Decomposition Theorems and Fine Estimates of Electrical Fields in the Presence of Close Inclusions

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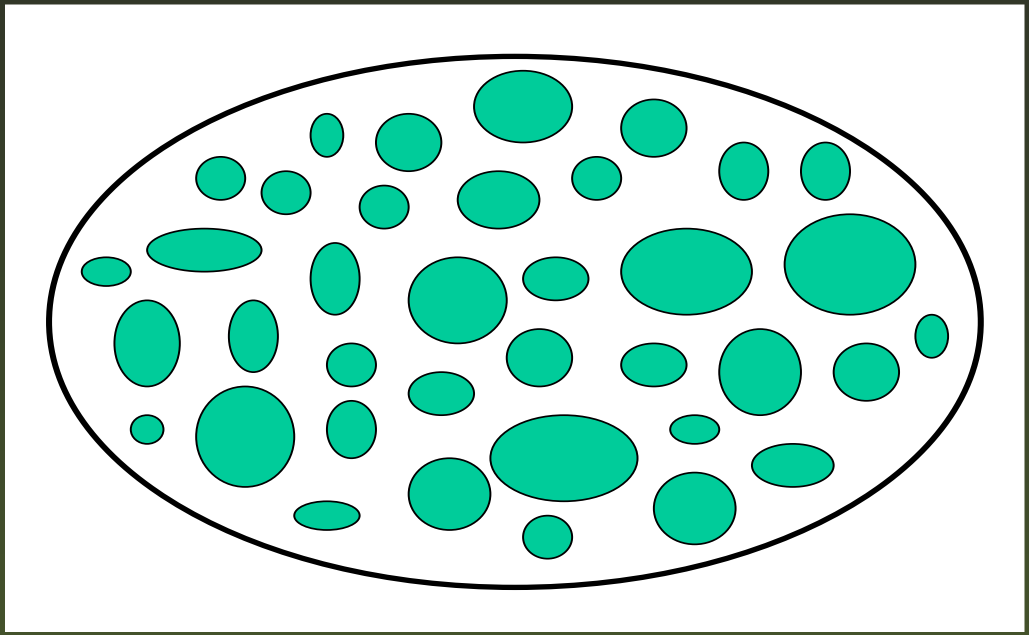
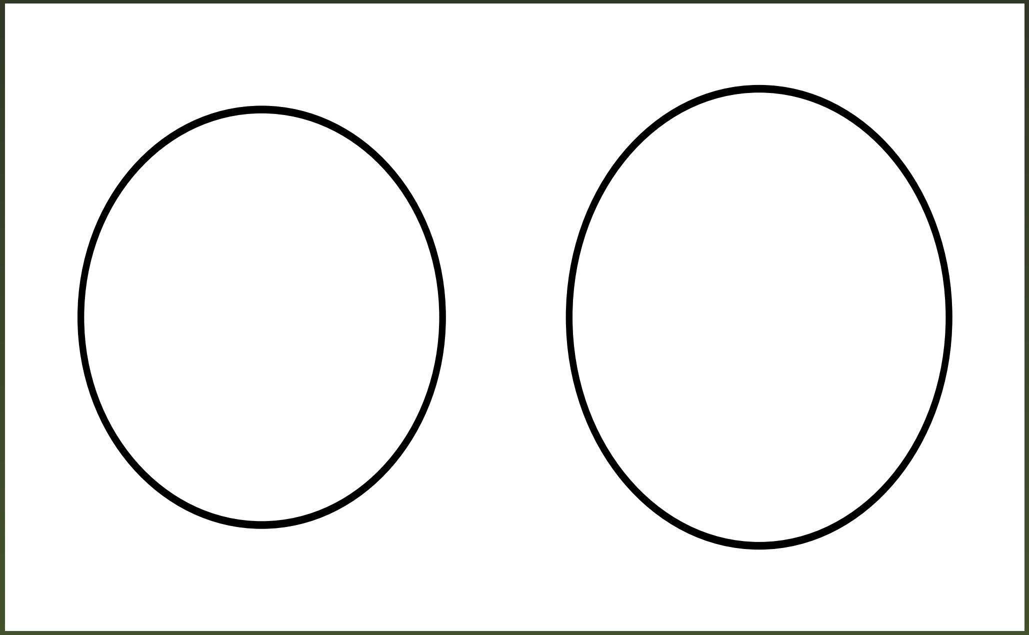
Stress in Composites



Two ways to consider the stress:

- Global feature (intensity dispersion ratio) (Berlyand)
- Local feature

Two Inclusions Model



Conductivity Equations (free space)

H: an entire harmonic function (in particular, $H(x) = a \cdot x$).

$$\begin{cases} \nabla \cdot \Big(\chi(\mathbb{R}^d \setminus \overline{B_1 \cup B_2}) + \sum_{j=1}^{2} \sigma_j \chi(B_j) \Big) \nabla u = 0 \quad \text{in } \mathbb{R}^d, \\ u(x) - H(x) = O(|x|^{1-d}) \quad \text{as } |x| \to \infty. \end{cases}$$

Equivalently,

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^d \setminus (\partial B_1 \cup \partial B_2), \\ u|_+ = u|_- & \text{on } \partial B_j, \ j = 1, 2, \\ \dfrac{\partial u}{\partial \nu}\Big|_+ = \sigma_j \dfrac{\partial u}{\partial \nu}\Big|_- & \text{on } \partial B_j, \ j = 1, 2, \\ u(x) - H(x) = O(|x|^{-1}) & \text{as } |x| \to \infty. \end{cases}$$

Conductivity Equations (bounded domain)

$$\begin{cases} \nabla\cdot\Big(\chi(\Omega\setminus\overline{B_1\cup B_2}) + \sum_{j=1}^{2}\sigma_j\chi(B_i)\Big)\nabla u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

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Estimates of $\|\nabla u\|_\infty$

Problem: Estimate $\|\nabla u\|_\infty$ on a bounded region including B_1 and B_2 as $\epsilon \longrightarrow 0$ where

$$\epsilon = \text{dist}\,(B_1, B_2).$$

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- If σ_j is bounded, then $\|\nabla u\|_{\infty(\Omega)} \leq C$ regardless of ϵ. (Li-Vogelius ARMA 00, Bonnetier-Vogelius SIMA 00).
- Extended the results to the linear (isotropic) elasticity (Li-Nirenberg CPAM 03).
- Computation of electric field in the presence of closely located circular inclusions (Cheng-Greengard, SIAP 98)

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Motivation:

- Estimates of the stress in the composites when grains are very close to each other.
- Computation of the electromagnetic field in the presence of close inclusions.

Conductivity Equations ($\sigma_j = 0$ or ∞)

$\sigma_j = 0$:

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$\sigma_j = \infty$:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \setminus \overline{B_1 \cup B_2}, \\ u = \lambda_j (constant) & \text{on } \partial B_j,\ j = 1, 2, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

(The constant λ_j will be specified.)

Conductivity Equations ($\sigma_j = 0$ or ∞)

Suppose $\sigma_1 = \sigma_2 = \sigma$.

- $u^\sigma \longrightarrow u^{\sigma_0}$ in H^1 as $\sigma_j \longrightarrow \sigma_0$ ($\sigma_0 = 0$ or ∞) (Friedman-Vogelius, ARMA 88)

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Question: Does $u^\sigma \to u^{\sigma_0}$ in $W^{1,\infty}$ as $\sigma_j \to \sigma_0$ *uniformly in* ϵ? (∂B_j is as smooth as one wishes, say $\mathcal{C}^{1,\alpha}$.)

Previous works

There have been some works (not rigorous) showing that when $\sigma = \infty$

$$\|\nabla u\|_{L^\infty} \geq \frac{C}{\sqrt{\epsilon}}$$

(Budiansky-Carrier JAM 84, Keller JAM 93, Markenscoff CM 96).

See Milton "The theory of composites" 10.10 for related works.

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Let $\epsilon := \text{dist}(B_1, B_2)$ and let $\nu^{(j)}$ and $T^{(j)}$, $j = 1, 2$, be the unit normal and tangential vector fields to ∂B_j, respectively.

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If $\sigma > 1$ and ϵ is sufficiently small, then

$$\frac{C_1|\langle \nabla H(X_j), \nu^{(j)}(X_j)\rangle|}{\frac{1}{\sigma}+\sqrt{\frac{\epsilon}{r}}} \leq \|\nabla u\|_{L^\infty(\Omega\setminus(B_1\cup B_2))} \leq \frac{C_2\|\nabla H\|_{L^\infty(\Omega)}}{\frac{1}{\sigma}+\sqrt{\frac{\epsilon}{r}}}, \quad j=1,2,$$

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If $\sigma < 1$, then

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If $r \approx \epsilon$, No blow-up.

Convex perfect conductors in 2D

Theorem 1 (Yun, SIAP 07). *B_1 and B_2 are convex 'with proper locations', $\sigma = \infty$, and $H(x,y) = x$. Then,*

$$|u|_{\partial B_1} - u|_{\partial B_2}| \geq C_1 \sqrt{\epsilon}, \quad \|\nabla u\|_{L^\infty} \leq \frac{C_2}{\sqrt{\epsilon}}.$$

Perfect conductors in 3D

B_1, B_2: convex perfect conductors ($\sigma = \infty$)

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \setminus \overline{B_1 \cup B_2}, \\ u = \lambda_j (constant) & \text{on } \partial B_j,\ j = 1, 2, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

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$$|\nabla u(x)| \geq \frac{C}{\epsilon} |\lambda_1 - \lambda_2|.$$

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$$\begin{cases} \Delta v_1 = 0 \quad \text{in } \Omega \setminus \overline{B_1 \cup B_2}, \\ u = 1 \quad \text{on } \partial B_1, \\ u = 0 \quad \text{on } \partial B_2, \\ u = 0 \quad \text{on } \partial \Omega. \end{cases} \qquad \begin{cases} \Delta v_2 = 0 \quad \text{in } \Omega \setminus \overline{B_1 \cup B_2}, \\ u = 0 \quad \text{on } \partial B_1, \\ u = 1 \quad \text{on } \partial B_2, \\ u = 0 \quad \text{on } \partial \Omega. \end{cases}$$

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$$u = \lambda_1 v_1 + \lambda_2 v_2 + v_0.$$

Perfect conductors in 3D

$$\lambda_1 \int_{\partial B_1} \frac{\partial v_1}{\partial \nu} + \lambda_2 \int_{\partial B_1} \frac{\partial v_2}{\partial \nu} + \int_{\partial B_1} \frac{\partial v_0}{\partial \nu} = 0,$$
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Suppose ∂B_j is given locally by

$$x_d = \psi_1(x') + \frac{\epsilon}{2} \quad \text{and} \quad x_d = -\psi_2(x') - \frac{\epsilon}{2} \quad |x'| < \delta.$$

Then, the blow-up rate is "determined" by

$$\int_{|x'|<\delta} \frac{1}{\psi_1(x') + \psi_2(x') + \epsilon} dx'.$$

(Bao-Li-Yin, ARMA to appear)

Perfect conductors in 3D

Theorem 2 (Bao-Li-Yin, ARMA to appear). *If ∂B_1 and ∂B_2 are strictly convex, then*

$$\|\nabla u\|_{L^\infty} \le C\|f\|_{C^2(\partial\Omega)} \begin{cases} \dfrac{1}{\sqrt{\epsilon}}, & d=2, \\ \dfrac{1}{\epsilon|\ln\epsilon|}, & d=3, \\ \dfrac{1}{\epsilon}, & d\ge 4, \end{cases}$$

and

$$\|\nabla u\|_{L^\infty} \ge C|Q[f]| \begin{cases} \dfrac{1}{\sqrt{\epsilon}}, & d=2, \\ \dfrac{1}{\epsilon|\ln\epsilon|}, & d=3, \\ \dfrac{1}{\epsilon}, & d\ge 4, \end{cases}$$

where

$$Q[f] = \int_{\partial B_1} \frac{\partial v_0}{\partial \nu} \int_{\partial\Omega} \frac{\partial v_2}{\partial \nu} - \int_{\partial B_2} \frac{\partial v_0}{\partial \nu} \int_{\partial\Omega} \frac{\partial v_1}{\partial \nu}.$$

Perfect conductors in 3D

If

$$\psi_1(x') + \psi_2(x') \approx |x'|^{2m}, \ \ m > 1,$$

then the blow-up rate is

$$\epsilon^{-1/2m}(2D), \quad \epsilon^{-1/m}(3D).$$

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Flat inclusions have lower blow-up rate!

Perfect conductors in 3D

A work of Lim-Yun ($\sigma = \infty$): B_1 and B_2 are balls in $\mathbb{R}^d$. Let h be a solution to

$$\begin{cases} \Delta u = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{B_1 \cup B_2}, \\ u(x) = O(|x|^{1-d}) \quad \text{as } |x| \to \infty, \\ u = k_j \text{ (constant)} \quad \text{on } \partial B_j,\ j = 1, 2, \\ \displaystyle\int_{\partial B_1} \frac{\partial h}{\partial \nu} d\sigma = 1, \quad \int_{\partial B_1} \frac{\partial h}{\partial \nu} d\sigma = -1. \end{cases}$$

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Lim-Yun constructed such an h when B_1 and B_2 are balls in $\mathbb{R}^d$ and obtain the same result for balls (independent work).

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In 2D,

$$h(x) = \frac{1}{2\pi} \big(\log|x - P_1| - \log|x - P_2| \big).$$

Circular Inclusions

Recall

$$\frac{C_1 |\langle \nabla H(X_j), T^{(j)}(X_j) \rangle|}{\sigma + \sqrt{\frac{\epsilon}{r}}} \le \|\nabla u\|_{L^\infty(\Omega \setminus (B_1 \cup B_2))} \le \frac{C_2 \|\nabla H\|_{L^\infty(\Omega)}}{\sigma + \sqrt{\frac{\epsilon}{r}}}.$$

This estimate is optimal in terms of ϵ, but not in terms of H.

Decomposition Theorem (preprint)

Theorem 3. *Suppose $\sigma < 1$. Let P the middle point of X_1X_2 and T be the unit vector orthogonal to X_1X_2. Let*

$$H_s(x) := (\nabla H(P)\cdot T)(T\cdot(x-P)), \quad H(x) = H_s(x) + H_r(x).$$

and

$$u = u_s + u_r$$

where u_s and u_r are the solutions corresponding to H_s and H_r, respectively. Then

$$\frac{C_1 \nabla H_s(P)\cdot T}{\sigma + \sqrt{\frac{\epsilon}{r}}} \le |\nabla u_s(X_1)| \le \frac{C_2 \nabla H_s(P)\cdot T}{\sigma + \sqrt{\frac{\epsilon}{r}}},$$

and

$$\|\nabla u_r\|_\infty \le C \ (\textit{independent of } \epsilon).$$

A similar estimate for the case $\sigma > 1$.

Circular Inclusions with nonzero permittivity

If permittivity of the inclusion is not zero, i.e., coefficient is given by $\sigma + i\omega\mu$, then current flows inside the inclusion even if $\sigma = 0$. Hence, a reduced blow-up is expected.

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In fact, if $H(x) = cT \cdot (x - P)$, then

$$\frac{C_1}{\sigma + |\omega\mu| + \sqrt{\dfrac{\epsilon}{r}}} \leq \|\nabla u\|_{L^\infty(\Omega \setminus (B_1 \cup B_2))} \leq \frac{C_2}{\sigma + |\omega\mu| + \sqrt{\dfrac{\epsilon}{r}}}.$$

Proofs

The solution u is represented as

$$u(x) = H(x) + \mathcal{S}_{B_1}[\varphi_1](x) + \mathcal{S}_{B_2}[\varphi_2](x), \quad x \in \mathbb{R}^2,$$

where

$$\varphi_1 = \frac{2(\sigma-1)}{\sigma+1} \sum_{m=0}^{\infty} \left(\frac{\sigma-1}{\sigma+1} \right)^m \frac{\partial}{\partial \nu^{(1)}} \left[(R_2 R_1)^m (I - \frac{\sigma-1}{\sigma+1} R_2) H \right] \Big|_{\partial B_1},$$

$$\varphi_2 = \frac{2(\sigma-1)}{\sigma+1} \sum_{m=0}^{\infty} \left(\frac{\sigma-1}{\sigma+1} \right)^m \frac{\partial}{\partial \nu^{(2)}} \left[(R_1 R_2)^m (I - \frac{\sigma-1}{\sigma+1} R_1) H \right] \Big|_{\partial B_2},$$

R_j, $j = 1, 2$, be the reflection with respect to ∂B_j, *i.e.*,

$$R_j(X) := \frac{r_j^2 (X - Z_j)}{|X - Z_j|^2} + Z_j, \quad j = 1, 2.$$

Implications for computation (Preliminary report)

Layer potential representation:

$$u(x) = H(x) + \mathcal{S}_{B_1}[\psi_1](x) + \mathcal{S}_{B_2}[\psi_2](x), \quad x \in \mathbb{R}^2,$$

with

$$A\psi := \begin{pmatrix} \frac{1}{2}I + \mathcal{K}^*_{B_1} & \left.\frac{\partial \mathcal{S}_{B_2}}{\partial \nu}\right|_{\partial B_1} \\ \left.\frac{\partial \mathcal{S}_{B_1}}{\partial \nu}\right|_{\partial B_2} & \frac{1}{2}I + \mathcal{K}^*_{B_2} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = - \begin{pmatrix} \left.\frac{\partial H}{\partial \nu}\right|_{\partial B_1} \\ \left.\frac{\partial H}{\partial \nu}\right|_{\partial B_2} \end{pmatrix}$$

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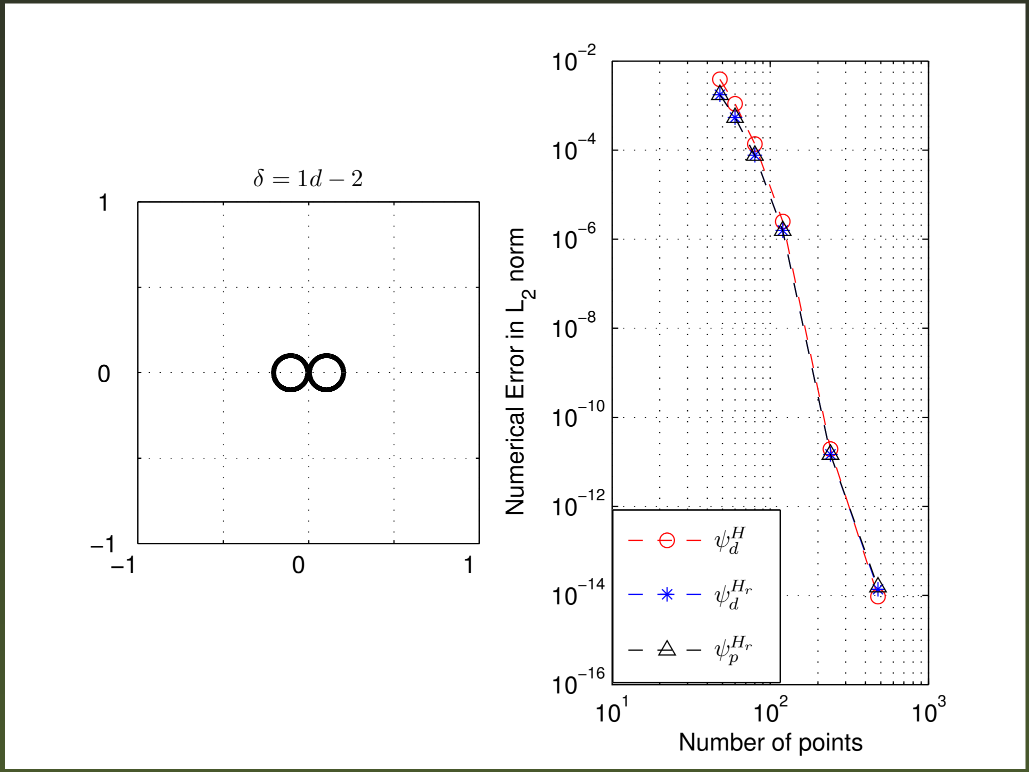
Decomposition theorem says that the operator A on $L^2(\partial B_1) \times L^2(\partial B_2)$ has a very small singular value (of order $\sqrt{\epsilon}$) of **rank 1**!

Implications for computation



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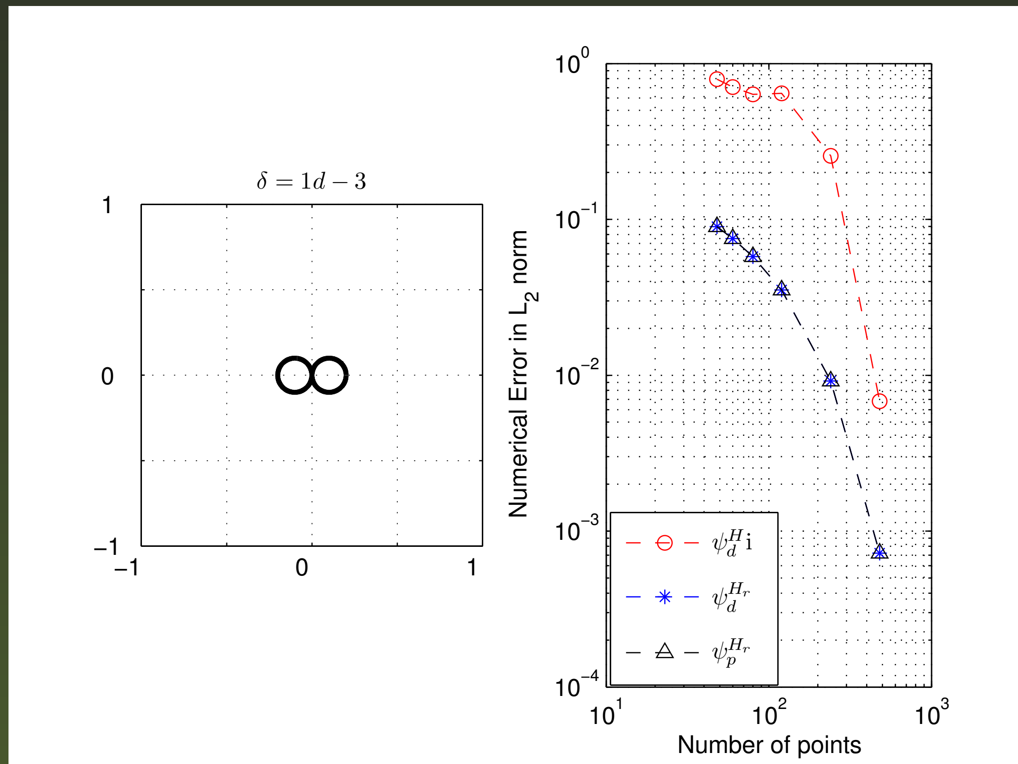

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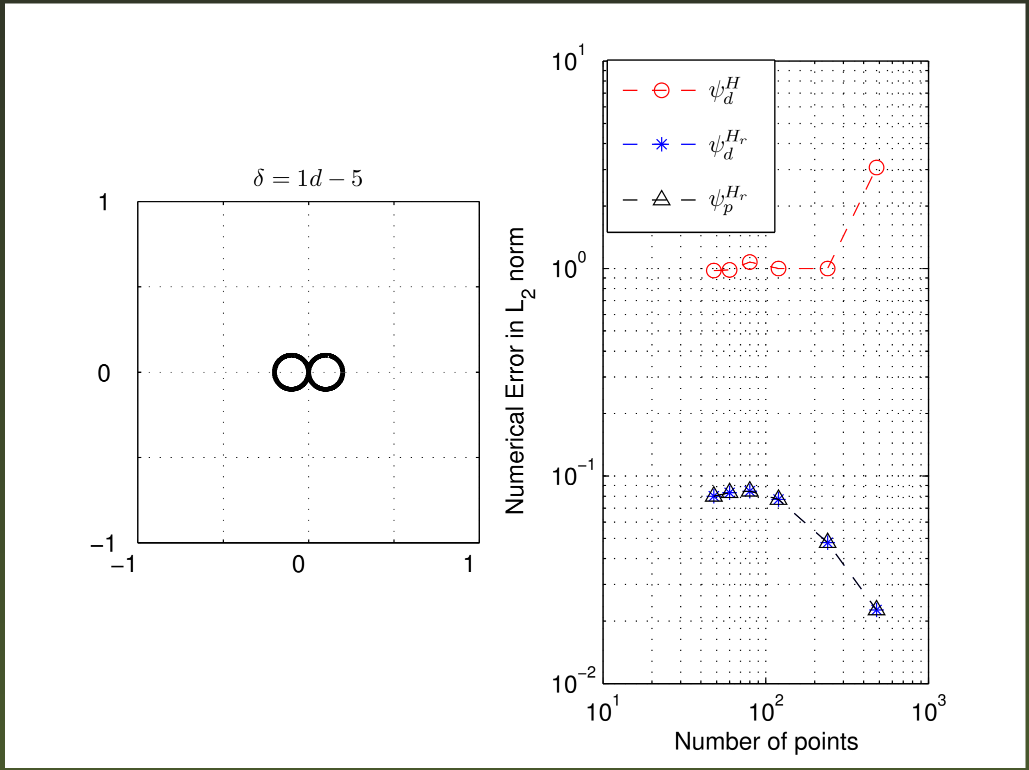

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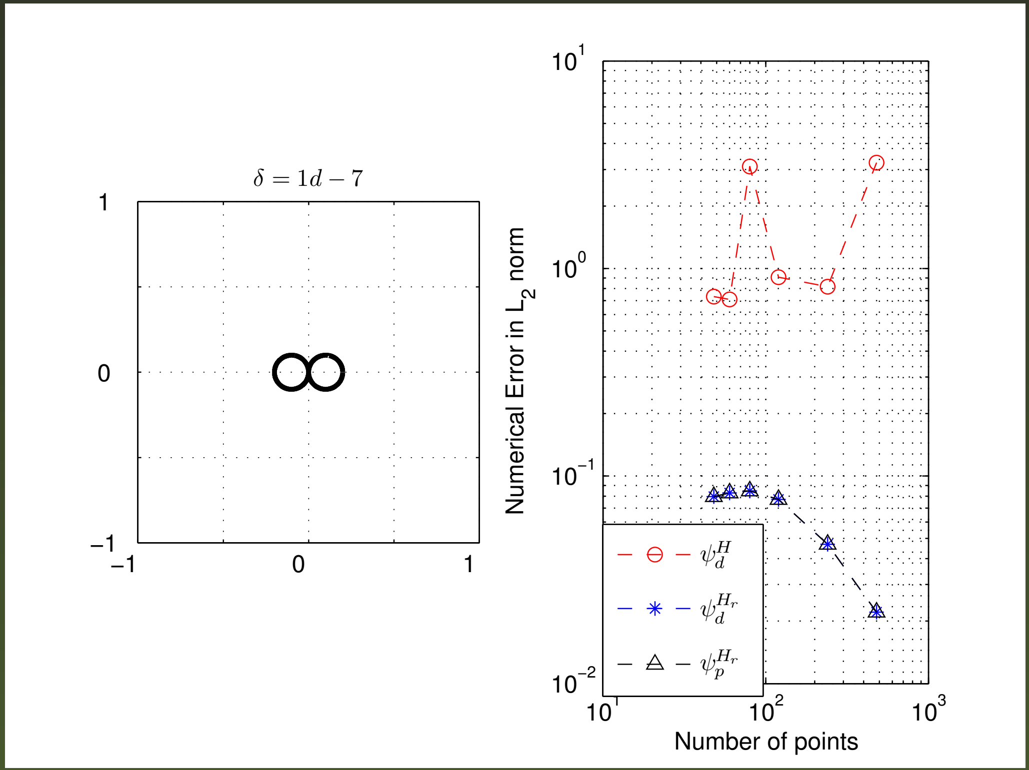

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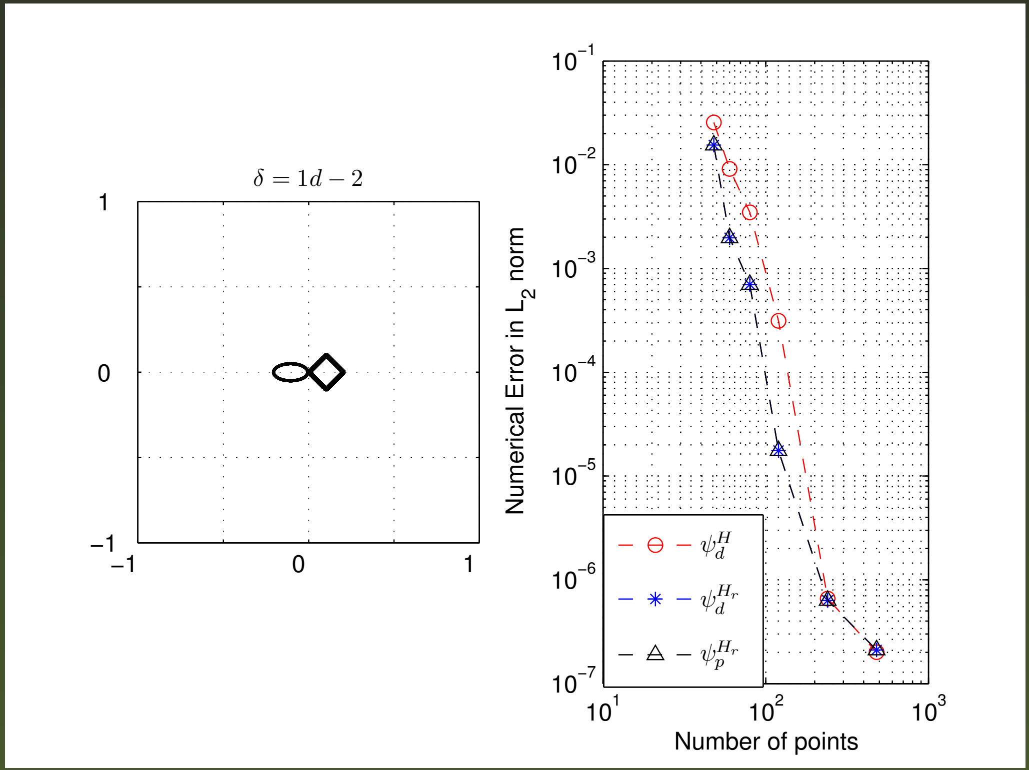

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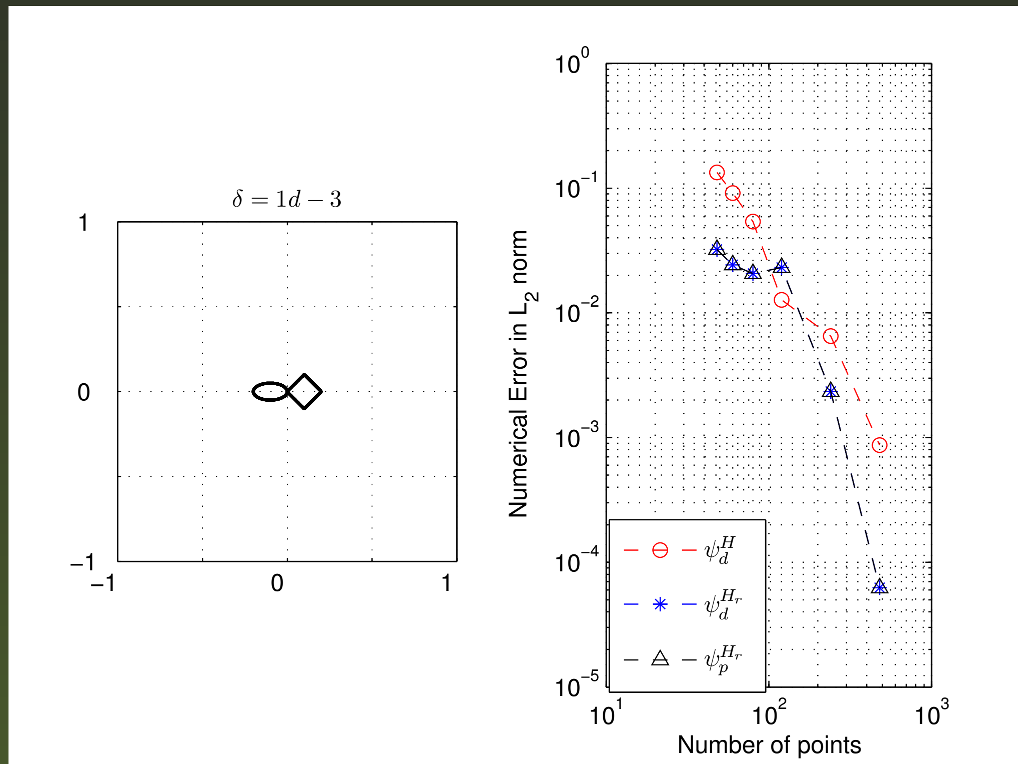

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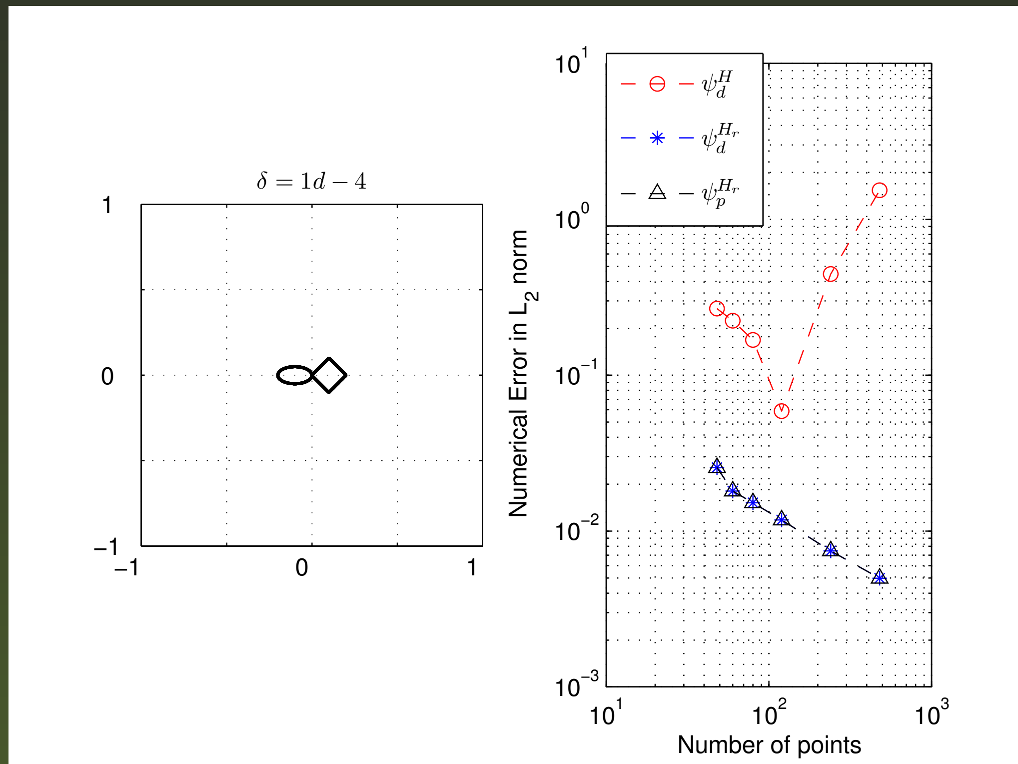

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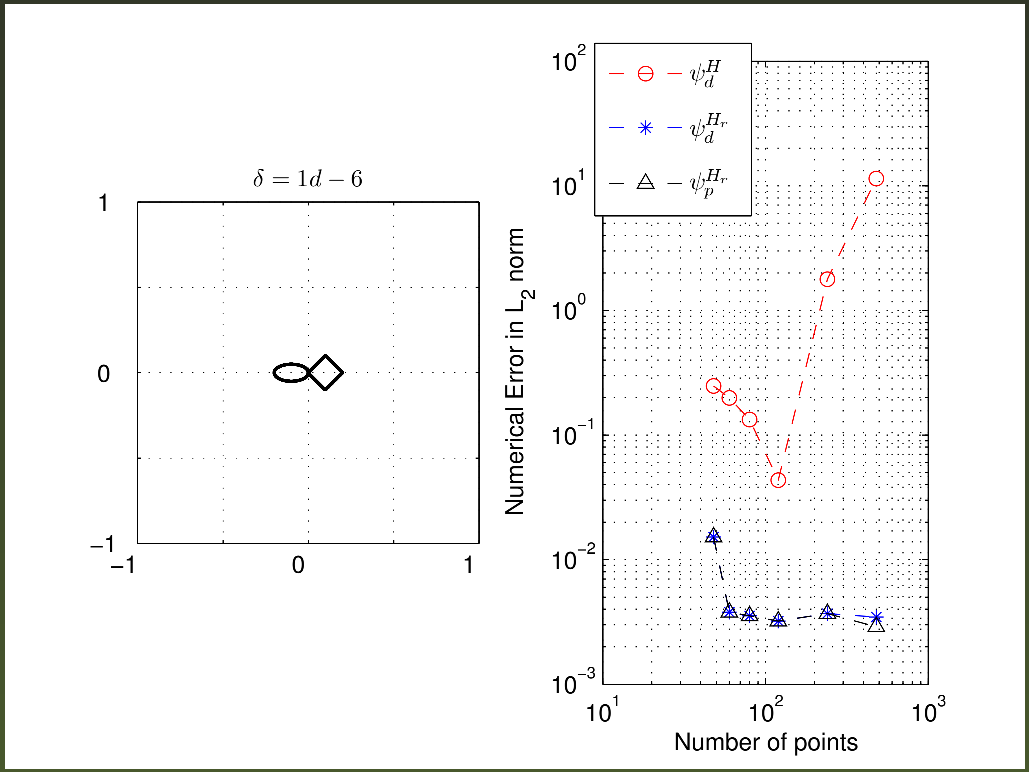

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