.$$

The proof is either indriect, using compactness, or direct using the fact that the second derivatives of u are in BMO, so a harmonic analysis approach will work.

Unstable case: Ideas of the proof

Ideas

One notices further that

$$\lim_{r_j \to 0} \frac{u(r_j X + X^0)}{\sup_{B_{r_j}(X^0)} |u|} = \lim_{r_j \to 0} \frac{\Pi(u, r_j, X^0)}{\sup_{B_1} |\Pi(u, r_j, X^0)|},$$

which follows from $u(r_j X + X^0) = (\Pi(u, r_j, X^0) + Z_p) r_j^2$ and that $\sup_{B_1} |u(rx)| = \sup_{B_1} \Pi(u(rx))$, along with the fact that the $r^2 / \sup_{B_{r_j}} |u|$ tend to zero.

So if we want to prove uniqueness of p it is enough to control how $\Pi(u, r, X^0)$ changes in r.

Unstable case: Ideas of the proof

Ideas

Specifically we would want to estimate

$$\left|\frac{\Pi(u,r,X^0)}{\sup_{B_1}|\Pi(u,r,X^0)|}-\frac{\Pi(u,r/2,X^0)}{\sup_{B_1}|\Pi(u,r/2,X^0)|}\right|,$$

that is how much the orientation of u changes when we pass from the ball $B_r(X^0)$ to $B_{r/2}(X^0)$.

Unstable case: Ideas of the proof

Ideas

To do this we notice that if

$$\Pi(u, r)/\tau_r \approx p(X)$$

for some polynomial p where $\tau_r = \sup_{B_1} |\Pi(u, r)|$ then

$$\Delta u \approx -\chi_{\{p>0\}} \qquad \text{in } B_r$$

and thus

$$u \approx r^2(\tau_r p + Z_p) \qquad \text{in } B_r.$$

Observer that τ_r tends to infinity with r tending to zero.

Unstable case: Ideas of the proof

Ideas

The main point here is that if $\Pi(u,r)/\tau_r \approx p(X)$ then

$$\Pi(u,r/2) \approx \Pi(\tau_r p + Z_p, 1/2) \approx \Pi(\tau_r p, 1/2) + \Pi(Z_p, 1/2)$$

$$= \tau_r p + \Pi(Z_p, 1/2) = \Pi(u,r) + \Pi(Z_p, 1/2)$$

so

$$|\Pi(u,r) - \Pi(u,r/2)| \approx |\Pi(Z_p, 1/2)|.$$

And therefore to estimate how much $\Pi(u,r)$ changes when we change r we need to be able to control $\Pi(Z_p,\cdot)$.

Unstable case: Ideas of the proof

Explicit computation of Z_p

Next one explicitly calculates $Z_p = q(X)\ln|X| + |X|^2\phi(X)$, when

$$p = xy \qquad \text{or} \qquad p = \pm\big((x^2+y^2)/2 - z^2\big).$$

Using these calculations we can show that at singular points and for small r we have

$$\sup_{B_r} |u(X+X^0)| \geq cr^2|\ln r|.$$

This is the first step in a series of EXACT estimates to come, and that leads us to the an accurate estimate of the turn of the polynomials (or u_r) in terms of r.

Existing results: Unstable case

Partial results

Monneau-Weiss, Chanillo-Kenig, Chanillo-Kenig-Tou, Andersson-Weiss, Sh., ...